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Functional variable method and its applications for finding exact solutions of nonlinear PDEs in mathematical physics

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The functional variable method is a powerful mathematical tool for obtaining exact solutions of nonlinear evolution equations in mathematical physics. In this paper, the functional variable method is used to establish exact solutions of the (2+1)-dimensional Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony (KP-BBM) equation, the (2+1)-dimensional Konopelchenko-Dubrovsky equation, the (3+1)-dimensional Burgers equation and the (3+1)- dimensional Jimbo-Miwa equation. The exact solutions of these four nonlinear equations including solitary wave solutions and periodic wave solutions are obtained. It is shown that the proposed method is effective and can be applied to many other nonlinear evolution equations. Comparison between our results obtained in this paper and the well-known results obtained by different authors using different methods are presented.

Key words: Functional variable method, nonlinear evolution equations, exact solutions, solitary wave solutions, Periodic wave solutions.

AMS Subject Classification Code: 35K99, 35P05.

INTRODUCTION

The investigation of exact solutions of nonlinear evolution equations that describe many physical phenomena help us to understand these phenomena better. These phenomena appear in various fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical physics, geochemistry and so on. With the development of soliton theory and the availability of computer symbolic system like Mathematica or Maple, many powerful methods for obtaining exact solutions of nonlinear evolution equations are presented, such as the inverse scattering method (Ablowitz and Clarson, 1991), the Bäcklund transformation method (Gu, 1995; Miura, 1978), the bilinear method (Hirota, 1971; Ma, 2011), the Painlevé method (Weiss et al., 1983), the tanh function method (Malfliet, 1992), the sine-cosine method (Yan, 1996), the homogeneous balance method (Wang, 1996), the homotopy perturbation method (He, 2005a), the variational method (He, 2005b), the exp-function method (He and Wu, 2006; Ma et al., 2010; Ma and Zhu 2012), the Adomain Padé approximation (Abassy et al., 2004), the algebraic method (Hu, 2005), the F-expansion method (Wang and Zhang, 2005), the Jacobi elliptic function method (Liu et al., 2001), the (G'/G)-expansion method (Liu et al., 2012; Wang et al., 2008), the modified simple equation method (Jawad et al., 2010; Zayed, 2011; Zayed and Ibrahim, 2012b), the functional variable method (Bekir and San, 2012; Zayed and Ibrahim, 2012a; Zayed et al., 2013; Zerarka et al., 2010; Zerarka and

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Ouamane, 2010; Zerarka et al., 2011), the generalized Riccati equation mapping method (Zhu, 2008), the local fractional variation iteration method (Yang and Baleanu, 2013), the local fractional series expansion method (Yang et al., 2013), the transformed rational function method (Ma and Fuchssteliner, 1996; Ma et al., 2007; Ma and Lee, 2009) and so on.

The authors (Bekir and San, 2012; Zayed and Ibrahim, 2012a; Zayed et al., 2013; Zerarka et al., 2010; Zerarka and Ouamane, 2010; Zerarka et al., 2011) have applied the functional variable method for finding exact solutions of real and complex nonlinear evolution equations in mathematical physics. The advantage of this method is that one treats nonlinear problems by essentially linear methods. That is based on which it is easy to construct the exact solutions such as soliton-like waves, compacton and noncompacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions and so on.

The objective of this paper is to apply the functional variable method to find the exact solutions of four nonlinear evolution equations, namely, the (2+1)-dimensional KP-BBM equation, the (2+1)-dimensional Konopelchenko-Dubrovsky equation, the (3+1)-dimensional Burgers equation and the (3+1)- dimensional Jimbo-Miwa equation. All these four nonlinear equations were not investigated before using that method.

The rest of this article is organized as follows: First is a description of the functional variable method. This is followed by illustration of applications of this method to the four nonlinear evolution equations indicated above; thereafter the paper is concluded.

DESCRIPTION OF THE FUNCTIONAL VARIABLE METHOD

Suppose we have a nonlinear evolution equation in the form:

$$F(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},...)=0, \tag{1}$$

where F is a polynomial in $u(x,y,t)$ and its partial derivatives. With reference to Bekir and San (2012), Zayed and Ibrahim (2012a), Zayed et al. (2013), Zerarka et al. (2010, 2011), and Zerarka and Ouamane (2010), the main steps of this method can be described as follows:

Step 1. We use the wave transformation

$$u(x,y,t)=u(\xi), \quad \xi=x+y-ct, \tag{2}$$

where c is a non zero constant, to reduce Equation (1) to the following ODE :

$$P(u,u_\xi,u_{\xi\xi},u_{\xi\xi\xi},...)=0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $u_\xi=du/d\xi,\ u_{\xi\xi}=d^2u/d\xi^2$ and so on.

Step 2. We make a transformation in which the unknown function $u(\xi)$ is considered as a functional variable in the form:

$$u_\xi=F(u), \tag{4}$$

and some successively derivatives of $u(\xi)$ are as follows:

$$\begin{aligned} u_{\xi\xi}&=\frac{1}{2}(F^2)', \\ u_{\xi\xi\xi}&=\frac{1}{2}(F^2)''\sqrt{F^2}, \\ u_{\xi\xi\xi\xi}&=\frac{1}{2}\left[(F^2)'''F^2+\frac{1}{2}(F^2)''(F^2)'\right], \end{aligned} \tag{5}$$

and so on, where $'=d/du$.

Step 3. We substitute (4) and (5) into (3) to reduce it to the following ODE:

$$R(u,F,F',F'',...)=0. \tag{6}$$

After integration, the Equation (6) provides the expression of *F*, and this in turn together with the Equation (4) give the appropriate solutions of the Equation (1). In order to illustrate how the proposed method works, we examine some examples treated by other methods. This matter is subsequently introduced.

APPLICATIONS

Here, we will apply the functional variable method to construct the exact solutions for the following four nonlinear evolution equations:

Example 1: The (2+1)-dimensional KP-BBM equation

This equation is well-known (Wazwaz, 2008; Yu and Ma, 2010; Zayed and Al-Joudi, 2010) and has the form:

$$\left(u_t+u_x-\alpha\left(u^2\right)_x-\beta u_{xxt}\right)_x+\gamma u_{yy}=0, \tag{7}$$

where α,β,γ are arbitrary constants. The solution of Equation (7) has been investigated in (Wazwaz, 2008)

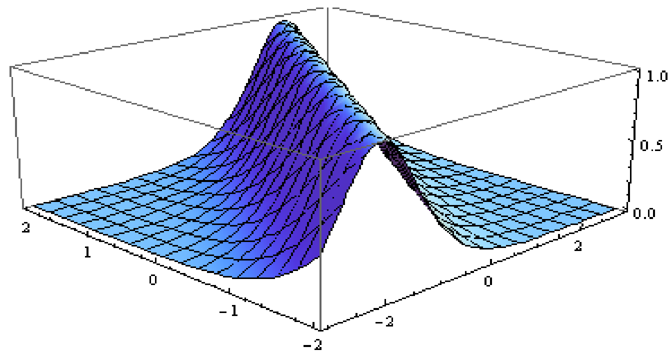

Figure 1. The plot of the solution (13) when $c=\gamma=1, \beta=\frac{-1}{4}, \alpha=\frac{3}{2}, \xi_0=y=0.$

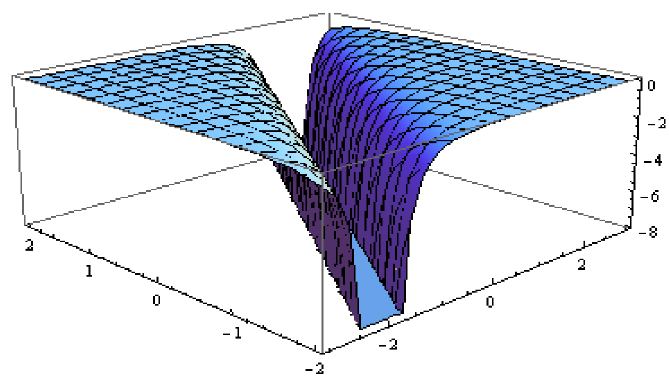

Figure 2. The plot of the solution (14) when $c=\gamma=1, \beta=\frac{-1}{4}, \alpha=\frac{3}{2}, \xi_0=y=0.$

using the extended tanh – function method and in (Yu and Ma, 2010) using the exp-function method and in (Zayed and Al-Joudi, 2010) using the auxiliary equation method respectively. Let us now solve Equation (7) using the aforementioned proposed method. To this end, we apply the wave transformation (2) to reduce Equation (7) into the following ODE:

$$\left((1-c)u_{\xi}-\alpha\left(u^2\right)_{\xi}+c\beta u_{\xi\xi\xi}\right)_{\xi}+\gamma u_{\xi\xi}=0. \tag{8}$$

Integrating the Equation (8) with respect to ξ twice, we get

$$(1+\gamma-c)u-\alpha u^2+c\beta u_{\xi\xi}=0. \tag{9}$$

with zero constants of integration. Substituting (5) into (9) we obtain

$$\left(F^2\right)'=\frac{2u}{c\beta}\left(\alpha u-(1+\gamma-c)\right). \tag{10}$$

Integrating the Equation (10) with respect to u, we have

$$F(u)=\sqrt{\frac{-2\alpha}{3c\beta}}\,u\sqrt{\frac{3(1+\gamma-c)}{2\alpha}-u}\,. \tag{11}$$

From (4) and (11) we deduce that

$$\int\frac{du}{u\sqrt{\frac{3(1+\gamma-c)}{2\alpha}-u}}=\sqrt{\frac{-2\alpha}{3c\beta}}(\xi+\xi_0), \tag{12}$$

where ξ_0 is a constant of integration. After integrating (12), we have the following exact solutions:

(i) If $\frac{1+\gamma-c}{c\beta}<0,$ we obtain the bell-shaped solitary wave solutions

$$u_1(x,y,t)=\frac{3(1+\gamma-c)}{2\alpha}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{-\left(\frac{1+\gamma-c}{c\beta}\right)}(x+y-ct+\xi_0)\right], \tag{13}$$

$$u_2(x,y,t)=\frac{-3(1+\gamma-c)}{2\alpha}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{-\left(\frac{1+\gamma-c}{c\beta}\right)}(x+y-ct+\xi_0)\right], \tag{14}$$

(ii) If $\frac{1+\gamma-c}{c\beta}>0,$ we obtain the periodic wave solutions

$$u_3(x,y,t)=\frac{3(1+\gamma-c)}{2\alpha}\sec^2\left[\frac{1}{2}\sqrt{\frac{1+\gamma-c}{c\beta}}(x+y-ct+\xi_0)\right], \tag{15}$$

$$u_4(x,y,t)=\frac{3(1+\gamma-c)}{2\alpha}\csc^2\left[\frac{1}{2}\sqrt{\frac{1+\gamma-c}{c\beta}}(x+y-ct+\xi_0)\right], \tag{16}$$

(iii) If $c=1+\gamma$, we obtain the rational solution

$$u_5(x,y,t)=\frac{6c\beta}{\alpha(x+y-(1+\gamma)t+\xi_0)^2}. \tag{17}$$

Remark 1: Equation (12) can be found using the direct integration method as follows: Multiply both sides of Equation (9) by u_{ξ} and integrate with zero constant of integration, we arrive at the Equation (12). Our solutions (13) to (16) are in agreement with the solutions obtained in Wazwaz (2008), Yu and Ma (2010), Zayed and Al-Joudi (2010) while the solution (17) is new. Figures 1 and 2 describe the behavior of the solutions (13) and (14) while Figures 3 and 4 describe the behavior of the

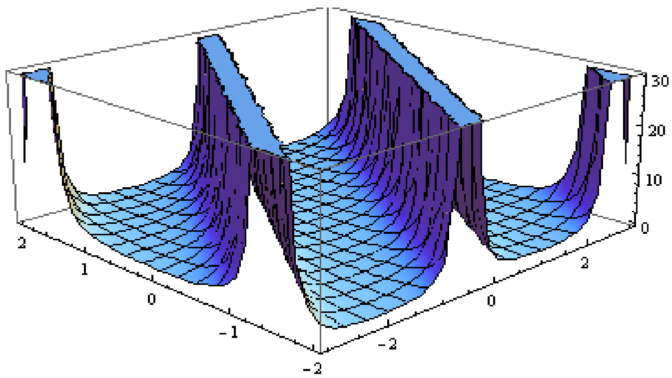

Figure 3. The plot of the solution (15) when $c=\gamma=1, \beta=\frac{1}{4}, \alpha=\frac{3}{2}, \xi_0=y=0.$

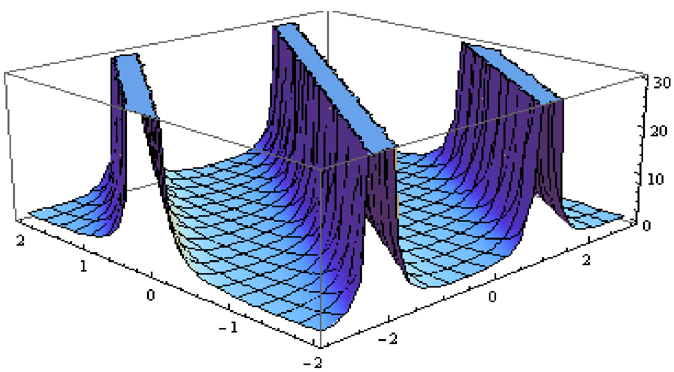

Figure 4. The plot of the solution (16) when $c=\gamma=1, \beta=\frac{1}{4}, \alpha=\frac{3}{2}, \xi_0=y=0.$

solutions (15) and (16).

Example 2: The (2+1)- dimensional Konopelchenko-Dubrovsky equation

This equation is well-known (Wang and Wei, 2010; Zhang and Xia, 2006) and has the form:

$$u_t-u_{xxx}-6\alpha uu_x+\frac{3}{2}\beta^2u^2u_x-3v_y+3\beta vu_x=0, \qquad (18)$$
$$u_y=v_x,$$

where α, β are non zero constants. The solution of the system (18) has been investigated using different methods (Wang and Wei, 2010; Zhang and Xia, 2006). Let us now solve system (18) using the aforementioned proposed method. To this end, we apply the wave transformations (2) to reduce the system (18) into the following ODE:

$$-(c+3)u_\xi-u_{\xi\xi\xi}+\frac{3}{2}\beta^2u^2u_\xi-3(2\alpha-\beta)uu_\xi=0, \qquad (19)$$

where $v=u$, with zero constant of integration. Integrating the Equation (19) once with respect to ξ, we get

$$-(c+3)u-u''+\frac{1}{2}\beta^2u^3-\frac{3}{2}(2\alpha-\beta)u^2=0, \qquad (20)$$

with zero constant of integration. Substituting (5) into (20) we obtain

$$\frac{1}{2}(F^2)'=\frac{1}{2}\beta^2u^3-\frac{3}{2}(2\alpha-\beta)u^2-(c+3)u \qquad (21)$$

Integrating the Equation (21) with respect to u, we have

$$F(u)=\frac{\beta}{2}u\sqrt{\left(u-\frac{2(2\alpha-\beta)}{\beta^2}\right)^2-\left(\frac{4(2\alpha-\beta)^2}{\beta^4}+\frac{4(c+3)}{\beta^2}\right)}, \qquad (22)$$

with zero constant of integration. From (4) and (22) we deduce that

$$\int\frac{du}{u\sqrt{(u-B)^2-A^2}}=\frac{\beta}{2}(\xi+\xi_0), \qquad (23)$$

where $A^2=B^2+\frac{4(c+3)}{\beta^2}, B=\frac{2(2\alpha-\beta)}{\beta^2}.$ After integrating (23) with zero constant of integration, we have the following exact solutions:

(i) If $c+3>0$, we have the periodic wave solutions

$$u_1(x,y,t)=\frac{(A+B)\sec^2\left[\frac{\sqrt{c+3}}{2}(x+y-ct+\xi_0)\right]}{1-\left(\frac{A+B}{A-B}\right)\tan^2\left[\frac{\sqrt{c+3}}{2}(x+y-ct+\xi_0)\right]}, \qquad (24)$$

$$u_2(x,y,t)=\frac{-(A-B)\csc^2\left[\frac{\sqrt{c+3}}{2}(x+y-ct+\xi_0)\right]}{1-\left(\frac{A-B}{A+B}\right)\cot^2\left[\frac{\sqrt{c+3}}{2}(x+y-ct+\xi_0)\right]}, \qquad (25)$$

(ii) If $c+3<0$, we have the solitary wave solutions

$$u_3(x,y,t)=\frac{(A+B)\operatorname{sech}^2\left[\frac{\sqrt{-(c+3)}}{2}(x+y-ct+\xi_0)\right]}{1+\left(\frac{A+B}{A-B}\right)\tanh^2\left[\frac{\sqrt{-(c+3)}}{2}(x+y-ct+\xi_0)\right]}, \qquad (26)$$

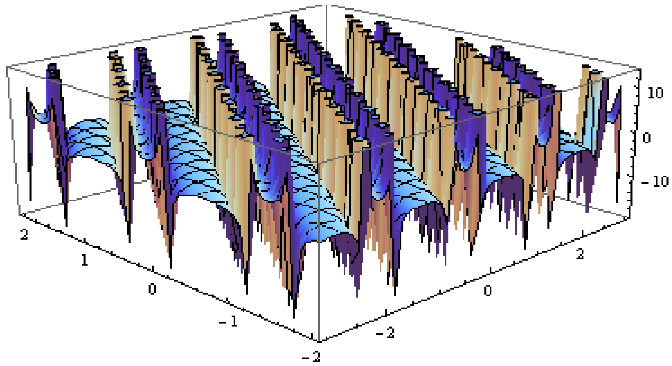

Figure 5. The plot of the solution (24) when $c = B = 1, \beta A = 2, \xi_0 = y = 0.$

$$u_4(x,y,t) = \frac{(A-B)\operatorname{csc}h^2\left[\frac{\sqrt{-(c+3)}}{2}(x+y-ct+\xi_0)\right]}{1+\left(\frac{A-B}{A+B}\right)\coth^2\left[\frac{\sqrt{-(c+3)}}{2}(x+y-ct+\xi_0)\right]} \quad (27)$$

(iii) If $c = -3$ and $B \neq 0$ we have the rational solution

$$u_5(x,y,t) = \frac{2B}{1-\frac{\beta^2 B^2}{4}(x+y+3t+\xi_0)}. \quad (28)$$

Remark 2: Equation (23) can be found using the direct integration method as follows: Multiply both sides of Equation (20) by u_ξ and integrate with zero constant of integration, we arrive at the Equation (23). Our exact solutions (24) to (28) of Equation (18) are new and not reported elsewhere using the proposed method. Figure 5 describes the behavior of the solutions (24).

Example 3: The (3+1)-dimensional Burgers equations

This equation is well known (Dai and Wang, 2009; Zhou et al., 2008) and has the form:

$$\begin{aligned} &u_t - 2uu_y - 2vu_x - 2wu_z - u_{xx} - u_{yy} - u_{zz} = 0, \\ &u_x = v_y, \quad u_z = w_y. \end{aligned} \quad (29)$$

The solution of the system (29) has been investigated using different methods (Dai and Wang, 2009; Zhou et al., 2008). Let us now solve system (29) using the aforementioned proposed method. To this end, we apply the wave transformation

$$u(x,y,z,t) = u(\xi), \quad \xi = x+y+z-ct, \quad (30)$$

to reduce system (29) into the following ODE:

$$-cu_\xi - 6uu_\xi - 3u_{\xi\xi} = 0, \quad (31)$$

where $v = u$ and $w = u$ with zero constants of integration. Integrating the Equation (31) once with respect to ξ, we get

$$cu + 3u^2 + 3u_\xi = 0, \quad (32)$$

with zero constant of integration. Substituting (5) into (32) we obtain

$$F(u) = -u\left(u + \frac{c}{3}\right). \quad (33)$$

From (4) and (33) we deduce that

$$\int \frac{-du}{u\left(u+\frac{c}{3}\right)} = \xi + \xi_0, \quad (34)$$

where ξ_0 is a constant of integration. After integrating (34), we have the following exact solutions:

$$u_1(x,y,z,t) = \frac{-c}{6}\left\{1-\tanh\left[\frac{c}{6}(x+y+z-ct+\xi_0)\right]\right\}, \quad (35)$$

$$u_2(x,y,z,t) = \frac{-c}{6}\left\{1-\coth\left[\frac{c}{6}(x+y+z-ct+\xi_0)\right]\right\}, \quad (36)$$

Remark 3: We can solve Equation (32) using the separation of variable method directly. Our exact solutions (35) and (36) are new and not reported elsewhere using the proposed method.

Example 4: The (3+1)-dimensional Jimbo-Miwa equation

This is well-known (Zhang et al., 2009) and has the form:

$$\begin{aligned} &u_{xxxy} + 6u_x u_y + 3uv_{xx} + 3u_{xx}v + 3u_{yt} - 3u_{zz} = 0, \\ &u_y = v_x. \end{aligned} \quad (37)$$

The solution of the system (37) has been investigated in Zhang et al. (2009) using the generalized F-expansion method. Let us now solve system (37) using the aforementioned proposed method. To this end, we apply the wave transformation (30) to reduce system (37) into the following ODE:

$$u_{\xi\xi\xi\xi} + 6\left(uu_{\xi}\right)_{\xi} - 3(c+1)u_{\xi\xi} = 0, \tag{38}$$

where $v = u$, with zero constant of integration. Integrating the equation (38) twice with respect to ξ, we get

$$u_{\xi\xi} + 3u^2 - 3(c+1)u = 0, \tag{39}$$

with zero constants of integration. Substituting (5) into (39) we obtain

$$\frac{1}{2}\left(F^2\right)' = 3(c+1)u - 3u^2, \tag{40}$$

Integrating the Equation (40) with respect to u, we have

$$F(u) = \sqrt{2}\,u\sqrt{\frac{3(c+1)}{2} - u}\,. \tag{41}$$

From (4) and (41) we deduce that

$$\int \frac{du}{u\sqrt{\frac{3(c+1)}{2} - u}} = \sqrt{2}(\xi + \xi_0), \tag{42}$$

where ξ_0 is a constant of integration. After integrating (42), we have the following exact solutions:

(i) If $c+1<0,$ we obtain the periodic wave solutions

$$u_1(x,y,z,t) = \frac{3(c+1)}{2}\sec^2\left[\frac{1}{2}\sqrt{-3(c+1)}(x+y+z-ct+\xi_0)\right], \tag{43}$$

$$u_2(x,y,z,t) = \frac{3(c+1)}{2}\csc^2\left[\frac{1}{2}\sqrt{-3(c+1)}(x+y+z-ct+\xi_0)\right], \tag{44}$$

(ii) If $c+1>0,$ we obtain the bell-shaped solitary wave solutions

$$u_3(x,y,z,t) = \frac{3(c+1)}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{3(c+1)}(x+y+z-ct+\xi_0)\right], \tag{45}$$

$$u_4(x,y,z,t) = \frac{3(c+1)}{2}\operatorname{csch}^2\left[\frac{1}{2}\sqrt{3(c+1)}(x+y+z-ct+\xi_0)\right], \tag{46}$$

(iii) If $c=-1,$ we obtain the rational solution

$$u_5(x,y,z,t) = \frac{-2}{(x+y+z+t+\xi_0)^2}. \tag{47}$$

Remark 4: Equation (42) can be found using the direct integration method as follows: Multiply both sides of the Equation (39) by u_{ξ} and integrate with zero constant of integration, we arrive at the Equation (42). Our exact solutions (43) to (47) are new and not reported elsewhere using the proposed method.

Conclusions

The functional variable method applied in this paper has been used to find the exact solutions of four nonlinear evolution equations, namely, the (2+1)-dimensional KP-BBM equation, the (2+1)-dimensional Konopelchenko-Dubrovsky equation, the (3+1)-dimensional Burgers equation and the (3+1)- dimensional Jimbo-Miwa equation, which were not discussed elsewhere using that method.

On comparing the proposed method in this article with the other methods used in Dai and Wang (2009), Wang and Wei (2010), Wazwaz (2008), Yu and Ma (2010), Zayed and Al-Joudi (2010), Zhang and Xia (2006), Zhang et al. (2009), and Zhou et al. (2008) we find that the functional variable method is simpler than those methods. Let us now compare between our obtained results and the well-known results obtained by other authors using different methods as follows: Our results (15) and (16) of the (2+1)-dimensional KP-BBM Equation (7) are in agreement with the results (20) and (21) obtained in Yu and Ma (2010) using the exp-function method, while our results (13) and (14) of the same Equation (7) are in agreement with the results (31) and (32) obtained in Wazwaz (2008) using the extended tanh- function method. Furthermore, our results (13) to (16) of the same Equation (7) are in agreement with the results $u_i,(i=1,2,3,4)$ obtained in Zayed and Al-Joudi (2010: 96) using an auxiliary ordinary differential equation. Our obtained solutions of the nonlinear Equations (18), (29), (37) are new and different from those obtained in Dai and Wang (2009), Zhang and Xia (2006), Zhang et al. (2009) and Zhou et al. (2008) which are not reported elsewhere.

In summary, we conclude that the proposed method used in this article allows us to produce easily exact solutions for several families of nonlinear evolution equations in mathematical physics. Finally, by using the Maple we have assured the correctness of the obtained solutions by putting them back into the original equations.

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