International Journal of the Physical Sciences Vol. 6(1), pp. 154-155, 4 January, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS10.658
ISSN 1992 - 1950 ©2011 Academic Journals

Short Communication

On the quasi elliptic functions

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Accepted 31 December, 2010

In this study, the function $\log\eta(z)$ was used instead of function Dedekind's $\eta(z)$. Then the value change of this function with the variables $\frac{z}{2^r}$ was investigated. Hence the quasi elliptic function $\phi_r(z)$ was established.

Key words: Elliptic function, lattice, period.

INTRODUCTİON

We will start our work the following definition.

Definition: A lattice Ω of complex numbers is an aggregate of complex numbers with the two properties:

(i) Ω is a group with repect to addition.
(ii) The absolute magnitudes of the non-zero elements are bounded below, that is, there is a real number $k \succ 0$ such that $|w| \geq k$ for all $w \neq 0$ in Ω (Duval, 1973).

Definition: A double periodic meromorphic function in the open z-plane is called an elliptic function. According to this definition, if a meromorphic function $f : C \to C$ satisfies

$$f(z+2w_1) = f(z)\ ,\ \ f(z+2w_2) = f(z) \qquad (1)$$

then $f(z)$ is called an elliptic function with respect to the periods $2w_1, 2w_2$. Where $2w_1$ *and* $2w_2$ are two complex numbers whose ratio is not a real number and z is a complex variable (Ocak ,1985). We denote 2-dimensionel lattice by

$$\Omega=\{m2w_1+n2w_2 : m,n\in Z,\ 2w_1,2w_2\in C \text{ and } \frac{2w_1}{2w_2}\neq \text{Real}\ \}$$

The Equations (1) are generally written as

$$f(z+\Omega) = f(z)$$

For some constant 2w, a function f(z) is called quasi-periodic and $2w$ is a quasi period if $f(z+2w)$ is some what trivially related to $f(z)$ though not indentically equal to, when the quasi-periods form a double lattice, the function is called quasi-double lattice, the function is called quasi-double periodic or quasi-elliptic (Kaplan, 2004).

Construction of Quasi-Elliptic Function

Here, we give information about the work before and the definition of function Dedekind's η.

Definition

The function

$$\eta(z) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}\left(1-e^{2\pi i z n}\right)$$

is called function Dedekind's η, where $z \in C$ and

$\mathrm{Im}(z) \succ 0$ (Chandrasekharan, 1985). The function $\log\eta\left(\sigma\left(\tau\right)\right)$ was used instead of function Dedekind's – $\eta\left(\tau\right)$ for

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL\left(2, Z\right),\ \sigma\left(\tau\right) = \frac{a\tau + b}{c\tau + d}$$

(Dedekind ,1930). Hence the quasi elliptic function $\psi_r\left(\tau\right)$ was established (Kaplan et al., 2001). Further, on transformation of the function $\log\theta_3$ some results was obtained by Kurt (Kurt ,1995).

Here, $\log\eta\left(z\right)$ function is written as follows:

$$\log\eta\left(z\right) = \frac{\pi iz}{12} + \sum_{n=1}^{\infty}\log(1 - e^{2\pi inz}) .$$

Let

$$\phi_r\left(z\right) = \log\eta(\frac{z}{2^r}) ,\ r = 1, 2, 3, ...$$

Then we have

$$\phi_1\left(z\right) = \log\eta(\frac{z}{2}) = \frac{\pi iz}{2.12} + \sum_{n=1}^{\infty}\log(1 - e^{2\pi in\frac{z}{2}}) \quad \text{for } 1 .$$

Otherwise,we have

$$\phi_1\left(z\right) = \frac{\pi iz}{2.12} - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{2\pi ikn\frac{z}{2}} \tag{2}$$

By using the expansion

$$\log(1 - e^{2\pi inz}) = -\sum_{k=1}^{\infty}\frac{1}{k}e^{2\pi inkz}$$

in (Chandrasekharan, 1985), for $n \in N^+$ and $\mathrm{Im}(z) \succ 0$. Then we expand the first of the double sum (2) to obtain

$$\phi_1(z) = \frac{\pi iz}{2.12} - \sum_{k=1}^{\infty}\frac{1}{k}\frac{e^{2\pi ik\frac{z}{2}}}{1 - e^{2\pi ik\frac{z}{2}}} .$$

In this equation , we take the partial sum of the infinite sum for $n \in N$, $n \prec \infty$ to get

$$\phi_1(z) = \frac{\pi iz}{2.12} - \sum_{k=1}^{n}\frac{1}{2k}\left(\frac{2e^{2\pi ik\frac{z}{2}}}{1 - e^{2\pi ik\frac{z}{2}}} + 1 - 1\right)$$

$$\phi_1(z) = \frac{\pi iz}{2.12} + i\sum_{k=1}^{n}\frac{1}{2k}[i\left(\frac{1 + e^{2\pi ik\frac{z}{2}}}{1 - e^{2\pi ik\frac{z}{2}}}\right) + i]$$

$$\phi_1(z) = \frac{\pi iz}{2.12} - i\sum_{k=1}^{n}\frac{1}{2k}(Co\,\mathrm{t}\,\pi k\frac{z}{2} - i)$$

$$\phi_1(z) = \frac{\pi iz}{2.12} - \sum_{k=1}^{n}\frac{1}{2k} - i\sum_{k=1}^{n}\frac{1}{2k}Cot\pi k\frac{z}{2} \tag{3}$$

If we repeat the same operations for $r = 2$, we hold of

$$\phi_2(z) = \frac{\pi iz}{2^2.12} - \sum_{k=1}^{n}\frac{1}{2k} - i\sum_{k=1}^{n}\frac{1}{2k}Cot\pi k\frac{z}{2^2} \tag{4}$$

Considering the Equations (3) and (4), we can write generelized function $\phi_r\left(z\right)$,

$$\phi_r\left(z\right) = \log\eta(\frac{z}{2^r}) = \frac{\pi iz}{2^r.12} - \sum_{k=1}^{n}\frac{1}{2k} - i\sum_{k=1}^{n}\frac{1}{2k}Cot\pi k\frac{z}{2^r} \tag{5}$$

Since the function $\phi_r\left(z\right)$ is meromorphic in upper half plane and it is quasiperiodic with periods $\frac{2^r}{k}$ and $2^r t$ for $k, n \in N$, $t \in Z$. $\phi_r\left(z\right)$ is a quasi-elliptic function. One can establish an elliptic function by using $\phi_r\left(z\right)$.

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