Full Length Research Paper

Optimum design of planar beams based on sensitivity analysis

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In this study, the size optimization problem of a beam, in which the beam heights are determined along the beam axis, is considered as a shape optimization problem of a plate in plane elasticity. To achieve the shape optimization, different principles such as structural analysis, sensitivity analysis and mathematical programming are interrelated. The ANSYS that is a reliable finite element package programme is used only for the static analysis. The objective of this optimization problem is to minimize the volume of the beam under the constraints that the maximum value of the von Mises stress in each node do not exceed a predefined value and the beam axis is to remain straight. Design variables are the vertical coordinates of the corner nodal points of design elements. The design sensitivities are calculated through the finite, difference method and linear programming technique is used for obtaining the final shape of a beam. Several examples are solved under different loadings and boundary conditions. The obtained optimum beam volumes and the design variables are given in tables and the optimum beam shapes are plotted. One of the results given is compared with the literature.

Key words: Shape optimization, sensitivity analysis, finite difference method.

INTRODUCTION

Studies in the structural design field have developed a new research area called structural shape optimization. Structural shape optimization is described in order to determine the best structural shape design satisfying a predefined objective function under geometric and behaviour restrictions among possible designs (Reaklaitis et al., 1983; Vanderplaats, 1984; Uysal, 2002). The work done by Zienkiewicz and Campbell (1973) was the first to determine the optimum shape of a structure. After then, many researchers have investigated the shape optimization of structures (Haftka and Prasad, 1979; Pedersen and Laursen, 1982; Choi and Haug, 1983; Ding, 1986; Chen, 1989; Yao and Choi, 1989; Gates and Accorsi, 1993; Pourazady and Fu, 1996). The growing interest on this subject reflects the importance of the effect of structural shape on structural performance. Structural optimization is composed of description of geometric model and mesh generation, definition of objective function, selection of design variables, use of sensitivity analysis and an optimization solution method. Depending on the selection of the design variables, the structural optimization can be classified as sizing optimization and shape optimization. The geometric shape of the structure remains unchanged for the sizing optimization case. For example the design variables are taken as plate thickness or laminate angles in the composite material structures (Haftka, 1979). In the case of shape optimization, the structural shape continually changes when the design variables are re-obtained in each optimization step. Thus, the shape optimization is apparently more complex than the sizing optimization (Yao and Choi, 1989; Gates and Accorsi, 1993).

After theoretical foundations have been (Zienkiewicz...
and Campbell, 1973; Reaklaitis et al., 1983; Vanderplaats 1984) made in the development of finite element-based shape optimization (Ding, 1986; Chen, 1989; Yao and Choi, 1989; Gates and Accorsi, 1993; Pourazady and Fu, 1996) in more recent works, it is more convenient to use a reliable finite element package program for structural analysis in optimization process (Holzleitner and Mahmoud, 1999; Duan and Sheppart, 2002; Asci et al., 2003; Lee and Lee, 2004; Uysal et al., 2007).

In this paper, the beam optimization is considered as a shape optimization of a plate in plane elasticity and then, the shape optimization of beam structures are achieved. To achieve the shape optimization, different principles such as structural analysis, sensitivity analysis and mathematical programming are interrelated. The objective is to minimize the volume of the beam under constraints that the maximum value of the von Mises stress in each element and move-limits for each design element do not exceed a predefined value and beam axis is to remain straight. The stability of beams is not considered. The optimum shapes of the beams subject to different boundary conditions and different loadings are determined.

SHAPE OPTIMIZATION

Mathematical programming representation

The shape optimization problem can be stated mathematically as:

Find \( \Rightarrow \) minimum \( F(S) \)

Subject to \( \Rightarrow g_i(S) \leq 0 \quad i=1,\ldots,m \)

\( h_j(S) = 0 \quad j=1,\ldots,l \)

\( S_k^l \leq S_k \leq S_k^u \quad k=1,\ldots,n_v \) (1)

where \( S = (S_1, S_2, \ldots, S_n) \) is the vector of design variables, \( F(S) \) is the objective function, \( g_i(S) \) are the inequality behaviours constraint functions, \( h_j(S) \) are the equality behaviours constraint functions, \( S_k^l \) and \( S_k^u \) are lower and upper limits of the shape design variable, \( S_k \), respectively. In addition, \( m \) is the number of inequality constraints, \( l \) is the number of equality constraints and \( n_v \) is the number of design variable.

In this study the volume is used as the objective function and sequential linear programming (SLP) is used to minimize the objective function with respect to inequality and equality behaviour constraint equations.

Selection of objective function and constraints functions

The objective function can be selected in many ways. In most cases, the weight or volume of a structure is chosen as the objective function. In this study, as indicated earlier, the volume is used as the objective function, that is:

\[ F(S) = \sum_{e=1}^{ne} V_e \] (2)

Where \( V_e \) is the volume \( e \) th finite element.

The maximum values of the von Mises stress in each element are taken as constraint functions. The stress constraints that the maximum values of the von Mises stress in each node \( \sigma_v \) do not exceed the allowable stress \( \sigma_0 \), which can be stated as follow:

\[ g_i(S) = \sigma_v = \sqrt{(\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)} \leq \sigma_0 \] (3)

Besides these stress constraints, for the beam axis to remain straight, some equality constraints are added. By consideration, the average of the two design variables on the same vertical have to be equal to the beam axis's ordinate. These equations are stated as follows:

\[ h_1(S) = \frac{S_1 + S_8}{2} - 115 = 0, \]

\[ h_2(S) = \frac{S_2 + S_9}{2} - 115 = 0, \]

\[ h_3(S) = \frac{S_3 + S_{10}}{2} - 115 = 0, \]

\[ h_4(S) = \frac{S_4 + S_{11}}{2} - 115 = 0, \]

\[ h_5(S) = \frac{S_5 + S_{12}}{2} - 115 = 0, \]
\[
\begin{align*}
    h_6(S) &= \frac{S_6 + S_{13}}{2} - 115 = 0, \\
    h_7(S) &= \frac{S_7 + S_{14}}{2} - 115 = 0 \\
    (4)
\end{align*}
\]

**Sensitivity analysis**

Design sensitivity analysis, that is, the calculation of quantitative information on how the response of a structure is affected by changes in the variables that define its shape, play an important role in structural shape optimization. There are three basic approaches to the calculation of sensitivities. The first approach is based on variation of the continuum equations (Choi and Huang, 1983; Haftka and Mroz, 1986), the second approach on differentiation of the finite element equations (Pedersen and Laursen, 1982; Pourazady and Fu, 1996), and the last approach on the method of finite differences.

The variation of continuum equations is based on a functional, which represents a continuum structural model. This approach appears to be a more general approach than the others, but finding the functional for a given structural is the major obstacle.

In the finite element method, the structural equilibrium equations are given by:

\[
[K] [U] = [P]
\]

(5)

Where \([K]\) is the structural stiffness matrix, \([U]\) is the unknown nodal displacement vector and \([P]\) is the generalized load vector. Using the direct differentiation method, the derivative of Equation (5) with respect to \(m\) th design variable \(S_m\) is given by

\[
[K] \left\{ \frac{\partial U}{\partial S_m} \right\} + \left\{ \frac{\partial K}{\partial S_m} \right\} [U] = \left\{ \frac{\partial P}{\partial S_m} \right\} \text{ or}
\]

(6)

\[
[K] \left\{ \frac{\partial U}{\partial S_m} \right\} = [R]
\]

Where \([R]\) is defined as:

\[
[R] = - \left\{ \frac{\partial K}{\partial S_m} \right\} [U] + \left\{ \frac{\partial P}{\partial S_m} \right\}
\]

Equations (5) and (6) are similar and have the same stiffness matrix \([K]\). First \([R]\) is evaluated and then \(\frac{\partial U}{\partial S_m}\) is obtained by solving Equation (6) to determine \([R]\), derivative of stiffness matrix \([K]\) and the force vector \([P]\) with respect to design variable \(S_m\) should be evaluated. In the finite element method, the stiffness matrix \([K]\) is obtained by assembling all the element stiffness matrices.

\[
[K] = \sum_{c=1}^{cl} [K]_c
\]

(7)

In the natural coordinate system, \([K]_c\) is defined by:

\[
[K]_c = \left\{ \int_{-1}^{1} \int_{-1}^{1} \left[ B \right]^T (\xi, \eta) \left[ D \right] (B) (\xi, \eta) [J] d\xi d\eta \right\}
\]

(8)

Assuming that the mechanical properties of a material do not change with the design variables, then the following equation is obtained:

\[
\frac{\partial K}{\partial S_n} = \sum_{j=1}^{n} \omega_j W^j \left\{ \left[ B \right]^T \left[ D \right] [B] + [B]^T \left[ (\frac{\partial B}{\partial S_m}) [B] \right] \right\}
\]

(9)

This equation can be evaluated numerically by using the Gauss quadrature method:

\[
\frac{\partial K}{\partial S_n} = \sum_{j=1}^{n} \omega_j W^j \left\{ \left[ B \right]^T \left[ D \right] [B] + [B]^T \left[ (\frac{\partial B}{\partial S_m}) [B] \right] \right\}
\]

(10)

Here \(W^j\) and \(W^k\) are the weighting factor and \(n_g\) is the number of sampling points. To find the derivative of the stiffness matrix \([K]\), the derivatives \(\frac{\partial [B]}{\partial S_m}\) and \(\frac{\partial [J]}{\partial S_m}\) should be evaluated first. The derivative of \([B]\) with respect to \(S_m\) is given by:

\[
\frac{\partial [B]}{\partial S_m} = \left\{ \frac{\partial B_1}{\partial S_m} \frac{\partial B_2}{\partial S_m} \frac{\partial B_3}{\partial S_m} \ldots \frac{\partial B_i}{\partial S_m} \ldots \frac{\partial B_n}{\partial S_m} \right\}
\]

(11)

Where
According to the chain rule differentiation, the derivatives of the shape functions \( N_i \) in the natural coordinate systems are related by:

\[
\frac{\partial N_i}{\partial x} = [J]^{-1} \frac{\partial N_i}{\partial \xi} \quad \frac{\partial N_i}{\partial y} = [J]^{-1} \frac{\partial N_i}{\partial \eta} \quad (13)
\]

Design variables are defined by the global coordinates of the corner coordinates of design elements called master nodes. Therefore, \( N_{i,x} \), \( N_{i,y} \) which are functions of local coordinates of a Gauss sampling points \((\xi, \eta)\) only, will not change with design variables. Thus:

\[
\frac{\partial}{\partial S_m} \left[ \frac{\partial N_i}{\partial x} \right] = \frac{\partial [J^{-1}]}{\partial S_m} \left[ \frac{\partial N_i}{\partial \xi} \right] = \frac{\partial [J]}{\partial S_m} \left[ \frac{\partial N_i}{\partial \xi} \right] \quad (14)
\]

Where

\[
\frac{\partial [J]}{\partial S_m} = \left[ \frac{\partial x}{\partial S_m} \frac{\partial y}{\partial S_m} \right] \quad (15)
\]

Next derivative of the stress vector is evaluated. For a plane stress problem, this vector is given as:

\[
\{\sigma\} = [D][\epsilon] = [D][B][U]_e \quad (16)
\]

Where \( \{U\} \) and \( \{\epsilon\} \) are the displacement and strain vectors, respectively. The derivative of the stress vector is given as:

\[
\frac{\partial \{\sigma\}}{\partial S_m} = [D] \left( \frac{\partial \{B\}}{\partial S_m} \right) \quad (17)
\]

These derivatives are used in the normalized constraint equations. For example, a von Mises stress constraint \( \sigma_c \), and its derivatives are expressed as:

\[
g_i = 1 - \frac{\sigma_c}{\sigma_0} > 0
\]

\[
\frac{\partial g_i}{\partial S_m} = \frac{\partial}{\partial S_m} \left( 1 - \frac{\sigma_c}{\sigma_0} \right) = -1 \frac{\partial \sigma_c}{\partial S_m} \quad (18)
\]

Where \( \sigma_0 \) is the allowable stress. For a plane stress problem \( \sigma_c \), which was defined previously in Equation (3), it is expressed as:

\[
(18)
\]

\[
\sigma_c = \left( \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right)^{1/2}
\]

Therefore

\[
\frac{\partial \sigma_c}{\partial S_m} = \frac{1}{2} \left[ (2\sigma_x - \sigma_y) \frac{\partial \sigma_x}{\partial S_m} + (2\sigma_y - \sigma_x) \frac{\partial \sigma_y}{\partial S_m} + 6\tau_{xy} \frac{\partial \tau_{xy}}{\partial S_m} \right].
\]

\[
\left( \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right)^{1/2} \quad (19)
\]

To find \( \frac{\partial g_i}{\partial S_m} \), Equation (19) is substituted into Equation

\[
\left( \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right)^{1/2} \quad (18)
\]

In the last method, the finite differences method is one of the most popular methods. This method has been considered as a reference method for linear elastic problems since it performs well for most cases (Pedersen and Laursen, 1982; Haftka and Mroz, 1986). However, it has serious limitations related to truncation and round off error. These errors can usually be minimized by using an appropriate step size of the design perturbation. Another drawback of the finite differences method is the
computational time: If the n design variables are used in the optimization, then n additional finite element analysis are required. Even though this method has several drawbacks, it is widely adopted because of its generality and simplicity (Lee and Hinton, 2000; Uysal et al., 2007).

In this study, the first-order forward finite difference is used to calculate the design sensitivities of the objective function and constraint functions.

The design sensitivities for objective function can be written as:

\[
\nabla_i F \approx \frac{F(S_1, ..., S_i + \Delta S_i, ..., S_n) - F(S_1, ..., S_n)}{\Delta S_i} \quad (20)
\]

and for constraint functions:

\[
\nabla_i g_j \approx \frac{g_j(S_1, ..., S_i + \Delta S_i, ..., S_n) - g_j(S_1, ..., S_n)}{\Delta S_i} \quad (21)
\]

Where \( \Delta S_i \) is a small perturbation in the variables \( S_i \).

Sequential linear programming methods and problem statement

In generally, the objective and constraint functions are non linear functions with respect to shape design variables. Sequential linear programming method involves linearizing the nonlinear objective function \( F(S) \) and nonlinear constraints \( g_j(S) \) about a design point. So, these functions are linearized properly in the neighborhood of design variable to simplify the problem and so, the nonlinear problem is approximately converted into a series of linear problems. Using the Taylor's expansion and in neglecting higher order terms, we will get the expanded form of these functions as follows:

\[
F = F^n + \sum_{j=1}^{n} \nabla_j F \Delta S_j = F^n + (\nabla_n F)^T \delta \{ S \} \quad (22)
\]

\[
g_i = g_i^n + \sum_{j=1}^{n} \nabla_{ij} g_j \Delta S_j = g_i^n + (\nabla_n g_i)^T \delta \{ S \} \quad (23)
\]

Combining Equations (22) and (23), Optimization problem can be stated as:

\[
\begin{align*}
\nabla_n^T F & \geq \min \\
\nabla_n^T g_1 & \geq -g_1^n \\
\nabla_n^T g_2 & \geq -g_2^n \\
\n\vdots & \vdots \\
\nabla_n^T g_n & \geq -g_n^n
\end{align*}
\]

For this approach the optimum solution is always obtained at one of the vertices formed by the design constraints (Reaklaitis et al., 1983; Vanderplaats, 1984; Pourazdy and Fu, 1996). Since there are finite numbers of such vertices in the feasible region, a systematic approach such as simplex method is used to search for the optimum solution among these vertices.

Move-limits to control convergence properties

In a sequential linear programming method, the result of each iteration is expected to be a better solution. Since the linear programming technique is used to find the optimum solution for a nonlinear problem, some measures have to be taken to improve the optimization process. In this study, to prevent discontinuity in the constraint functions and reduce the irregular movement of the design variable, extra constraint equations called move-limits are introduced to control the changes in the design variables. These equations are given as:

\[
|\delta S_i| \leq \bar{\delta}_i \quad i = 1, ..., n_v \quad (24)
\]

where \( \bar{\delta}_i \) is the limit for the movement of the design variable \( S_i \). Equations (25) can be expressed by two inequalities:

\[
\delta S_i \leq \bar{\delta}_i \quad \text{or} \quad S_i \leq S_i^n + \bar{\delta}_i \quad (26)
\]

\[
\delta S_i \geq -\bar{\delta}_i \quad \text{or} \quad S_i \geq S_i^n - \bar{\delta}_i \quad (27)
\]

The success of the optimization process depends on the selection of the move-limits value. No general rule can be stated for different problems. Generally, large move-limits values should be used in first few iterations and value should be reduced as the number of iterations increased.
Table 1. Initial values of design variables (mm).

<table>
<thead>
<tr>
<th>Design variable number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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</thead>
<tbody>
<tr>
<td>x coordinate</td>
<td>0</td>
<td>30</td>
<td>60</td>
<td>90</td>
<td>120</td>
<td>150</td>
<td>180</td>
<td>0</td>
<td>30</td>
<td>60</td>
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<td>y coordinate</td>
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<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

EXAMPLES

For the all examples, the dimensions of the initial beams are taken as $180 \times 30 \times 1$ mm, which is same as the beam in Pourazdy and Fu's (1996) study and the corner nodal coordinates of the design elements that are the initial values of the design variables are given in Table 1. The mesh size is determined according to Pourazdy and Fu's (1996) study. Each beam is modelled by six design element (Figure 1) and each design element is then subdivided into quadrilateral elements by using ANSYS's plane 82 element. Thus the finite element mesh is generated. Elastic modules, Poisson ratio and allowable stresses are $2.07E5 \text{ N/mm}^2$, 0.3 and $20 \text{ N/mm}^2$, respectively. The following loads and support conditions are considered in examples:

1. Concentrated load at the mid section.
2. Uniformly distributed loading.
3. Uniformly distributed loading in the first half part.

The support conditions are:

1. Both ends of the beam are fixed.
2. Both ends of the beam are hinged, the hinges are at the bottom edge of the cross section.
3. Both ends of the beam are hinged; the hinges are at the neutral axis of the cross section.

The optimum shapes of the beams are obtained for those boundary conditions and loadings. In these examples, the optimum shapes of the beams are shown in Figures 2, 4, 6 and the volume iteration histories are shown in Figures 3, 5 and 7. The optimum values of the design variables and the volume at optimum state are given in Table 2 for each example.

Example 1: A beam under the concentrated load at the mid section

For this case, the beams subject to concentrated load at the mid span and different support conditions are considered. The loading and support types of beams are shown in Figure 2. For this loading condition, four examples are solved and the obtained optimum beam shapes and the volume iteration histories are shown in Figures 2 and 3, respectively. Depending on support type,
Figure 2. Initial and optimum beam shapes for different support conditions under the concentrated load at the mid section.

Figure 3. Volume history for the beams under the concentrated load at the mid section.
Figure 4. Initial and optimum beam shapes for different support conditions under the uniform load.

Figure 5. Volume history for the beams under the uniform load.
Figure 6. Initial and optimum beam shapes for different support conditions under the uniform load in the first half or the second one.

Figure 7. Volume history for the beams under the uniform load in the first half or the second one of the beam.

Table 2. Optimum values of design variables and beam volume for all examples.

<table>
<thead>
<tr>
<th>Design variable number</th>
<th>Optimum values of design variables (mm)</th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Example 1</td>
<td>Example 2</td>
<td>Example 3</td>
<td>Example 3</td>
</tr>
<tr>
<td>1</td>
<td>137.26 118.05 124.60 128.25 125.70 146.21 119.09 125.51 136.54 118.63</td>
<td>137.26 118.05 124.60 128.25 125.70 146.21 119.09 125.51 136.54 118.63</td>
<td>137.26 118.05 124.60 128.25 125.70 146.21 119.09 125.51 136.54 118.63</td>
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<td>117.44 122.89 120.78 122.85 123.50 122.03 123.61 121.30 117.12 122.28</td>
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</tr>
<tr>
<td>3</td>
<td>125.99 125.67 120.27 123.17 123.50 118.39 125.57 117.57 120.92 123.56</td>
<td>125.99 125.67 120.27 123.17 123.50 118.39 125.57 117.57 120.92 123.56</td>
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<tr>
<td>4</td>
<td>126.33 128.29 124.82 129.25 126.80 120.18 126.18 119.40 119.81 123.22</td>
<td>126.33 128.29 124.82 129.25 126.80 120.18 126.18 119.40 119.81 123.22</td>
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</tr>
<tr>
<td>5</td>
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<td>137.25 118.05 124.59 128.38 125.70 146.20 119.09 125.50 130.65 116.78</td>
<td>137.25 118.05 124.59 128.38 125.70 146.20 119.09 125.50 130.65 116.78</td>
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<tr>
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<td>92.74 111.95 105.40 101.75 101.70 83.79 110.91 104.49 93.46 111.37</td>
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<td>104.01 104.33 109.73 106.83 106.50 111.61 104.43 112.43 109.08 106.44</td>
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Table 2. Optimum values of design variables and beam volume for all examples (Continued).

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<th>Optimum volume (mm$^3$)</th>
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<td>2490.9</td>
<td>3582.8</td>
<td>3510</td>
</tr>
</tbody>
</table>

The results under Ref. column are belongs to the results of the work done by Pourazady and Fu (1996).

Example 2: A beam under the uniformly distributed load

The beams subject to uniform distributed load and different support conditions are considered secondary (Figure 4). Three examples are solved and the obtained optimum beam shapes and the volume of iteration histories are shown in Figures 4 and 5, respectively. The volume of the beam reduced from 5400 - 1959.1 mm$^3$ when the beam supports are fixed. The final values of the design variables are given in Table 2.

Example 3: A beam under the uniformly distributed load in the first half of the beam

For this case, the beams subject to uniform distributed load at the first half part of the beam and different supports are considered. The loading and support types of beams are shown in Figure 6. For this loading condition, two examples are solved and the obtained optimum beam shapes and the volume iteration histories are shown in Figures 6 and 7, respectively.

The results in Pourazady and Fu's work (1996), which correspond to the Figure 1 (Example 4) in this study, are given and compared in Figure 8 and Table 2. In Figure 8, these optimum beams are also compared with the result of Asci et al. (2003).
Conclusions

The problem of determining the beam heights along the beam axis has been considered as a shape optimization of a plate in plane elasticity and different principles such as structural analysis, sensitivity analysis and mathematical programming are interrelated to achieve the shape optimization. The active constraints reach the equality case at the optimum state. The selection of different beam dimensions at the beginning of optimization does not seriously affect the obtained optimum beam; rather it affects only the iteration number. The beam sections subjected to the larger bending moments have higher depths than the other sections. When the supports are fixed or hinge supports are at the bottom edge of the beam cross section, the depths of the obtained optimum shapes is high in this cross section at which these supports have been. In these support case, the horizontal force parallel to the beam axis at these supports, results in larger bending moment at the centroid of the cross section and so, the depth of the cross sections which are subjected to the larger bending moment is higher than the others. Similarly, this statement has been reported in the work by Uysal et al. (2004) in which the shape optimization problem of a beam is studied and in the work by Daloglu et al. (1997) and Uzman et al. (1999) in which the sized optimization problems of an arch is examined.

The results in Pourazady and Fu (1996) and the present paper have been given together in Figure 8 and Table 2. They are very similar. Because of the arch form, the volume of the beam in Pourazady and Fu’s study (1996) is less than the beam in this study. The optimization problem in which the beam axis was not forced to remain straight as a constraint was differently resulted and instead of a beam, an arch was obtained as an optimum shape (Asci et al., 2003).

REFERENCES


Uysal H, Asci N, Uzman U (2004). Structural shape optimization of a beam considered as a two dimensional elasticity problem, Sixth International Conference on Advances in Civil Engineering, Istanbul, Turkey, pp. 843-851.


