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A daunting task for pre-service mathematics teachers: Developing students' mathematical thinking

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The purpose of this study was to examine pre-service teachers' teaching practice in terms of providing suitable conditions for developing students' mathematical thinking in the frame of the Advancing Children's Thinking framework. In the study, Advancing Children's Thinking framework developed by Fraivillig et al. was adopted as theoretical framework. Case study was used and participants were determined as four pre-service mathematics teachers. Four lessons for each participant were observed via video camera. The data were analyzed by using descriptive analysis technique within framework components. It was found that pre-service mathematics teachers often elicited students' mathematical thinking but less often supported and extended. Although they had some theoretical knowledge about the mathematical thinking, they reflected this knowledge in practice for the first time. In this sense, it can be said that the pre-service teachers made important efforts in the development of the mathematical thinking and tried to realize a suitable instruction in the context of the framework.

Key words: Mathematical thinking, teaching, pre-service mathematics teacher, developing mathematical thinking.

INTRODUCTION

Mathematical thinking (MT) is considered one of the most important targets in mathematics education. Various definitions of MT have been put forward by different researchers. For example, Liu and Niess (2006) define MT as a combination of complicated processes involving guessing, induction, deduction, specification, generalization, analogy, reasoning, and verification. According to Mason et al. (2010) MT is a dynamic process which, by enabling us to increase the complexity of ideas we can handle, expands our understanding. Wilson (1993) states that MT involves using mathematically rich thinking skills to understand ideas, discover relationships among the ideas, draw or support conditions about the ideas and their relationships and solve problems involving the ideas (cited in Lutfiyya, 1998, p. 55-56). By considering these definitions, MT can be defined as a dynamic process that expands our understanding and involves using mathematically rich thinking skills such as guessing, induction, deduction, specification, generalization, analogy, reasoning, and verification. Burton (1984) claims that MT is not thinking about the subject matter of mathematics but a style of thinking that is a function of particular

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operations, processes, and dynamics that are recognizably mathematical. Therefore, it can be said that MT is a skill that an (any) individual, not only mathematicians, should gain.

MT is one of the basic skills emphasized by standards and programs developed for mathematics learning and teaching. The National Council of Teachers of Mathematics (NCTM) (2000) states that, just as the level of mathematics needed for intelligent citizenship has increased dramatically, so too has the level of MT and problem solving needed in the workplace and in professional areas ranging from health care to graphic design. This change has also taken place in the objectives of mathematics education in the mathematics curriculum in Turkey.

In 2005, a radical change was made in the Mathematics Curriculum in our country. A change towards a modern approach from the traditional approach was implemented in the program. In 2011, the program was revised considering the problems in practice. MT has been incorporated into the skills targeted to be developed by this curriculum (Ministry of National Education [MNE], 2005, 2011).

It is stated that the activities teachers bring to practice in the classroom within the framework of the mathematical instruction program must be towards students' gaining high level MT skills, such as analyzing, synthesising, assessment, connection, classification, generalization and deduction (MNE, 2005). Moreover, the statement in the MNE (2011) mathematics course instruction program goals "*MT, problem solving, association, being able to use mathematics as a language of communication and modeling skills are the basic elements of learning and doing mathematics.*" shows the emphasis on MT in the program.

Besides, focusing on the importance of MT; according to NCTM (2000), effective teaching includes observing the students, listening to their ideas and explanations carefully, having mathematical goals, and using this knowledge when taking instructional decisions. Teachers using these applications motivate the students to engage them in MT and reasoning and provide learning opportunities that challenge students at all levels of understanding (NCTM, 2000, p. 19).

Even and Tirosh (2008, p. 219) report that *"It is widely accepted today that teachers should be aware of and knowledgeable about students' mathematical learning. It is believed that such awareness and knowledge significantly contribute to various aspects of the practice of teaching."* As it is seen in the literature, understanding students' (mathematical) thinking is important, therefore it is of importance too how well teachers can do this in their teaching.

Teachers' knowledge of students' thinking

Teachers' knowledge of students' (mathematical) thinking has drawn the interest of many researchers (An et al., 2004; Ball et al., 2008; Grossman, 1990; Hill et al., 2008; Kovarik, 2008; Magnusson et al., 1999; Park and Oliver, 2008; Shulman, 1986; 1987). Shulman (1986) stated that this knowledge includes an understanding of what makes learning a specific topic easy or difficult, and the conceptions and preconceptions that students of different ages and backgrounds bring with them to those most frequently taught topics and lessons. Ball et al. (2008) defined one domain of teacher knowledge as *knowledge of content and students.* They stated that this knowledge combines knowing about students and knowing about mathematics. According to Ball et al. (2008), this domain of teacher knowledge includes anticipating what students are likely to think and what they will find confusing; predicting what students will find interesting and motivating when choosing an example; anticipating what students are likely to do with it and whether they will find it easy or hard; being able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language when assigning a task.

An et al. (2004) classified knowing students' thinking in four categories: *Addressing students' misconceptions, engaging students in math learning, promoting students' thinking mathematics, building on students' math ideas.* According to these authors, an effective teacher attends to students' MT: preparing instruction according to students' needs, delivering instruction consistent with students' levels of understanding, addressing students' misconceptions with specific strategies, engaging students in activities and problems that focus on important mathematical ideas, and providing opportunities for students to revise and extend their mathematical ideas (Kulm et al., 2001; cited in An et al., 2004, p. 148).

Hughes (2006) emphasizes that *"teachers should have knowledge of how students think about and learn specific mathematics content; including knowledge of how students acquire new mathematical content, the possible solution strategies or processes students might employ, and the likely preconceptions and misconceptions that students will have"* (Hughes, 2006, p. 3). No matter what components are dealt with, for effective teaching teachers' understanding and attending students' MT is critical. Franke and Kazemi (2001) see focusing on students' MT as a powerful mechanism for bringing pedagogy, mathematics and students understanding together. Because if teachers had knowledge of students they would use it in their instructional decision-making, so that learning would be improved (Fennema and Franke, 1992). Cooper (2009) indicates that the teacher can arrange a more individualized education and thus increase the learning of the students by focusing on their MT. Crespo (2000) also suggests that analyzing the students' MT will help the teachers in taking more appropriate decisions and developing their practice in their classrooms.

In this context, it can be said that teachers must know the students' mathematical ideas and develop instruction within the frame of these ideas (Olkun and Toluk, 2004). Although teachers' understanding of and attending to students' MT is essential for effective teaching, they have some difficulties in using this knowledge in their teaching process (Chamberlin, 2002; Hughes, 2006). Even, Hughes (2006) stated that teachers who have this knowledge find it challenging to make use of it in the process of teaching.

Researchers put forth that the interest of pre-service teachers in the students' MT also contributes to the development of their teaching. For example, taking an interest in students' MT allows pre-service teachers to question their mathematical knowledge and learning (McLeman and Cavell 2009; Philipp, 2008). However, it has been observed that even though teachers who successfully make use of students' thinking in their teaching process are expert teachers, for beginner teachers it is seen as a daunting task (Hughes, 2006). We are of the opinion that it would be useful to make pre-service teachers practice and let them think over those practices to be successful in this challenging task before they start first year of professional teaching. This is the duty of the institutions which train the teachers.

THEORETICAL FRAMEWORK

Fraivillig et al. (1999) presented and described a pedagogical framework supporting the development of conceptual mathematical understanding of the students in their study. They synthesized Advancing Children's Thinking (ACT) framework from an in-depth analysis of observed and reported data from one skillful first grade teacher. Then this framework became a guide for the authors to make a cross-teacher analysis over five additional first grade teachers.

Cengiz et al. (2011) also used the ACT framework to build a new framework for examining whole-group discussions based on students' existing mathematical thinking. They focused on how teachers' mathematical knowledge for teaching supports them in their efforts to extend students' thinking. They examined the teaching of six experienced elementary school teachers and found that all teachers created opportunities for extending student thinking about important mathematical ideas and solution methods during group discussions. Bobis et al. (2005) derived from ACT framework to create a professional development program in their larger Project study. They found that the teachers identified considerable personal professional growth in their knowledge of children's learning in mathematics and and an understanding of how such growth could be facilitated.

As can be seen, ACT framework was used to create new frameworks or examining/developing teachers' knowledge in the previous studies. Differently in this study, we adopted the ACT framework as the theoretical framework both to support pre-service teachers' reflection on their teaching practice and to assist the researchers organizing the data. This framework was preferred because it not only suggested that students' MT should be developed and supported but also revealed a concrete way in which the teachers could manage to do this.

The framework consists of three separable, though overlapping components. The first component of the ACT framework is Eliciting Children's Solution Methods. Fraivillig et al. (1999) describe eliciting as "the teacher's efforts to provide students with the opportunity and necessary encouragement to express their ideas about mathematics". The second component is Supporting Children's Conceptual Understanding. This component is about teacher's pedagogical decisions and treatment of elicited responses. It regards the instructional strategies that support students in carrying out solution methods. The third component of the ACT framework, Extending Children's MT, is a bit different from the first and the second component, because "*The instructional components of Eliciting and Supporting involve instructional strategies for accessing and facilitating children's thinking about solution methods with which they are already familiar"* (Fraivillig et al., 1999). The authors describe that these framework components hadn't captured methods teachers employ to challenge or extend children's thinking. The strategies for advancing children's progress through their zones of proximal development (Vygotsky, 1978), in other words, areas in which they could learn with assistance, comprise the third component of the framework (cited in Fraivillig et al., 1999, p. 160). According to the ACT Framework, the instructional strategies which must be used by the teacher for developing students' MT in a questioning classroom environment revealing the thoughts and solutions of the students are presented in Table 1.

Purpose of the study

The purpose of this study was to examine pre-service mathematics teachers' teaching practice in terms of providing suitable conditions for developing students' MT in the frame of the Advancing Children's Thinking

Table 1. Examples of instructional strategies of ACT framework (from Fravillig et al.,1999, p. 155).

Eliciting	Supporting	Extending
Facilitates students' responding Elicits many solution methods for one problem from the entire class Wait for and listen to students' descriptions of solution methods Encourages elaboration of students' responses Conveys accepting attitude toward students' errors and problem solving efforts Promotes collaborative problem solving **Orchestrates classroom discussions** Uses students' explanation for lesson's content Monitors students' levels of engagement Decides which students need opportunities to speak publicly or which methods should be discussed	**Supports describers' thinking** Reminds students of conceptually similar problem situations Provides background knowledge Directs group help for an individual student Assists individual students in clarifying their own solution methods. **Supports listeners' thinking** Provides teacher-led instant replays. Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method **Supports describers' and listeners' thinking** Records symbolic representation of each solution method on the chalkboard Asks a different student to explain a peer's method Supports individuals in private help sessions Encourages the students to request assistance (Only when needed)	**Maintains high standards and expectations for all students** Asks all students to attempt to solve difficult problems and to try various solution methods **Encourages mathematical reflection** Encourages students to analyze, compare, and generalize mathematical concepts Encourages students to consider and discuss interrelationships among concepts Lists all solution methods on the chalkboard to promote reflection **Goes beyond initial solution methods** Pushes individual students to try alternative solution methods for one problem situation Promotes use of more efficient solution methods for all students Uses students' responses, questions, and problems as core lesson Cultivates love of challenge

framework?

Method

Case study is preferred when "how" or "why" questions are posed, when the investigator has little control over events, and when the focus is on a contemporary phenomenon within some real-life context (Yin, 1984, p. 13). In this study, it was searched that how four mathematics pre-service teachers tried to improve students' MT in their teaching experience. So case study was chosen from among the qualitative research methods for use in the study.

Participants

Convenience sampling was used in the determination of the participants. Participants of the study were determined as four volunteer pre-service mathematics teachers receiving education in the senior class of a faculty of education Academic grade point average (GPA) of the participants in courses regarding content knowledge (pure mathematics courses), pedagogical content knowledge (courses regarding teaching and teaching mathematics) and general academic GPA are given in Table 2. This information is given because it will be helpful for us to discuss the results of the study.

Participants were told that their real names would be undisclosed and were asked to determine pseudonyms for themselves. Only Aslı determined the pseudonym for herself, others stated it would not matter which pseudonym was used. So the authors decided to use the pseudonyms "Ege, Aslı, Arda and Irem".

Procedure

In the faculty of education, where this research was conducted, pre-service teachers took courses regarding teaching the field (mathematics) such as Special Teaching Methods, Instructional Technology and Material Design, Mathematical Thinking, Mathematical Modeling. The Mathematical Thinking Course was given in the last term of the teacher education program, three hours in a week. They also took two courses regarding classroom practice. In the first term of their senior class they took "School Experience". In this course they went to secondary schools and observed their mentor teachers for four hours a week. Then they came to the faculty and shared their observations. In the second term they took the course called "Teaching Practice". In the context of this course pre-service teachers went to secondary schools and at first, observe their mentor teacher for six hours a week. Then they planned and taught their lessons. Due to the big numbers of pre-service teachers going the same school, one pre-service teacher could teach his/her lesson once in a term, usually for four hours.

This study was carried out within the frame of teaching practice within the scope of the Teaching Practice Course of pre-service teachers at an education faculty in Turkey. Ege, Arda and Irem went to an Anatolian High School and Aslı went to a Vocational

Table 2. Grade point average of pre-service mathematics teachers.

Pseudonyms (gender) GPA	Ege (male)	Aslı (female)	Arda (male)	Irem (female)
GPA of the coursesregarding content knowledge	2,48	3,23	2,80	1,78
GPA of the coursesregarding pedagogical content knowledge	3,38	3,71	3,60	3,71
General academic GPA	2,82	3,44	3,06	2,49

High School as training schools. Students aged between 14 and 17 study at these schools. There are different types of schools for this age group in Turkey. In order to gain entry to these schools, students must pass the entry examinations. The type of high school students enter is based on the scores they receive from the nationwide common exam by the Ministry of Education and schools they prefer. School types are (from high to low according to scores) science high schools, social sciences high schools, Anatolian high schools and vocational high schools. At the beginning of the study, individually semi-structured interviews were performed with pre-service teachers about MT and mathematics teaching. Subsequently, participants and two of the researchers came together and discussed the answers given by pre-service teachers to the interview questions.

The purpose of this discussion was to support the pre-service teachers in terms of theoretical knowledge about MT and to reach consensus how to develop it while teaching. Then two exemplary videos of math classes were watched and participants were asked to evaluate these lessons in the context of MT.

The purpose of this group meeting was to provide participants with a consensus on MT and its importance for math education. Later on, a section of the studies about MT was presented to the pre-service teachers. In this scope, the ACT Framework was also presented to them. Following that, pre-service teachers were asked to prepare a lesson plan (4 h lessons) that developed students' MT. Each pre-service teacher examined his/her plan by meeting with a researcher when he/she prepared the lesson plan, and made some changes in line with the feedback. Then, pre-service teachers performed the lessons they prepared at their training schools. These four lessons were observed by using a video camera. These four hours were not one piece. Mathematics lessons are usually taught for two hours or one hour in a day in Turkey. So the observed four lessons were as 2+2 or 2+1+1. After they had completed their teaching, a researcher watched the videos and re-interviewed the pre-service teachers and asked them to evaluate their performance.

The focus of this study was observations of the pre-service mathematics teachers' teaching practice. Ege carried out the instruction of Conics in the 11th class at an Anatolian High School. Aslı performed the topic of Inverse Trigonometric Functions in 10th class at a Vocational High School. Arda taught matrices in an 11th grade class in an Anatolian High School. Irem, taught the subject of Trigonometric Functions in a 10th grade in an Anatolian High School.

Data sources

Data of the research were collected by means of observations. Four lessons taught by each of the participants were observed via a video camera.

Data analysis

Data obtained from observations were analyzed by using the descriptive analysis technique in the frame of the ACT Framework developed by Fraivillig et al. (1999). Firstly, we watched the lessons individually and took notes in accordance with the framework, then came together to discuss coding and reached a consensus. We have assessed the courses separately for three components of the ACT framework and examined each component in terms of their sub-components. When required, we presented some sections of the dialogues and screen quotations.

Validity and reliability

In this research, even though we only focused on the observations, collecting data by using interviews and observations provided data triangulation as well as evidence for the validity of the research. The observations were directly conducted by one of the researchers to create a valid and reliable class environment. In the results section, direct quotations were also given to present evidence for the reliability of the research.

RESULTS

Pre-service teachers' teaching practice in the frame of developing students' MT

In this section, results were presented case by case in the scope of framework components.

Case of Ege

Eliciting the solutions of the students

Generally, reaching only one solution for one question was sufficient for Ege. So he did not reveal different solutions during his instruction; he did not elicit many solution methods for one problem from the entire class. When Ege asked the students questions or asked a student to come to the blackboard, he gave the student a sufficient amount of time to explain his/her thought or solution and listened to them. He waited for and listened to students descriptions. However, when the student could not explain a particular point he directly explained what to do. Furthermore, he sometimes helped students who had difficulty in determining the type of cone by indicating the graphic of the cone with his hand. Also, he did not support the students in explaining their thoughts in detail and did not attempt to elicit further explanations from them. He did not question the answers by asking why or how, and focused only on the correct answer. A part of the instruction of Ege for this situation is as follows:

Ege: I drew two cones; they are symmetrical according to this point. Right cone, its base is circle. If we intersect it with a plane like this (*shows by hand*), what can we obtain? What kind of shapes?

Student 1: Triangle

Ege: Triangle?

Student 1: Can't we obtain a triangle?

Student 2: Ellipse is obtained.

Student 1: Ellipse, sorry, ellipse is obtained.

Student 3: Circle

Ege: Circle, we obtain a circle of the simplest form. Now I'll intersect this with a plane. If I intersect it with a plane parallel with the circle on that base (*drawing*), of course our drawings are not that good, we were relying on the projection, but we'll continue anyway, I'll obtain a circle. If I intersect it in a manner parallel to the base, again, likewise, I'll obtain a circle since I have a circle at the base. Only its radius will be smaller. What else? This is a parallel intersection. What if I intersect it with a little slope?

Student 2: Semi thing... ellipse

Student 4: Trapezoid.

Ege: Let's extend it like this, guys (*draws an inclined planed intersecting the conic*). This time, it becomes an ellipse guys, even if we do not see it visually. Something like that will occur (*drawing an ellipse*). The first one, the previous one was like that (*drawing a circle*) this also seemed like an ellipse anyway but I tried to draw a circle below. The first one is a circle and the second one is an ellipse. It is somewhat elliptical, only a little more oblate than the circle.

In this dialogue, it is seen that Ege did not question student answers that were wrong, such as triangle and trapezoid, and continued his lesson by considering correct answers like circle and ellipse. It can be said Ege did not encourage students to elaborate on their responses. Ege gave responses to student questions with alternative explanations during his lessons. He continued his explanations until clearing the confusion in the minds of the students. However, he did not support the students in reaching the correct answer on their own. This situation can be considered as an indicator of an accepting attitude toward students' errors and problem solving efforts in eliciting the component of the ACT. Furthermore, Ege exhibited an approach supporting the collaborative problem solving in his teaching. However, he conducted only one group work session during his four lessons. He allowed the students to work in groups consisting of three and four persons by distributing work sheets containing the questions and some graphics provided for the solution.

He started exercising this group work to allow the class to question whether the cones have common characteristics. However, no relation could be established with this purpose in the examination phase of the questions. Ege used student explanations for the content of the lesson and continued the lessons by focusing on the comments of those who gave a correct answer. Ege did not determine the participation levels of the students. When a question was asked or a wrong answer was given to the question, he did not orientate the students towards thinking about the question or the thought. He preferred giving the correct answer himself. This also prevented the entire classroom from engaging in the lesson.

Therefore he did not monitor students' levels of engagement. He also tried to bring different students to the blackboard; however, since the students did not volunteer, he conducted his lessons with actively and voluntarily participating students. Ege was not successful in deciding which students need opportunities to speak publicly or which methods should be discussed.

Supporting the students' conceptual understanding

Ege was content with showing only one solution to the questions he solved or he wanted students to solve during his instruction. He did not make any comments about whether different solutions existed. That is to say, he did not induce students to perceive that there may be different solutions. He did not ask whether anyone had a different solution either. Ege made instant replays on points needed by students during his lessons. The information he highlighted most frequently was the determination of the type of cone according to the value of the eccentricity. An exemplar video part for this situation is below:

Ege dictates a question to the students. The student cleaning the blackboard notes the data given in the question on the blackboard: "*Please determine the type* *of cone with focus F (-3, 2), directrix 3x-2y-6=0, and passing through point P (0,6)."* The student draws a coordinate axis on the blackboard. Ege again summarizes the data given in the question and directs the question to the classroom.

Ege: Just remember, how do we determine the type of conicity?

Student: Now it has a focus, it has a directrix, so this is an ellipse.

Ege: You can't know. It may be hyperbola, parabola. As you see, during the previous lesson, it is the most important one of the section we've seen until now.

Another student: You see, we were telling it by looking at "e (eccentricity)".

Ege: We were looking at the eccentricity. What was the eccentricity? It was the proportion of the distances from the focus and from the line of the point.

This part is an example of the evidence of the fact that Ege highlights previous knowledge with instant replays as well as the approach of non-consideration of student's wrong answer and not helping the student in explaining his/her individual thought indicated by eliciting component. It cannot be said that he encouraged the students a lot to ask for help when they needed it. He monitored student progress only by using questions such as "*Do you understand?*" and also gave answers to individually asked questions. A comfortable environment could not be created for the students in terms of asking whether they understood it or not.

Extending the students' MT

When Ege's instruction was analyzed within the frame of the ACT, positive findings could not be obtained for instructional components at extending level; because Ege did not ask students different and challenging problems and did not encourage them to think from different aspects during his lessons. He did not give students the opportunity to analyze, compare or generalize mathematical concepts. He played an active role in reaching the general equations of cones, making comparisons between conic types, but did not ensure the participation of the students. He asked questions such as "*What is a circle?*", "*What is a geometric locus?*" in the first lesson to establish relations between the concepts; however, when he could not get an answer, he made the definition of geometric locus, circle, and line without changing tack. Subsequently, he went on to talk about cones and explained that the circle and the line are also a cone. He tried to correlate the concepts of ellipse and circle. However, here again he explained the relation without compelling students to think.

Case of Aslı

Eliciting the solutions of the students

In her lessons Aslı did not give the solution herself when studying on a question or a problem and wanted students to share their solutions. She tried to elicit different solutions for one problem from the entire class by means of questions such as "Who solved it in a different way?", "Did anyone do it differently?" She asked if there were different solutions to the solution of the student she brought to the blackboard, and if any, she wanted the students to share them. For example, after having examined if the function f: $\mathcal{R} \rightarrow [-1, 1]$, f(x) = sinx whose graphic was given in her first activity is a bijection, she moved on to the question "*Is there any interval where this function is bijective? If any, please show it*". The student she brought to the blackboard wrote: $[0, \pi] \rightarrow [-1, 1]$.

Student 1: Is it right, teacher?

Aslı: If you thought something different, come and write that, too.

Student 2: Teacher, my friend has also done it as $[\frac{\pi}{2}, \pi]$.

Student 1: $\frac{\pi}{2}$ is also there, teacher.

Aslı: Okay. You come and write it too, let's have a look and see if it's correct.

Student 1: No teacher, no need if it's correct.

Student 2: Please tell me teacher, is it correct?

Aslı: Guys, if you're making another interval, let's talk about that, too. For example, did you say $[\frac{\pi}{2}, \pi]$?

Aslı wanted the students to explain the solution, waited for and listened to them. She always questioned the answers given by the students and expected detailed explanations from them. So she could encourage the students to elaborate on their responses. She asked questions such as "*Why yes?*" or "*Why no?*" to the students giving yes/no answers. She did not directly say correct or wrong in response to students' answers and appreciated all of the opinions. Thus she was able to determine what the students thought and to take measures against possible mistakes. She listened to the explanations of the students giving wrong or irrelevant answers and made remedial explanations to eliminate the existing difficulty. She provided a comfortable classroom environment for the students so that students could ask about points they did not understand without hesitation with questions such as "*Has anyone had any difficulties so far?*", "*Is there a point you haven't understood?*" Aslı's approach also showed that she has an understanding attitude towards student mistakes. She conveyed an accepting attitude toward students' errors and problem solving efforts. During her lessons, Aslı motivated the students in a collaborative working environment with four activities and one worksheet. During this process, she continuously walked between the desks. She took care of almost all of the groups, answered the questions, and guided the groups in reaching solutions. Questions included in the activities focused not only on the operational skills of the students but also their conceptual knowledge. She shaped her lessons according to the approaches of the students and used students' explanations for the content of the lesson. Aslı tried to engage the students in the lesson by using expressions such as "*Look at the blackboard, did you do it like that?*", "*Are you thinking as your friend thinks?*" So she could monitor students' levels of engagement. She was careful to bring different students to the blackboard to show the solutions or explain their opinions so that every student had the chance to speak.

Supporting the students' conceptual understanding

In her lessons, Aslı lead students to establish interrelations in the definition of inverse functions of sine, cosine, tangent, and cotangent functions and reminded them of conceptually similar aspects. For example, she expected from the students to learn arcsine function to write x = arccosine y if y = cosine x for arccosine function. She called a student to the blackboard. The student wrote y^{-1} under y = cosine x expression, and then arccosine and then arccosine y after a warning from Aslı. When the student got stuck on this section, the class shouted out $f^{-1}(x)$ to help them. Then the student wrote arccosine y = -x. Meanwhile Aslı made the following explanation by noticing that the student was experiencing difficulty: "*What's going on guys? (The student) she changed its place. What were we doing while writing the definition and the range sets? What did we do while writing its inverse? We've changed the place of the function. We're also changing the place while writing these."* Aslı reminded the students of previous (background) knowledge when necessary. For example, at the beginning of the lesson she started a classroom discussion about what conditions must be satisfied so that inverse of function can exist. And then, she found it necessary to remind them what the function was. However, in the 4th lesson, she created a discussion environment about how the factorization while transitioning to sum and difference formulas can be used in trigonometry.

Also, since the students could not continue to study because they could not remember the Sine Theorem in the 4th activity, she reminded them of the Sine Theorem by calling a student to the blackboard and guiding the student. She helped each student in the explanation of individual solutions in the discussion of the activities by the classroom. She also made instant replays in line with the explanations or questions of the students. Aslı did not adhere to only one solution and expressed that she's open to different solutions during her instruction.

She showed her own solution where students experienced difficulty. She ensured that all of the students see the different answers she got verbally from the students by noting them on the blackboard. She frequently asked whether there is any point that is not understood and encouraged the students to ask for help when they needed.

Extending the Students' MT

It was observed that Aslı confronted her students with questions that might be different for them and of a type they are not accustomed to in the activities. In this sense, the questions were challenging for her students. Aslı asked each student to solve these questions. She supported the students in trying ways that might compel them individually. She took the answers and the solutions of the students to be the center of the lessons and guided her lessons in this direction. She encouraged the students to analyze the concepts, to make comparisons, and to generalize during her instruction. At the same time, she supported the students in establishing relations between the concepts. For example, she tried to enable the students to reach the sum formula for the sine function in her 4th activity.

Here she asked the students to find the area of OAB, OAP, and OBP triangles with the help of the Sine Theorem and to show the relation between these areas. Thus, a formula for sine (α+β) was obtained together with the students (Figure 1).

Case of Arda

Eliciting the Solutions of the Students

Arda shared multiple ways of reaching a solution to a problem in the classroom. He supported students' different ways of reaching a solution. For instance, in finding a determinant of a matrix, he demonstrated both his solution and two other students' solutions. Also, he waited for the students to explain their solutions regarding the questions they asked and he listened to them. However, he did not encourage the students to explain their responses in detail.

Arda had a tolerant approach towards students'

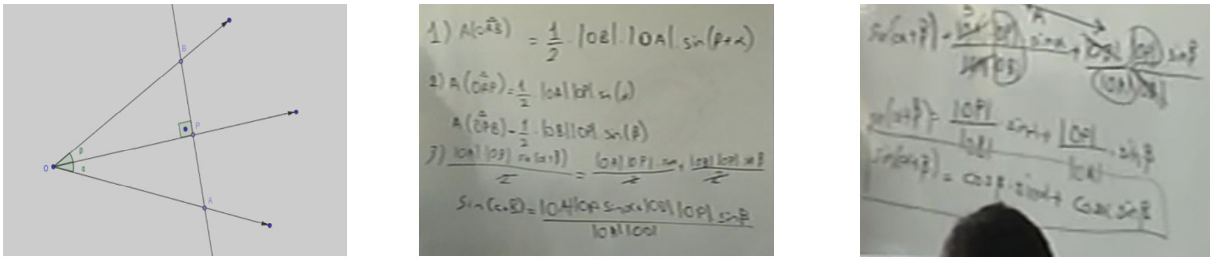


Figure 1. A figure and screen extractions from an activity performed by Aslı.

mistakes and challenges in his class. For instance, he asked the students to find inverse of the matrix in his first class.

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here, students needed to multiply the two matrices initially. At this point, Arda asked the students whether they had difficulty in the multiplication operation. When one of the students stated that he was confused, Arda said: "*Then, we'll practice solving the problem with you*".

Arda supported collaborative work in his classes. He divided the class into groups and let them do worksheets. Also, Arda shaped his class with student explanations that arose from time to time in classroom discussions he created. For instance:

Arda: In the end, the 2x2 matrix transformed the square into a rhomboid. Then, what can we name this matrix?

Student 1: A transformable matrix.

Arda: Here, is the matrix transforming or transformer?

All students replied with transformer.

Arda: Now, friends what does the C point refer to?

Student 2: Vector.

Arda: It also indicates a vector. What have we done? We rotated or pushed that vector with that matrix. Whichever was appropriate? Then, our 2x2 square matrix turned a point on the line into another point.

All through his classes, Arda tried to engage the students in the class with the question-answer technique. He let different students show their explanations or solution ways. In particular, he supported the students who were having difficulty in understanding the subject.

Supporting the students' conceptual understanding

Arda tried to remind the students of conceptually similar problems. For instance, while they were trying to work on how to find the inverseof a matrix, he asked "*How do you find inverse of 5 in multiplication?*" When students replied with 1/5, Arda asked why they did it that way. However, without waiting for students to think and reply, he gave an explanation. "*For instance, what should we multiply by five to get the unit element. What do we need for this? 1/5. Then, here, with the same rationale, we will try to get the inverse matrix*".

Meanwhile, Arda reminded the students of previous knowledge. For example, while trying to show that determinant is a rule of function that matches the set of real numbers with the set of matrix, Arda reminded the students of the concept of function.

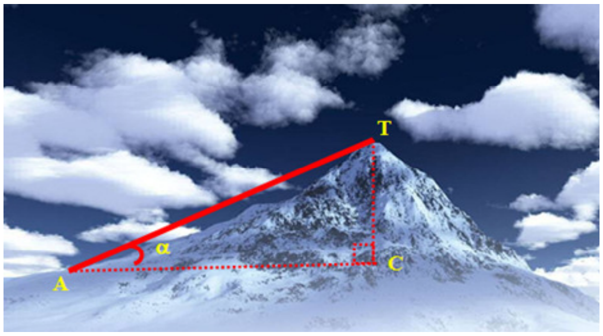
While performing group studies, he did not guide the individuals in the group to help each other. During his teaching, no different individual solution was offered by the students. Student solutions were generally as expected.

Arda helped the students explain their solutions. When students struggled to reach a solution, or for the purpose of reinforcing some information, he made repetitions. In finding a determinant of a matrix, although he showed two students' solutions on the board, Arda showed his own solution to the students, too. This approach is an example that Arda shows the solution he had chosen without using single method.

Arda did not ask a different student to explain his friend's solution. With some statements like "*Is there anyone who is having trouble in multiplication in matrix?*", "*Let's deal with anything you do not understand right now?*" he encouraged students to ask for help whenever they needed or whenever they had a problem.

Extending the students' MT

Unlike the traditional approach, Arda made sure the students arrived at the information themselves. In this respect, he encouraged students to analyze the concepts and make comparisons and generalizations. For instance, when Arda asked how the inverse of amatrix could be found, a student replied immediately. Arda said instead of



Mr. Ahmet is located at point T. The height of the mountain peak is 2500 m. His friend Mehmet is at Point A and his height from sea level is 1850 m. Since Mehmet at Point A looks at his friend Ahmet at a 12 degree angle, how do we find the distance between A and T?

Figure 2. An activity from Irem's lesson.

providing the answer from memory, they would focus on why. Before finding the inversematrix, Arda made them work on an example. The example he gave later was about a matrix withoutthe inverse. When they saw this matrix did not have aninverse, Arda asked the students to find a general statement for any 2x2 matrix. He showed that some matrices did not have an inverse. This kind of teaching, which the students were not familiar with, was also challenging for them. In this respect, Arda's approach in classes can be regarded as a positive finding for the sub-component asking students to work on solving difficult problems and try different solution methods. Moreover, Arda made necessary studies for students to consider the relationships among the concepts. For instance, he encouraged students to notice the relationships of determinant and function concepts and matrix and transformation.

Case of Irem

Eliciting the solutions of the students

Irem could not show multiple solutions to a problem. For instance, she asked the students to find the angle she put on a mountain in one of her activities (Figure 2).

She called on of the students to the board for this activity. When the student stated that he found the tangent of the angle, Irem asked: "*Why did you find the tangent?*" The student replied: "*Because I cannot find the angle*". Irem then turned back to the classroom and said: "*Your friend found the tangent*". Although the class misunderstood that the only way to find the angle in that problem was to find the tangent, Irem moved on to another activity as soon as the student completed his solution. She did not mention the fact that the angle could have been found by some other methods. Neither did she ask the class whether anyone had used another method.

When Irem's instruction was assessed within the ACT framework, we observed that Irem only focused on the correct answer in each component. Irem waited for and listened to students' methods of reaching a solution, but when she heard a wrong answer from a student, she moved on to a student who had the correct answer, instead of paying attention to the student answered incorrectly. She did not encourage students to explain their reasoning in detail. To illustrate, in an activity where she tried to associate trigonometric functions with daily life, Irem stated in her presentation that "*One of the basic problems of trigonometry is to define the height of an unreachable object*". And she asked how this could happen. One student replied with "*similarity*" and Irem moved on to the next slide without waiting for the student to explain his answer. In another activity, she asked why Cot 0 was undefined and the student replied, "*0 or 180*" and added that he did not understand the logic. So, Irem put the question to the class by saying "*Is there anyone who knows the logic*?" and looked for someone to reply but she did not try to draw out the students' responses. She did not provide any opportunity for the student to explain her idea and resolve the problems she encountered. Irem's approach shows that she did not have a positive approach towards students' mistakes and difficulties because, as stated above, Irem focused on the correct answers and ignored the wrong ones. She did not try to help students who made mistakes and instead she paid attention to the student who replied correctly. Thus, it can be said that Irem could not direct course content by using student explanations. For instance, Irem asked the students in the 3rd lesson: "*What does sin (-θ) refer to?*" One student responded, "*the inverse*" while another one responded, "*if we subtract θ from 360*". However Irem, ignoring these two responses, immediately gave the correct answer herself. Although in fact the student responses provided were good ideas to reach the exact answer, Irem did not give any feedback on these responses. Thus, it can be said that Irem is not very successful in determining students' level of engagement in the course.

Irem instructed her classes by using a presentation she had previously prepared and reflected on the board making use of technology. She sometimes divided the students into groups of two and distributed activity worksheets to them. In this respect, it can be said that she tried to promote collaborative problem solving. However, Irem only moved around the groups. She did not interact with the groups and check what they were doing. Besides, Irem tried to call different students to the board but she did not make any effort to choose students who needed assistance to come to the board.

Supporting the students' conceptual understanding

While defining the trigonometric functions in the first class, Irem asked students to remind her of the concept of functions.

Irem: When doesn't function exist?

Student 1: When there is an unknown.

Student 2: When there is a number.

Irem: When there is a number. Where do we get those numbers to use this number?

Student 3: Natural numbers.

Irem: What do natural numbers define?

Student 4: Sets.

Irem: They define sets.

Irem: They define natural number sets. That is, unless there is a set, there is no function. Ok. How do we use that set in functions? We use it while defining. Is there anyone with an answer?

As seen in these dialogues, Irem tried to remind the students of previous knowledge with regards to function and set concepts. Next, she stated the definition set and value set for trigonometric functions.

Irem made use of group studies in her classes. However, within this process she did not direct the students in this group to help each other. For instance while working on an activity, one student did not understand anything from an activity and asked the teacher.

Irem: What is the expression that corresponds to length C? We have just done it, remember?

Student: Well, I'll understand after the example.

Irem: Try to do it.

At this point, Irem could have directed the other students in the group to help this student but she did not do that.

Irem did not quite assist the students in explaining their individual solutions. For instance, when one of the students said he had found a mistake in the solution his friends made, Irem said: "*Yes, we have made a mistake*" and then called the student: "*Would you like to come up, too?*" Then, she let the student who made the mistake sit down. If she had asked the student to explain his solution, maybe the student could have noticed his mistake and corrected it.

Irem did not do any necessary repetitions in her teaching. For instance, while working on an activity, one of the students asked: "*Is grad degree divided by 180*?" Irem only replied, "*What was that?*" Meanwhile, the student who asked the question learnt the answer from one of his friends and kept on working. Irem did not feel like repeating it since there was no more problem.

Extending the students' MT

When Irem's classes were examined within the framework of the ACT, positive findings could not be obtained for the extending component. In fact, the activities Irem prepared were quite proper for students to analyze the mathematical concepts, compare them and generalize. However, Irem could not manage these activities well. Her purpose in activities was always to find the correct answer. Instead of a teaching method that takes student's ideas into account through effective questioning, Irem followed a traditional approach in her classes and focused on the result not the process. Due to this approach, she did not attempt to develop students' MT.

DISCUSSION AND CONCLUSION

The purpose of this study was to examine the teaching practice of pre-service mathematics teachers in terms of providing suitable conditions for developing students' MT in the frame of the ACT Framework developed by Fraivillig et al. (1999).

In the eliciting component of the ACT Framework, Aslı put students' opinions in the center of her teaching and prompted students' to explain their reasoning/solutions. While Arda considered different solutions, he was insufficient in supporting students' explanations in their solutions or opinions. Ege and Irem focused on only the right answer in their classes. While they were putting questions to students and getting some answers, they did not try to draw out student responses or learn the source of their thoughts. In facilitating students' responding of the eliciting component, all participants conveyed an accepting attitude toward students' errors and problem solving. This may stem from being a pre-service teacher. Because these teaching process were their initial attempts for teaching mathematics and it was an exciting experience for them. So, they could be more patient and tolerant. Within the scope of eliciting, all of the pre-service teachers encouraged cooperative problem solving. By distributing students activities and worksheets, they made students do group study. However their way of practice was different. Aslı and Arda considered it important that activities were finalized by students; listened to different students and shaped the class with students' explanations. After getting the correct answers in the activities, Ege and Irem passed onto the next step without listening to the other responses. In fact, in a group study, the teacher should not focus on the correct answers but become an observer who facilitates the interaction in the group (Baki, 2008, p. 185-186). But, Ege and Irem were not successful to achieve this.

The second component of the ACT framework was supporting.Ege, Aslı and Arda had instant replays at the necessary moments. On the other hand, Irem did not do any instant replays although she reminded the students of previous knowledge. Aslı and Arda reminded students of conceptually similar problems and tried to provide them with some clues regarding the solution of the problem. Ege, Arda and Irem were teaching in an Anatolian High School while Aslı was in a Vocational High School. Despite this (mathematics achievement is lower in Vocational schools than in others), the most comfortable classroom setting that enabled students to ask questions or explain their ideas was Aslı's classroom. Aslı encouraged her students to state their ideas even if they were wrong. In this respect, Aslı was the participant who had the best pedagogical skills in terms of encouraging students' to express themselves. Even though she was the most successful participant in ensuring the conditions for supporting component, Aslı did not direct the group to help an individual student or ask a different student to explain a peer's method. The grade point average of Aslı's students was not very good and the students were not accustomed to doing these activities in a lesson. So, these may be the reasons for her challenges. She could not assist individuals in private help sessions. The reason for this may be being a pre-service teacher and not their regular teacher. Aslı and Arda reminded students of conceptually similar problems and helped students explain their individual solutions. In contrast, Ege and Irem did not teach in this way.

The last component of the ACT framework was extending. Fraivillig et al. (1999) emphasized that in the first two components, problem solutions, which students were formerly used to, were emphasized but in the last component attempts were made to challenge students and draw out their answers. In the extending component, no findings were found in Ege's and Irem's classes because neither Ege nor Irem showed students any condition which might make students think in some other ways. Aslı and Arda ensured that there were conditions when students were challenged. They supported the students in examining mathematical concepts, making comparisons and reaching generalizations. They tried to form connections among mathematical concepts. This might be the result of the fact that Aslı and Arda had a particularly good level of content knowledge because it is seen that the average achievement level of Aslı and Arda in the pure mathematics courses they attended at university was higher than the averages of Ege and Irem. Also, considering their GPA, Aslı and Arda had higher GPA scores than the other pre-service teachers. In particular, since Aslı and Arda had good levels of content knowledge, it was observed that they knew the relationships among mathematical concepts really well in teaching towards developing mathematical thinking. We think this is why they could ask questions that might make students realize the relationships among concepts comfortably and direct their classes without hesitating on questions that might be asked by students. Due to the low academic and content knowledge achievement scores of Ege and Irem, they focused on the right answers of the students and avoided examining these correct answers or wrong ones in detail. This was mostly apparent in the third component of the ACT Framework. While Ege and Arda did nothing about this component because of their lack of knowledge, Aslı and Arda tried to get students to work on difficult problems.These findings of this study are consistent with the previous research results that emphasize the importance of the content knowledge for teaching (Ball et al., 2008; Kahan et al., 2003). As a result, it can be said that Aslı was the most skillful pre-service mathematics teacher in terms of providing suitable conditions in developing students' MT in the frame of ACT framework in her teaching. Pre-service mathematics teachers of this research often elicited students' MT but less often supported and extended. Although not all of them exhibited the same level of skills, pre-service teachers identified the current ideas of students before giving the concepts or principles and tried to construct new information on this old information. According to Fraivillig et al. (1999), learning what children know and how they think about academic concepts is a critical factor for developing children's thinking. By revealing children's responses, teachers measure children's individual thinking and arrange learning opportunities for all students (Yackel, 1995; cited in Fraivillig et al., 1999, p. 154).

Differently, Fraivillig et al. (1999) found that their participants had often supported students' MT. They determined eliciting and then using student descriptions of MT as a complex and time-consuming task requiring patience, skill and high levels of knowledge about individual children and about typical solution methods in major mathematical areas. This difference might result from the fact that the grade levels taught were different. Similar to the study by Fraivillig et al. (1999), extending was the least observed component in our study, too.

Although the participant pre-service teachers got theoretical knowledge about MT during their teacher education program, they reflected this knowledge in practice for the first time in this study. In this sense, it can be said that the pre-service teachers made significant efforts in terms of developing students' MT in their teaching and tried to realize an instruction suitable for the ACT framework. Similarly to the results of this study, Hughes (2006) also determined that ten mathematics pre-service teachers learned to deal with the MT of the students in lesson planning before and after a lesson they took. It is reported that they showed a meaningful development in terms of their skills for dealing with the MT of the students from the beginning until the end of a lesson they took at university. Similarly to the study of Hughes (2006), this study can also be performed by monitoring the teaching practice of pre-service teachers before and after the study process and comparing the results. Furthermore, handling the teaching practice of the pre-service teachers in the same concept may create different results. Another study could be carried out with teachers from different faculties, teaching the same subject topic (for example: four more teachers teaching conics). Further studies could compare the results with pre-service teachers from different faculties and also with teachers with more experience. It would be appropriate to match the topics worked by the pre-service teachers in a further study.

In conclusion, in line with the results obtained in this study, it is thought that it will be useful if pre-service teachers are informed about the ways of developing students' thinking in detail, and gain experience about reflecting the knowledge they have theoretically, and this is included in the curriculum. Furthermore, the preparation and application of the lesson plans related to how the lessons that could contribute to components of supporting and extending MT must also be included in the process by teacher education institutions.

Conflict of Interests

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