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Stability analysis of stochastic model of stock market price

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A stability analysis of stochastic model of price change at the floor of a stock market is considered. Precise conditions are obtained which determine the equilibrium price and growth rate of the stock shares in a particular case. In general, stability in a numerical procedure for the Black-Schole’s partial differential equation is established. An illustrative example is provided.

Key words: Stability analysis, stochastic model, stock price changes, Black-Schole’s PDE, mathematics

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INTRODUCTION

We consider the problem of the stability of the solution of the general Black-Scholes partial differential equation (PDE). Black-Scholes (1991) obtained option pricing model with a constant volatility. It is well known that this model is not consistent with observed option prices. One possible remedy for this is to make the volatility a function of time and the strike price. The price of an option based on option pricing model can be obtained as the solution of a parabolic differential equation. Heston (1993) introduced a model which assumed that the volatility of the price process is stochastic. Dragulescu and Yakovenko (2002) showed that the time-dependent probability distribution of stock price changes generated by Heston’s model is consistent with the Dow-Jones index after the calibration of the few parameters appearing in the model. Brennan and Schwartz (1977) considered a finite difference method for pricing American options for the Black-Scholes model leading to one dimensional parabolic partial differential inequality. Clarke and Parrott (1999) considered the discretization of a partial differential inequality for the price of an American option using finite differences for space derivatives and a slightly stabilized Crank-Nicolson method for the time derivative.

Ikonen and Toivanen (2004) proposed operator splitting method for pricing American options using Heston’s stochastic volatility model. This they did by studying operator-splitting methods for performing time stepping after finite difference space discretization is done.

On the other hand, Ugbebor et al. (2001) have considered a stochastic model of price changes at the floor of a stock market. Here the equilibrium price and market growth rate of shares are determined. Subsequently, there have been some works with considerable extensions and constraints (Osu and Okoroafor, 2007).

The aim of this paper is first: to establish a dynamic stochastic model aimed at determining the equilibrium price and growth rate of the asset under certain conditions, and second to analyze the stability of a solution of the Black-Scholes PDE in a general case.

The outline of the paper is the following. In section 2, the stock price model is introduced. The equilibrium price is determined in the next section. Section 3 presents our stability analysis. In the end of this paper, we give some illustrative examples and some conclusions.

Stock price model

In order to determine the value of an economic asset, such as stock, we need to take into account at least two aspects of random variability: the growth of the asset and its price. Let $S_t$ be the unit price of the stock and $K$, the capacity of the asset both at time $t$. If the dividends are declared at time $t$, then the revenue is $R_t = S_t K_t$, which is called the “aggregate intrinsic value” of the stock. Many authors have extensively addressed the problem associated with random behaviour of stock price (see for example, Black and Karasinski, 1991).

It is well known that (Black and Scholes, 1973;
Etheridge, 2002) the price \( S_t \) of the risky asset such as stock evolves according to the stochastic differential equation

\[
dS_t = \alpha S_t dt + \sigma S_t dB_t, \quad (1)
\]

where; \( B_t \) is a standard Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \( \mathcal{F} \) is a \( \sigma \)-algebra generated by \( B_t, t \geq 0 \), (Lim and Choi, 2009) It is not hard to see using Ito formula that

\[
S_t = S_0 \exp\left\{ \sigma B_t + \left( \alpha - \frac{1}{2} \sigma^2 \right) t \right\}, \quad \forall \ t \in [0,1], \quad (2)
\]

starting from \( S_0 \) at time 0. If we consider a short trading period, where new dividends will not have been declared and no new assets have been purchased, then the stock price follows the process (Osu, 2008).

\[
dS_t = \hat{\alpha} S_t dt + \sigma S_t dB_t, \quad \hat{\alpha} = \alpha + \lambda, \quad (3)
\]

where; \( \lambda \) is the market price of risk.

The stock pricing PDE is then the backward Black-Scholes PDE given (in one variable) as (Osu, 2008):

\[
\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + (\alpha - \lambda) S \frac{\partial W}{\partial S} - r_t W = 0. \quad (4)
\]

Assume \( \lambda = 0 \), then we have (4) as:

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + \alpha S \frac{\partial W}{\partial S} - r_t W = -\frac{\partial W}{\partial t}. \quad (5)
\]

We take \( W = W(S_t, K_t) \) as the investment output and \( K_t \) the investment over a period \( t \). \( r_t \) (the interest rate) is assumed a linear function. Further, we write the right hand side of (5) (Osu and Okoroafor, 2007) as:

\[
\frac{\partial W(S_t, K_t)}{\partial t} = p(S_t, K_t) - c(S_t, K_t), \quad (6)
\]

Where; \( p \) is the production rate and \( c \) the consumption rate.

If \( c(S_t, K_t) \to 0 \) as \( t \to 0 \) and since the rate of change of the out put value depends on the investment output at present time \( t \), we write (6) as:

\[
\frac{dW}{dt} = p(K_t) = S_t, \quad (7)
\]

For short trading period (Busca, 2002). Equation (5) now reduces to an ordinary differential equation of the form:

\[
\frac{1}{2} \sigma^2 S^2 \frac{d^2 W(S)}{dS^2} + \alpha S \frac{dW(S)}{dS} - r_t W(S) = -S, \quad (8)
\]

Since the differentiation of \( W \) on the left hand side of (5) is with respect to \( S \) (and not \( K \)).

**Determination of equilibrium price in particular case**

We now derive the expected equilibrium price of primary security. In what follows, we state **Theorem 1**. For a twice continuously differentiable function \( W(S) \), the solution of the time-homogeneous investment equation

\[
\frac{1}{2} \sigma^2 S^2 \frac{d^2 W(S)}{dS^2} + \alpha S \frac{dW(S)}{dS} - r_t W(S) = -S, \quad (8)
\]

is given as:

\[
W(S) = \frac{A}{2} (1 + S) + BS \hat{A} + \frac{S}{(r_t - \alpha)}, \quad (9)
\]

with

\[
W(0) = \begin{cases} 
0 & \text{for } A = 0 \\
\frac{A}{2} & \text{for } A \neq 0
\end{cases} \quad (10)
\]

and

\[
\frac{dW(S)}{dS} = \frac{A}{2} (1 + \hat{S}) + B \hat{A} \hat{S} \hat{A} + \frac{\hat{S}}{(r_t - \alpha)} = 0. \quad (11)
\]

A and B are constants, \( \hat{S} \) is the expected equilibrium price of security for a period \( t \).

\[
\hat{\lambda}_1 = \frac{1}{2} \left\{ 1 - \frac{2\alpha}{\sigma^2} + \sqrt{\left( \frac{2\alpha}{\sigma^2} - 1 \right)^2 + \frac{8r_t}{\sigma^2}} \right\} \quad (12a)
\]

and

\[
\hat{\lambda}_2 = \frac{1}{2} \left\{ 1 - \frac{2\alpha}{\sigma^2} - \sqrt{\left( \frac{2\alpha}{\sigma^2} - 1 \right)^2 + \frac{8r_t}{\sigma^2}} \right\} \quad (12b)
\]
are the positive and negative characteristics roots of (8). Here;
\[ r_\pm = \alpha + \sigma \tau, (\alpha > 0, 0 \leq \sigma \leq 1) \]  
(13)

Is a decreasing (or an increasing) linear function of \( t \) as \( t \) increases. The parameter \( \alpha \) in (13) is the initial amount. It determines the speed of mean-reversion to the stationary level (Anderson and Lund, 1997). A situation where current interest rates are high, imply a negative drift until rates revert of the long-run value, and low current rates are associated with positive expected drift. It can be noted that \( \sigma \) (the acceleration coefficient in (13)) which is the variance of the process is proportional to the level of the interest rate; as the interest rate moves towards 0, the variance decreases.

**Proof**

By the method of change of independent variable, (8) becomes:
\[ \frac{1}{2} \sigma^2 D^2 W + \left( \alpha - \frac{1}{2} \sigma^2 \right) DW - r_\tau W = e^\tau. \]  
(14)

For the homogeneous part of (14), we let \( \sigma = 0 \) to obtain
\[ \alpha DW - r_\tau W = 0, \]
Which is a first order differential equation with solution;
\[ W = A \exp \left\{ \frac{1}{2} \int_0^t r_\tau ds \right\}. \]  
(15)

This suggests that (15) is a solution of the homogeneous part of (14) and we get for \( r_\tau = 0 \),
\[ W = A \]
and for \( r_\tau = \alpha \)
\[ W = Ae^\alpha. \]  
(16b)

We combine (16a) and (16b) to get
\[ W(S) = \frac{A}{2} (1 + S). \]  
(17)

Using Euler’s substitution and solving by variation of parameter for \( \sigma \neq 0 \), we get from (14) with (17):
\[ W(S) = BS^{\lambda} + \frac{S}{r_\tau - \alpha} + \frac{A}{2} (1 + S) \]
as required.

**Remark:** For \( A=0 \), we get the result in Ugberbor et al. 2001
Since the discounted profit from a unit capacity for the expected equilibrium price \( \hat{S} \) must be equal to the expected unit cost \( \overline{S} \) of risky security options; we have by (11),
\[ W(\hat{S}) = \frac{A}{2} (1 + \hat{S}) + BS^{\lambda} + \frac{\hat{S}}{r_\tau - \alpha} = \hat{S}. \]  
(18)

For \( A = 0 \) in (18) we get,
\[ B = -\frac{S}{(r_\tau - \alpha)\lambda S^{\lambda}}. \]  
(19)

For \( A = 0 \) in (11) we get,
\[ B = \frac{(r_\tau - \alpha) \overline{S} - \hat{S}}{(r_\tau - \alpha) S^{\lambda}}. \]  
(20)

Equating (19) and (20) gives:
\[ \hat{S} = \frac{S}{\lambda (r_\tau - \alpha)} \frac{S^{\lambda}}{\lambda - 1}, \]  
(21)
as in Ugbebor et al. 2001.
For \( B = 0 \) in (11), we get:
\[ A = \frac{-2\hat{S}}{(r_\tau - \alpha)(1 + \hat{S})}. \]  
(22)

For \( B = 0 \) in (18), we get:
\[ A = \frac{2\left[ (r_\tau - \alpha) \overline{S} - \hat{S} \right]}{(r_\tau - \alpha)(1 + \hat{S})}. \]  
(23)

Equating (22) and (23) gives,
\[ r_\tau = \alpha. \]  
(24)

Again for \( B \neq 0 \) and \( A \neq 0 \), solving equation (11) and (18) for \( \hat{S} \) gives:
\[ \hat{S} = \left( \frac{S}{(1 - \lambda B)} \right)^{\frac{1}{\lambda}}. \]  
(25)
For \( r \) as in (13), the equilibrium price (21) becomes
\[
\hat{S} = \frac{\hat{\lambda}_1(\sigma t)S}{\hat{\lambda}_1 - 1}.
\] (26)

Note: When \( \hat{\lambda}_1 = 1 \) equilibrium prices \( \hat{S} \) of (21), (25) and (26) are indeterminate.

**Stability analysis of the solution of Black-Scholes PDE**

The question of stability in numerical procedures for partial differential equations arises in almost all problems involving time as an independent variable. This is natural, since solutions over long time intervals maybe of interest. Here we consider a method of stability analysis ascribed to Von Neumann, which can also be termed the 'Fourier method'. In this method, we try to find a solution of the finite-difference equation having the form:
\[
V_{ji} = e^{ij\sigma h}e^{iak} \quad (i = \sqrt{-1}).
\] (27)

(Here we use \( j \) for the first subscript instead of \( i \) so that it will not be confused with the complex number \( i = \sqrt{-1} \)).

Once this is done, by a suitable choice of \( a \), the behaviour of this solution is examined as \( t \to \infty \) or \( n \to \infty \). This obviously depends on the fact that \( e^{iak} = (e^{ak})^n \) in (27). Consider the equation:
\[
\frac{\partial W(t, S)}{\partial t} + \frac{1}{2} \sigma^2 S \frac{\partial^2 W(t, S)}{\partial S^2} - rW(t, S) + rS \frac{\partial W(t, S)}{\partial S} = 0, \quad \forall (t, S) \in (0, \infty) \times (0, T)
\] (28)

with terminal condition \( W(S, t) = g(S) \). This is the famous Black-Scholes parabolic PDE. The fact that the price can be expressed as the solution to a PDE is a consequence of the deep connection between stochastic differential equations and the Black-Scholes PDE.

We define the function (Etheridge, 2002)
\[
\tilde{W}(t, S) = e^{-rt}W(t, Se^\theta), \quad \tilde{V}_i = \tilde{W}(t, X_i).
\]

We observe that (28) becomes:
\[
\frac{\partial \tilde{W}(t, S)}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{W}(t, S)}{\partial S^2} = 0.
\] (29)

Using the finite-difference method, after an initial discretization of the domain as follows:
\[
\begin{aligned}
& t_j = jk \quad j \geq 0 \\
& S_j = ih \quad 0 \leq i \leq n = 1
\end{aligned}
\] (30)

with \( t \) and \( S \) having different step sizes \((k, h)\) respectively, we have
\[
\frac{k}{h^2} [V_{i+1,j} - 2V_{ij} + V_{i-1,j}] = -\frac{2}{\sigma^2 S^2} [V_{i,j+1} - V_{ij}]
\]
or
\[
V_{i,j+1} = -\tilde{\xi} [V_{i+1,j} - 2V_{ij} + V_{i-1,j}] + V_{ij},
\] (31)

where \( \tilde{\xi} = \frac{k \sigma^2 S^2}{2h^2} \). Applying the Fourier method, we have:
\[
e^{ij\sigma h}e^{i(n+1)ak} = \tilde{\xi}e^{i(j-1)ak} + (1 + 2\tilde{\xi})e^{iak}e^{iak} - \tilde{\xi}e^{i(j+1)ak} e^{iak},
\]

which gives after removing the factor \( e^{iak} \)
\[
e^{iak} = -\tilde{\xi} e^{-iak} + 1 + 2\tilde{\xi} - \tilde{\xi} e^{iak}
\]
\[
= -2\tilde{\xi} \cos \sigma h + 1 + 2\tilde{\xi}
\]
\[
= 1 + 2\tilde{\xi}(1 - \cos \sigma h)
\]
\[
= 1 + 4\tilde{\xi} \sin^2 \frac{\sigma h}{2}.
\] (32)

(We have used the familiar equation \( e^{i\theta} = \cos \theta + i \sin \theta, 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \)). In (32) \( \tilde{\xi} \geq 1 \) (as long \( \sigma \) is real). For stability, we require also that \( \tilde{\xi} \leq 1 \) and this leads to the restriction \( \tilde{\xi} \sin^2 \left(\frac{\sigma h}{2}\right) \leq -\frac{1}{2} \). Since \( \sin^2 \frac{\sigma h}{2} \) can be close to 1, we must have \( \tilde{\xi} \leq -\frac{1}{2} \) for stability.

**Parameter estimation and empirical example**

The linear stochastic differential equation has been given previously as \( dS_t = \alpha dt + \sigma dB_t \). We want to estimate \( \alpha \). The log likelihood is
\[
\frac{\alpha \sigma}{\sigma} \left[ \int_0^T dS_t - \frac{\alpha \sigma}{2\sigma} \int_0^T S_t dt \right] = \frac{\alpha}{\sigma} \left\{ \alpha \left( S(T) - S(0) \right) \right\} - \frac{1}{2} \left( S_T - S_0 \right),
\] (33)
Table 1. UACN quotation on NSE market.

<table>
<thead>
<tr>
<th>S/No. (Q)</th>
<th>Deals (No.)</th>
<th>Quantity</th>
<th>Value (₦)</th>
<th>Share price (S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>3,750</td>
<td>21,000.00</td>
<td>5.60</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>129,783</td>
<td>726,784.80</td>
<td>5.60</td>
</tr>
<tr>
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<td>4</td>
<td>511,751</td>
<td>2,737,867.85</td>
<td>5.35</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>34,362</td>
<td>183,836.70</td>
<td>5.35</td>
</tr>
<tr>
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<td>5</td>
<td>6,498</td>
<td>34,114.50</td>
<td>5.25</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>11,189</td>
<td>58,742.25</td>
<td>5.25</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>22,324</td>
<td>117,201.00</td>
<td>5.25</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>5,062</td>
<td>18,729.40</td>
<td>3.70</td>
</tr>
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<td>7</td>
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<td>235,235.00</td>
<td>4.70</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>25,202</td>
<td>107,108.50</td>
<td>4.25</td>
</tr>
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<td>11</td>
<td>29</td>
<td>77,285</td>
<td>241,902.05</td>
<td>3.13</td>
</tr>
</tbody>
</table>

whence the maximum likelihood estimates of $\alpha$ is

$$\hat{\alpha} = \frac{S_T - S_0}{\int S_i di} = \frac{S_T - S_0}{\sum S_i}. \quad (34)$$

This approximation is a Riemann sum using the observed values $S_1, ..., S_T$. The approximate variance of $\hat{\alpha}$ is given by

$$\sigma^2 = \frac{1}{T} \sum_{i=1}^{T} \left( \frac{S(i2^{-n})}{S((i-1)2^{-n})} \right). \quad (36)$$

The data are only observed to precision $n=0$, yielding the approximate estimate

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{i=1}^{T} \log^2 \left[ \frac{S_i}{S_{i+1}} \right]. \quad (37)$$

By (11), we have

$$S \frac{dW}{dS} = W. \quad (38)$$

Putting (38) into (8) with

$$S = S \text{ gives } \alpha \frac{dW}{dS} - r_t \frac{dW}{dS} \text{ or } \frac{dW}{dS} = \frac{1}{\sigma} \frac{1}{r_t - \alpha} \quad \text{(using (13)).} \quad (39)$$

Equation (39) is called the market growth rate of the stock shares.

We shall now use the following data to interpret the model. The data below in Table 1 are some of the United African Company of Nigeria (UACN) quotations on the Nigeria Stock Exchange (NSE), Lagos, between February and August, 1999. Table 2 are some of the industrial and domestic products quotations on the Nigeria Stock Exchange (NSE) in Financial Standard (FS), Vol.17 No.18, February 20, 2006.

Fitting analysis for the models

The fitness of Table 1, we take trading time $t$ in days.

$T=180$ days, with $t = \frac{n}{T}$. The parameter estimated

$t = 0.061, \alpha = 0.0462$ (using (34)), $\sigma^2 = 0.00262$ (using (37)), $r_t = 0.0493$ (using (13)), $\lambda_t = 1.065088$ (using (12a)). The fitness of Table 2, we take trading time $t$ in days.

$T=28$ days, with $t = \frac{n}{T}$, $t = 0.429, \alpha = 0.00036$ using (34)), $\sigma^2 = 0.15489$ (using (37)). $r_t = 0.169199$ (using (13)), $\lambda_t = 1.975772$

Conclusion

It is observed that whenever $\alpha > r_t$ then $\lambda_t$ is such that $0 < \lambda_t < 1$, so that $\hat{S}$ in (25) is always positive. But when-
ever $\alpha < r$, then $\lambda < 1$. This guarantees that $\hat{S}$ is always positive in (21) and (26). From another perspective, it is observed that the growth rate of the stock market (using (39)) depends on the value of $\sigma$.

Further more, if $|k^{\text{vol}}| > 1$, the solution in (32) becomes unbounded. Since any numerical solution will eventually be contaminated by all unwanted extraneous solution, these unbounded solutions will dominate the true solution, which is decaying exponentially.

REFERENCES

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