Full Length Research Paper

Optimal capital structure decision with credit rating management

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This paper examines the impact of credit rating management on determining optimal capital structure. The models capture twofold empirical behavior of credit rating management: the using of rating-linked coupon scheme and target minimum rating policy. Numerical results showed that as long as the rating at issuing time is not too low, tax shields of the rating-linked coupon debt are larger than those of standard debt with same par, and hence, optimal leverage usage of the firm with rating-linked coupon scheme is greater. Further, the behavior of targeting a minimum rating causes a mean-reverting leverage dynamic. Following a downgrade from target minimum rating, managers appear to make over-repurchased choices for adjusting the current ratings back to the initial target.

Key words: Optimal capital structure, credit rating, rating-linked coupon scheme, target minimum rating policy.

INTRODUCTION

“Credit rating” is always the most prevailing and significant measurement for predicting corporate debt’s default risk in making capital structure decision.1 Motivated by the strong linkage between leverage and credit rating, there is a growing amount of literature with making efforts to the understanding of firms’ credit rating management for financing activities. Kisgen (2006a) proposes credit rating-capital structure hypothesis (henceforth denoted by “CR-CS”) that shows how credit ratings directly affect how the firm’s capital structure decisions are made by managers. Kisgen (2006b) observes that the firms’ exhibit behavior is consistent with targeting minimum ratings. The behavior following the receipt of downgrades and upgrades is asymmetric. Firms experiencing deterioration in ratings will undertake a leverage-reducing activity to regain the previous better rating, while upgraded firms do not significantly undertake such an activity. Manso et al. (2010) join with Acharya et al. (2002) and Lando and Mortensen (2005) to concentrate on issues of performance-sensitive debt.2 They old that

1Standard and Poor’s (2001) explicitly defines issue credit ratings to be an opinion of the creditworthiness of an obligor with respect to a specific financial obligation. A firm credit rating provides potential investors with its own information on credit quality beyond publicly available information. Hence, credit ratings can release significant and sensitive information that is not public if the firm is unwilling to disclose private information that may compromise its strategic programs, in particular with regard to its opponents. A CFO survey by Graham and Harvey (2001) indicates that credit rating is the second greatest concern for CFOs when considering debt issuance. Following Graham and Harvey, Molina (2005) also suggests that leverage and credit rating should be jointly considered during analysis, and thus, uses the debt’s rating as a proxy for default risk in measuring leverage impact on default probability.

2Performance-sensitive debt is the class of debt obligation whose interest payment is linked with some measure of borrower’s performance, such as credit rating or financial ratios. In practice, credit-sensitive notes (Acharya et al., 2002) and step-up bonds (Lando and Mortensen, 2005) are both special cases of performance-sensitive debt. For notational convenience in this study, all these types of debt will be termed “rating-linked coupon debt”. The idea of linking credit quality to debt’s cashflow is not new. This type of credit-sensitive derivative has been openly traded since the late 1980s. For more details on their development, see Das and Tufano (1996), Acharya et al. (2002), Lando and Mortensen (2005), and Manso et al. (2010).
relative to a fix-coupon debt, default time of the issuer of performance-sensitive debt is earlier. Higher bankruptcy costs thus cause a smaller initial market value of the equity, suggesting that the behavior of linking performance measure (for example, credit rating) to the debt’s cash flow seems inefficient.

Value-maximization is a common objective for managers to optimally make the debt-financing choices. Issues that show how firms reach the capital structure optimization thus naturally receive considerable attention from corporate finance studies.3 aforementioned sentiments on the credit rating management, however, are not considered and not explained by traditional capital structure theories. As a result, a question that remains open is how credit rating management affects firm’s optimal capital structure decisions.

The first contribution of this research is to construct a firm-value based credit migration framework, and to show its application on modeling the capital structure decision problem. The models we propose allow for examining the impacts of credit rating management on determining optimal financing policies. Differing with a class of existing Markov Chain models (Jarrow et al., 1997), the innovation lies in the dimension that considers a system of rating transition multi-boundaries. Such a firm-value based idea prevents from the appearance of the conflict between intensity-based approaches and fundamentals of capital structure theory in the model.4 To ensure the existence of a critical value boundary for each credit rating, this study assumes that there exists a mapping between the rating of the reference asset and the value of firm as the issuer of the reference asset. A numerical example for calculating firm-value based rating transition probabilities is also provided.5

Several aspects of the benefit of this multi-boundaries system are as follow. First, the evolution of credit rating is according to fluctuations in the asset value; since the system is operated in continuous time, the estimated accuracy for probability with long migrating distance is improved.6 Further, via rating transition boundaries and instantaneous asset value, investors can easily observe a firm’s current credit rating; not having to recheck credit rating data saves a tremendous amount of money. Finally, the series of rating transition boundaries is uniquely dependent on the leverage choice (Graham and Harvey, 2001; Molina, 2005), but also, rating transited probability can truly reflect firm-specific information (for example asset value volatility).

This paper proposes two extended models that combine the capital structure problem with rating-related empirical behavior. One is the case of the firm selling a debt with rating-linked coupon scheme, which not only implements the idea of linking credit rating to debt’s cashflow in practice, but also considers direct benefits (costs) to a firm receiving rating upgrades (downgrades) that arise from a decrement (increment) in its coupon rate. The other is debt with a rating-dependent callable option derived from target minimum rating policy. In this case, there is only one chance for managers to adjust the firm’s capital structure and credit rating. Repurchase activities are under-taken only when the rating fails to remain in the target range during the debt’s life. The size of the repurchase is predetermined and depends on firm credit rating policies. Via this repurchase, the firm will temporarily regain previous higher ratings after receiving a downgrade from the target minimum rating.

The main findings contributed to the literature in two ways. Firstly, in considering the behavior of linking the firm’s credit to the promised coupon, the result indicates that as long as the debt’s rating at issuing time is not too low, the firm using rating-linked coupon debt has larger tax shields relative to the case of using standard debt with same par. Managers, thus, are motivated to issue more rating-linked coupon debt, implying that the corresponded optimal leverage usage is greater also. Secondly, if the firm exhibits the behavior as targeting a minimum rating, debt-repurchase activities required by the credit rating policy will generate a mean-reverting leverage dynamic. Following a downgrade from target minimum rating, managers appear to make over-repurchase choices to adjust their current rating back to the initial target.

A FIRM-VALUE-BASED FRAMEWORK OF CREDIT MIGRATION

Merton (1974) is the earliest example of serving total assets’ value as the firm’s solvency, and defines that bankruptcy is triggered at the debt’s maturity only when the solvency is insufficient to meet current obligation. The idea of bankruptcy-triggered single boundary is the core of his pricing theory for corporate debt. Black and Cox (1976) improve Merton to develop a model that can capture the rights of creditors to force a firm into bankruptcy once its asset value drops too low to violate the covenant. Such a model is subsequently termed “first-passage-time model.” Based on these two articles, the study expands the idea of single bankruptcy-triggered boundary to become a system of rating transition multi-boundaries.
The model

Without loss of generality, firstly consider a circumstance in a continuous trading economy where a reference unlevered firm tends to issue a debt with finite maturity $T$. The reference firm’s assets are assumed to have total value unaffected by capital structure, which is denoted at time-$t$, by $V(t), t \in R_+$ following a diffusion process:

$$dV(t)/V(t) = (\mu - \delta)dt + \sigma dW_p(t) \quad (1)$$

Where $\mu$ represents appreciation rate; $\delta$ stands for payout ratio of the total assets' value; and $\sigma$ plays a constant volatility. $W_p(\cdot)$ denotes a single dimensional Weiner process defined over a filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t \geq 0}, P)$. $\Omega$ is the sample space. Sigma algebra, $\mathcal{F}_t$, collects the information generated by the observation of asset value up to time-$t$ and available to all agents in the economy. $P$ is the so-called real measure for historical probability. To examine this framework in a risk neutral world, the study still supposes that default-free bonds are allowed for trading in this economy and pay a constant interest rate, $r$.

Now consider the definition of firm’s rating process. Let $N = \{1, 2, ..., n\}$ be the space of all possible credit states, where state $n$ denotes the highest credit rating state (that is, AAA rate), state 2 denotes the lowest credit rating, and state 1 is the default rate. Given a debt rating system consisting of AAA, AA, A, BBB, BB, B, CCC, and D, the total number of possible states $n$ thus equals to 8. Also let $D^i_k, i, k \in N, k \neq 1$ denotes the rating $i$’s lower transition boundary under initial credit rating $k$. Each boundary is predetermined by capital decision and implicates a specific least-required level of solvency (asset value). At the moment in which the path of asset value hits the boundary, the firm will receive a rating downgrade or upgrade. Since a higher rating always requires a stronger solvency, the level of the boundary must be monotonically increasing with the intended rating; namely, $D^i_k > D^j_k$ for $i > j$. Further, the lowest boundary $D^1_k$ is assigned as zero, implying that there is no requirement for solvency when retaining the default rate. The boundary $D^i_k$ is gifted with the function of bankruptcy-trigger, and hence can be set as the par of the firm’s debt.

Assume that the credit rating of the firm’s debt $\eta^i(\cdot)$ follows a Markov process on finite space $N$ with initial state, $k$. The firm-value-based representation for the rating process is given as: (i) when $s = 0$

$$[\{ \omega : \eta^k(\omega, s) = k \} = \{ \omega : V(\omega, s) \in [D^1_k, \infty) \}, k = n]$$

$$[\{ \omega : \eta^k(\omega, s) = k \} = \{ \omega : V(\omega, s) \in [D^1_k, D^{k+1}_k] \}, k = 2, ..., n-1] \quad (2)$$

And (ii) when $0 < s \leq T$:

$$[\{ \omega : \eta^k(\omega, s) = j \} = \{ \omega : \inf_{0 \leq \omega \leq s} V(\omega, s) \in [D^j_k, \infty) \}, V(\omega, s) \in [D^j_k, D^{j+1}_k] \}, j = n]$$

$$[\{ \omega : \eta^k(\omega, s) = j \} = \{ \omega : \inf_{0 \leq \omega \leq s} V(\omega, s) \in [D^j_k, \infty) \}, V(\omega, s) \in [D^j_k, D^{j+1}_k] \}, j = 2, ..., n-1]$$

$$[\{ \omega : \eta^k(\omega, s) = j \} = \{ \omega : \inf_{0 \leq \omega \leq s} V(\omega, s) \in [0, D^j_k] \} \}, j = 1] \quad (3)$$

Where $\inf_{0 \leq \omega \leq s} V(\omega)$ denotes the minimum of the path of asset value over the period $[0, s]$.

Applying the probability theory with above expressions in Equations 2 and 3, the rating transition at any time during the debt's trading period can be formed as a $n$ by $n$ continuous time probability matrix:

$$P_{n,n} = \begin{pmatrix} p_{1,1}(0,s) & p_{1,2}(0,s) & ... & p_{1,n}(0,s) \\ p_{2,1}(0,s) & p_{2,2}(0,s) & ... & p_{2,n}(0,s) \\ . & . & . & . \\ . & . & . & . \\ p_{n,1}(0,s) & p_{n,2}(0,s) & ... & p_{n,n}(0,s) \\ 0 & 0 & ... & 1 \end{pmatrix} \quad (4)$$

Note that each entry of Equations 4:

$$p_j^i(0,s) = P(\eta^i(s) = j \mid \eta^i(0) = k)$$

For $j, k \in N$ symbolizes the transition likelihood of going from rating $k$ at time-0 to $j$ at time-$s$ under historical measure $P$. The sum of each entry in the same row must equal one. The last row in the matrix must satisfy the property that the default state is absorbing.

To solve the matrix (4) under the risk-neutral measure $Q$, the use of Girsanov’s theorem and of reflection principle is required. For technical details to the derivation for
risk-neutral rating transition probabilities, one can refer to appendix. The study now compiles the formulas as the following proposition:

\[
q_i^j(0,s) = \begin{cases} 
N(\theta_2) - (D_i^j / V(0))^2 \quad & N(\theta_j) \quad , j = n \\
N(\theta_4) - N(\theta_1) - (D_i^j / V(0))^2 \quad [N(\theta_j) - N(\theta_2)] \quad , j = 2, \ldots, n-1 \\
(D_i^j / V(0))^2 \quad N(\theta_4) + 1 - N(\theta_j) \quad , j = 1
\end{cases}
\]

Where, \( s \in [0,T] \), \( \chi = 2\lambda / \sigma^2 \), and \( \lambda = r - \delta - (1/2)\sigma^2 \). The arguments for \( N(.) \), the standard normal cumulative distribution, are given by

\[
\theta_1 = f(2,-1,0,-1) ; \theta_2 = f(0,1,0,-1) ; \theta_3 = f(2,-1,0,-1) ; \theta_4 = f(0,1,0,-1) \quad \theta_5 = f(2,-1,0,-1) ; \theta_6 = f(1,1,0,0) ; \theta_7 = f(-1,1,0,0)
\]

and the compound function is expressed as:

\[
f(a, b, c, d) = (\sigma s^\delta)^{-3} \left( a \ln D_i^j + b \ln V(0) + c \ln D_{ji+1}^j + d \ln D_i^j + \chi s \right)
\]

Via the formulas in Proposition 1, the evolution of the credit rating during the debt’s lifespan can be examined. Subsequently, these will help us value defaultable contingent claims derived from the firm’s capital structure decision, such as corporate debt, tax benefits, and bankruptcy costs.

**A numerical example**

Here, a numerical example of estimating rating transition boundaries and calculating transition probabilities is subsequently given. For tractability, let \( V(0) = $100 \) \( r = 5\% \), \( \delta = 3.75\% \), \( \sigma = 38.02\% \), and \( T = 5 \). The estimation of rating transition boundaries is roughly achieved in five steps. In the first step, the study takes an economy-wide five-year rating transition probability matrix that can be estimated from market data (for example, Standard and Poor’s special report). Given this probability matrix and the combination of the above parameters, the study figures out a rating transition boundary matrix based on an economic-wide credit migration as the second step. In the third step, the study divides the boundary matrix by its last column to obtain a new matrix, termed the economic-wide rating transition structure. Each element in this matrix denotes a specific multiplier of the lowest required solvency that corresponds to its own migration distance. Each series composed of multipliers in same row represents the rating transition structure over the estimated period under a certain initial credit state. Via the firm assets’ value and the diagonal of transition structure matrix, the implied relation between the initial credit states and debt-issuing amount (as measured by its par) is clarified in the fourth step. Importantly, this step helps establish the rule of determining the start credit state. In the final step, multiplying the chosen debt’s face value by the corresponding rating transition structure brings the series of rating transition boundaries.

Recall the restriction on computing the probabilities with long rating migration distance in existing literatures. These probabilities are often omitted as zero and thus are underestimated. Further, this will lead to the failure in solving the rating transition boundaries. Fortunately, a continuous time hidden Markov chain model, introduced by Christensen et al. (2004), can address this issue. The main advantage of using their model in the estimation of rating transition probabilities is to improve the estimated accuracy for those with special focus on the rare events. For simplicity, here the study directly borrows a five-year transition probability matrix from that article and compile it as the Panel A of Table 1 after making some adjustments. The adjustments to the probability matrix are made to ensure that the ranking order obeys the following properties: (i) each element in the matrix is strictly non-negative, and the sum of elements in the same row is always equal to one; (ii) better ratings should never have greater chance of bankruptcy; (iii) as the migration distance is longer, the chance of migration should become less; and (iv) the possibility of migrating to a given rating will be larger for more closely adjacent rating categories.

The economic-wide rating transition boundary matrix is compiled in the Panel B of Table 2. Via the weighted operation, Panel C of Table 1 gives the economic-wide rating transition structure. The size of the multiplier in the transition structure matrix determines the required intensity for solvency. The fact that the multiplier in same row is monotonically increasing with the intended rating implies

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1 The value of the assets’ volatility is taken from Ju et al. (2005).
Table 1. Economy-wide 5-year transition for credit rating. Panel A: transition probabilities.

<table>
<thead>
<tr>
<th>ICR</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: transition probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA</td>
<td>9.61E-01</td>
<td>3.64E-02</td>
<td>2.11E-03</td>
<td>1.38E-04</td>
<td>3.14E-06</td>
<td>1.39E-07</td>
<td>1.06E-08</td>
<td>1.13E-08</td>
</tr>
<tr>
<td>AA</td>
<td>7.76E-03</td>
<td>8.86E-01</td>
<td>1.02E-01</td>
<td>4.06E-03</td>
<td>2.32E-04</td>
<td>1.25E-05</td>
<td>1.44E-06</td>
<td>1.06E-06</td>
</tr>
<tr>
<td>A</td>
<td>1.50E-03</td>
<td>1.97E-02</td>
<td>8.78E-01</td>
<td>7.97E-02</td>
<td>1.52E-02</td>
<td>5.37E-03</td>
<td>4.33E-04</td>
<td>2.84E-05</td>
</tr>
<tr>
<td>BBB</td>
<td>1.34E-03</td>
<td>1.96E-03</td>
<td>7.22E-02</td>
<td>8.58E-01</td>
<td>5.44E-02</td>
<td>6.24E-03</td>
<td>8.97E-04</td>
<td>3.55E-04</td>
</tr>
<tr>
<td>BB</td>
<td>1.19E-04</td>
<td>2.62E-04</td>
<td>1.44E-02</td>
<td>1.49E-01</td>
<td>7.54E-01</td>
<td>7.43E-02</td>
<td>6.28E-03</td>
<td>1.61E-03</td>
</tr>
<tr>
<td>B</td>
<td>1.65E-05</td>
<td>2.33E-04</td>
<td>4.59E-03</td>
<td>2.23E-02</td>
<td>3.57E-02</td>
<td>6.96E-01</td>
<td>1.80E-01</td>
<td>6.04E-02</td>
</tr>
<tr>
<td>CCC</td>
<td>2.80E-07</td>
<td>1.72E-06</td>
<td>1.41E-04</td>
<td>9.11E-04</td>
<td>3.96E-03</td>
<td>8.17E-04</td>
<td>5.87E-01</td>
<td>3.26E-01</td>
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<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>1</td>
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</table>

Panel B: transition boundaries

<table>
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<tr>
<th>ICR</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>16.51</td>
<td>6.63</td>
<td>3.39</td>
<td>1.61</td>
<td>0.96</td>
<td>0.70</td>
<td>0.58</td>
<td>0</td>
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<td>AA</td>
<td>580.47</td>
<td>25.71</td>
<td>7.95</td>
<td>3.84</td>
<td>2.13</td>
<td>1.52</td>
<td>1.18</td>
<td>0</td>
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<tr>
<td>A</td>
<td>924.08</td>
<td>416.56</td>
<td>25.03</td>
<td>13.17</td>
<td>8.69</td>
<td>4.44</td>
<td>2.14</td>
<td>0</td>
</tr>
<tr>
<td>BBB</td>
<td>951.90</td>
<td>746.61</td>
<td>244.16</td>
<td>20.04</td>
<td>9.37</td>
<td>5.66</td>
<td>3.62</td>
<td>0</td>
</tr>
<tr>
<td>BB</td>
<td>1686.59</td>
<td>1297.39</td>
<td>471.98</td>
<td>170.66</td>
<td>22.75</td>
<td>9.51</td>
<td>5.18</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>2529.76</td>
<td>1430.70</td>
<td>669.03</td>
<td>380.78</td>
<td>272.56</td>
<td>40.69</td>
<td>15.71</td>
<td>0</td>
</tr>
<tr>
<td>CCC</td>
<td>5223.39</td>
<td>3739.36</td>
<td>1620.28</td>
<td>1012.47</td>
<td>662.21</td>
<td>35.23</td>
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<td></td>
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<tr>
<td>D</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
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</table>

Panel C: transition structure

<table>
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<tr>
<th>ICR</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
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<tbody>
<tr>
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<td>28.346</td>
<td>11.3811</td>
<td>5.8115</td>
<td>2.7617</td>
<td>1.6522</td>
<td>1.2014</td>
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<td>AA</td>
<td>490.8550</td>
<td>21.7397</td>
<td>6.7188</td>
<td>3.2436</td>
<td>1.8048</td>
<td>1.2860</td>
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<tr>
<td>A</td>
<td>431.7437</td>
<td>194.6243</td>
<td>11.6963</td>
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<td>4.0587</td>
<td>2.0723</td>
<td>1.0</td>
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<tr>
<td>BBB</td>
<td>262.7213</td>
<td>206.0619</td>
<td>67.3879</td>
<td>5.5299</td>
<td>2.5871</td>
<td>1.5624</td>
<td>1.0</td>
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<tr>
<td>BB</td>
<td>325.4149</td>
<td>250.3215</td>
<td>91.0655</td>
<td>32.9285</td>
<td>4.3885</td>
<td>1.8356</td>
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</tr>
<tr>
<td>B</td>
<td>161.0033</td>
<td>91.0553</td>
<td>42.5798</td>
<td>24.2344</td>
<td>17.3466</td>
<td>2.5895</td>
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<td>CCC</td>
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<td>45.9863</td>
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<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
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</tr>
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</table>

Note: ICR denotes initial credit rating.

Table 2. Relation between initial rating and the range of debt issuing amount.

<table>
<thead>
<tr>
<th>Initial Credit Ratings (Investment-Grade)</th>
<th>Variable</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Credit Ratings (Speculative-Grade)</td>
<td>Variable</td>
<td>BB</td>
<td>B</td>
<td>CCC</td>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FV ($)</td>
<td>0~3.5280</td>
<td>3.5280~4.5998</td>
<td>4.5998~8.5497</td>
<td>8.5497~18.0834</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FV ($)</td>
<td>18.0834~22.7867</td>
<td>22.7867~38.6175</td>
<td>38.6175~99.9999</td>
<td>N/A</td>
<td></td>
<td></td>
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</tbody>
</table>

Note: FV denotes face value.

that if a firm tends to get a better credit rating, the minimum required level of its solvency becomes greater.

Table 2 displays implied relations between initial credit ratings and debt issuing amount. As the amount of issuance is larger, the initial credit will be determined at a lower rating. Such relation is consistent with much of empirical evidence, including Huang and Huang (2003), Klock et al. (2005), Molina (2005), Kisgen (2006a), and Guttler and Wahrenburg (2007). Based on the result exhibited in Table 2, managers can optimally make the decision of leverage.
usage for each target credit rating. The value of the firm’s rating transition probabilities are calculated and compiled as a matrix in Table 3. Each row of this matrix represents, under a certain initial rating, the probability distribution of the rating transition at debt’s maturity. The value of each element measures the possibility of the credit rating finally entering into a corresponded state from a given initial state. Intuitively, if the initial credit rating is lower, the series of transition boundaries becomes greater to reflect more difficulty in standing on better ratings and a higher potential for going to bankrupt. Due to this fact, the firm with the lower initial rating finally has a higher chance of obtaining worse credit ratings and greater default frequencies.

An application of capital structure problem (the base case)

Here, the static contingent claim model for capital structure proposed by Ju et al. (2005) is rebuilt. Recall the circumstance where the reference firm tends to issue a finite maturity debt. At initial time, assume that managers will determine the debt’s rating at a certain non-default level consistent with the long-term target to implement the target initial rating policy. Based on the choice of target rating, firm’s leverage usage is bounded in a corresponded range. The optimization of the capital structure choice is reached by maximizing the wealth of equity-holders conditioned on active credit rating policy.

The debt has a series of rating transition boundaries \( D^k_l, \ldots, D^k_1 \), determined by its face value \( F^k_L \). The debt continuously pays the coupon at a constant annualized rate \( C_L^k \) that is depended on the initial rating solely. The coupon payment shields income from taxes at effective rate \( \beta^k \), and tax benefits enjoyed by the debt have the value \( TB^k(0) \) at initial time. The debt requires a protective minimum rating covenant that if the credit rating falls to state D at any time during the life of debt, the firm is forced into bankruptcy. The default-triggering time thus can be expressed as:

\[
\tau_L^k = \inf\{ s \in [0, T], \eta^k(s) = 1 \} = \inf\{ s \in [0, T], V(s) \leq F^k_L \}
\]

Once the bankruptcy is declared, equity (stock) becomes valueless and only a portion \( 1 - \alpha \) of the levered value of assets can be redeemed by debt holders. The fractional loss of the assets’ value is supposed to be expended in the bankruptcy process. Bankruptcy costs thus are the present value of expected losses in bankruptcy and can be denoted by \( BC_L^k(0) \).

Following Leland and Toft (1996) and Ju et al. (2005), the value of debt at the initial time sums a contribution from the coupon, a contribution from the recovered payment to debt holders if the firm bankrupts, and the repayment of par if bankruptcy does not occur until to maturity \( T \), that is,

\[
D^k_L(0) = E^k_0\left( \int_0^T C_L^k e^{-\theta t} 1_{\tau_L^k \leq t} dt + \frac{TV^k_L(0)}{V(0)}(1 - \alpha) V(T_L^k) e^{-\theta T_L^k} 1_{\tau_L^k > T_L^k} + F^k_L e^{-\theta T_L^k} 1_{\tau_L^k > T_L^k} \right)
\]

\(5\)

\( \tau_L^k = \inf\{ s \in [0, T], \eta^k(s) = 1 \} = \inf\{ s \in [0, T], V(s) \leq F^k_L \} \)

\(1\) For the comparability with the subsequent case of rating-linked coupon debt, let the determination of the coupon rate be exogenous via observing average spreads on market defaultable bonds whose current ratings are the same as the firm’s initial rating. Such a doing still captures the natural relation between debt’s credit and the interest payments, although it seems inconsistent with most traditional models with endogenous coupon rates.

\(12\) For more details of refinancing, see Ju et al. (2005).
Where \( E^Q_0(\cdot) \) denotes the expectation operator conditioned on \( \mathcal{F}_0 \) under risk-neutral measure, \( TV_L^k(0) = V(0) + TB_L^k(0) - BC_L^k(0) \) is the total levered value of the firm at time zero, and \( 1_{\cdot} \) denotes the indicator function. The factor \( TV_L^k(0)/V(0) \) implements the modeling decision that, upon bankruptcy, the firm reorganizes and debt holders are allowed to takeover and become new shareholders. After the bankruptcy process, new shareholders will optimally raise the new debt and receive \( TV_L^k(0)/V(0) \) of the remaining asset value \((1-\alpha) V(\tau^k_L)\). By the use of Proposition 1, the explicit solution for \( D_L^k(0) \) is as follows:

\[
D_L^k(0) = \int_0^T C_L^k e^{-\alpha} \sum_{j=2}^n q_j^k(0,s) ds + \int_0^T TV_L^k(0)(1-\alpha) F_L^k(V(0))^{-1} e^{-rs} f_{\tau^k_L}(s) ds
\]

where \( f_{\tau^k_L}(s) = \partial q_j^k(0,s)/\partial s \) denotes the probability density for firstly entering the default state at time-s. It is known that the bankruptcy process consumes \( \alpha V(\tau^k_L) \), and hence the present value of bankruptcy costs is:

\[
BC_L^k(0) = E^Q_0(\alpha V(\tau^k_L)e^{-rs} 1_{(\tau^k_L \leq T)}) = \int_0^T \alpha F_L^k e^{-rs} f_{\tau^k_L}(s) ds
\]

Notice that, in (6), the value of the remaining asset the study uses is levered, but the value of the lost assets used for computing bankruptcy cost is unlevered. This is because the value in (7) corresponds to the cost to the original stockholders before the firm is levered. The interest tax shields of debt accrue to the firm as long as it does not go to bankruptcy. Thus the current value of tax benefits can be computed by:

\[
TB_L^k(0) = E^Q_0(\int_0^T \beta C_L^k e^{-rs} 1_{(\tau^k_L \leq s)} ds) = \int_0^T \beta C_L^k e^{-rs} \sum_{j=2}^n q_j^k(0,s) ds
\]

The equity’s value equals total levered value of the firm less the value of debt, that is,

\[
E_L^k(0) = V(0) + TB_L^k(0) - BC_L^k(0) - D_L^k(0)
\]

For the objective of maximizing the equity value, one can differentiate Equation 9 with respect to the debt’s face value, set this expression equal to zero, and then solve the optimal capital structure variables.

**DEBT WITH RATING-LINKED COUPON SCHEME**

The spread between the coupon and the risk-less interest rate symbolizes the compensation for the risk debt-holders bear. Usually, the changes in risk bearing are observed by tracking transitions in the credit rating. If so, the assumption in the base case where the firm determines its coupon at a constant rate solely dependent on the initial rating is insufficient. Such doing only considers the current rating but ignores the possibility of the future rating’s transition, since the firm’s credit can always improve or deteriorate during the life of debt. To capture the credit sensitivity on a coupon rate, the rating-dependent effects with tradeoff theory was compared to extend the base model in the way that replaces the scheme of a constant coupon with a rating-linked coupon.

**The model**

Replicate the situation in the base case to omit having to repeatedly introduce the main economic setup. Thus the subscript \( f \) can be substituted for \( L \) when referring to the quantities.

Then the study restricts the attention on the most significant feature of the model, “rating-linked coupon scheme.” While a firm exhibits behavior as linking its credit to the promised coupons, interest payment carried by debt is no longer fixing, and would jump with the change in credit rating during the debt’s lifespan. That is, for \( 0 \leq s \leq T \):

\[
C^f_j(s) = C^{AAA}_1 q_{(1)}^{(j,AAA)} + C^{AAA}_1 q_{(1)}^{(j,AAA)} + C^{A+1}_1 q_{(1)}^{(j,A+1)} + C^{BBB}_1 q_{(1)}^{(j,BBB)} + C^{BB}_1 q_{(1)}^{(j,BB)} + C^{CCC}_1 q_{(1)}^{(j,CCC)} = \sum_{j=2}^n C_j^{1}(q_{(1)}^{(j,r)} j)
\]
Notice that there leaves no role for $C^D$ to play in expression (10). This is because the debt’s protective covenant may force the firm to go to bankrupt once the rating falls to the D level. For comparability, the spreads between the coupon and the risk-free interest rate are assumed to be consistent with those in the base case; namely, \( C^k_F(s) / F^k_s - r = C^k_L / F^k_s - r \) conditional on a certain non-default state \( \eta^k(s) = k \).

Now consider the initial value of the debt paying the rating-linked coupon. The study rewrites Equation 5 by replacing $C^k_L$ with Equation 10 and then derives as
\[
D^k_f(0) = \sum_{k=0}^{\infty} \int_0^T C^k e^{-rs} q^k_f(0,s) ds + \int_0^T TV^k_f(0)(1-\alpha) F^k_s(V(0))^{-1} e^{-rs} f^k_c(s) ds 
\]
\[
+ F^k_s e^{-rT} \sum_{k=0}^{\infty} q^k_f(0,T) 
\]

When compared to the base case, the mere discrepancy between Equations 6 and 11 appears on the contribution from the coupon payment. While the rate of coupon no longer depends on the initial rating solely, the structure of the coupon value becomes more complicated. Given different instantaneous ratings, the coupon paid by the firm has an individual present value. In pricing the total contribution to the debt from the coupon payment, these values are separately computed with the corresponded coupon rate first, and then summed up to yield the total value of the rating-linked coupon. To further study the difference in the value of rating-linked and constant coupon, a closer inspection is given subsequently.

Following the spirit of Equations 7 and 8, the bankruptcy costs and the tax benefits in this model have respective current value:
\[
BC^k_f(0) = \int_0^T \alpha F^k_s e^{-rs} f^k_c(s) ds 
\]
\[
TB^k_f(0) = \sum_{k=0}^{\infty} \int_0^T \beta C^k e^{-rs} q^k_f(0,s) ds 
\]

Note that Equation 12 is almost equivalent to Equation 7. This is because there is not any influence on the debt’s default-triggered protective covenant, no matter how rating transitions drive the coupon rate. The value of the interest tax shields becomes more sensitive to the rating’s migration when compared to Equation 8. This reflects the fact that the coupon in Equation 13 is dependent on the instantaneous rating level rather than fixing. The equity has initial value equaling the unlevered value of the assets plus the value of the tax benefits from the debt minus the bankruptcy costs and the value of the debt,
\[
E^k_f(0) = V(0) + TB^k_f(0) - BC^k_f(0) - D^k_f(0). 
\]

The impact of rating-linked coupon scheme on optimal capital structure

Here, the study jointly explores the issues of how the behavior of linking the firm’s rating to the promised coupons impacts the optimal capital structure decision, and why a firm in practice issues rating-linked coupon debt. At the beginning of the analysis, firstly select the value of some relevant parameters. For simplicity, let the economic-wide rating transition structure follow Table 1. Assume that the total assets value of the unlevered reference firm is normalized to equal $100 at the initial time, which is divided among one hundred shares, each worth $1. The dynamics of the total asset value has constant payout ratio \( \delta = 3.75\% \) and volatility \( \sigma = 38.02\% \). The maturity of debt sold by firm \( T = 5 \), the effective tax rate \( \beta = 40\% \), the bankruptcy cost parameter \( \alpha = 30\% \), the risk-free interest rate \( r = 5\% \), and the five-year rating-dependent average spread on the corporate debt’s coupons (The choices of spreads are broadly referred to Huang and Huang (2003) and Moody’s special report. The negative relation between average credit spreads and target ratings is consistent with the empirical findings of most existing literatures, including Longstaff and Schwartz (1995), Kan (1998), Huang and Huang (2003), Lando and Mortensen (2005), and Kisgen (2006b)).:
\[
C^k_f(s) / F^k_s - r = 1_{(q^k_f(x)=AAA)} 40bps + 1_{(q^k_f(x)=AA)} 55bps + 1_{(q^k_f(x)=A)} 120bps + 1_{(q^k_f(x)=BBB)} 210bps 
\]
\[
+ 1_{(q^k_f(x)=BB)} 330bps + 1_{(q^k_f(x)=B)} 470bps + 1_{(q^k_f(x)=CCC)} 620bps. 
\]

Given the combination of the reasonable range of model’s parameters, outputs of the model with rating-linked/constant coupon scheme are compiled in Table 4.

Observe that along different target initial credit ratings,
the relation of optimal debt ratios between two cases is asymmetric. When initially targeting at investment-grade or BB level, the optimal ratios of debt to total capital in the case of rating-linked coupon scheme are smaller than those with constant coupon scheme (the contrast between numbers in Columns 1 and 6). This is reversed, however, if the target initial rating is lower than BB. To explain this fact, the study moves the attention to the comparison of tax benefits and bankruptcy costs between two cases. Since the values of coupon payment and tax benefits are strongly positive-correlated, an in-depth inspection to the structure of the rating-linked/constant coupon value is required and helpful.

From Table 5, observe that their values are identical only when an instantaneous rating is consistent with the initial target. Otherwise, the values of the rating-linked coupon are always larger (smaller) if the instantaneous rating is inferior (superior) to its initial target. Except for the case of targeting the initial rating at CCC, the rating-linked coupon payments usually have greater total present values. There are two reasons for explaining such results. First is the nonlinear negative relation between the credit rating and the required credit spreads: more clearly, the increment in spreads is increasing as the rating falls. Second is the asymmetry in the estimation of the rating transition probability.

Figures 1 and 2 provide clear checks to such an asymmetry. In a joint view of these two figures, the probabilities of the firm’s credit remaining in ratings lower than the initial target during the debt’s life are significantly greater, while those remaining in ratings higher than the initial target usually close to zero. This has an interesting implication: a levered firm is always more likely to receive downgrades than upgrades no matter what the initial rating is inferior (superior) to its initial target. Except for the case of targeting the initial rating at CCC, the rating-linked coupon payments usually have greater total present values. There are two reasons for explaining such results. First is the nonlinear negative relation between the credit rating and the required credit costs. More clearly, the increment in spreads is increasing as the rating falls. Second is the asymmetry in the estimation of the rating transition probability.

The values of bankruptcy costs in the case with rating-linked/constant coupon scheme are given in Columns 3 and 8 of Table 4, respectively, and appear to be almost alike. A notable exception, however, arises in the case where the target initial rating policy is annulled (or the case where the initial rating is targeted at B). It is observable that the value of bankruptcy costs of debt with rating-linked coupon scheme equalling $1.3331 is larger than that with constant coupon scheme equalling $1.1303, suggesting that the optimal usage of the former is greater. This is because constraints on the debt usage are released in the absence of target rating policy. Optimal debt-issuing amount depends on the tradeoff between tax benefits and bankruptcy costs. Based on tradeoff theory, the account of firm using more rating-linked coupon debt relative to standard debt, in such case, is due to the cognition that the former commonly yields bigger tax benefits under the same face value and bankruptcy loss.

Lump the analysis of tax benefits and bankruptcy costs together to have the explanation for the pattern of optimal debt ratios. When making under-levered choices to target the initial rating at higher levels, the behavior of linking the credit rating to the promised coupons aids the firm in obtaining a greater total leveraged value and a smaller optimal debt ratio via increasing tax benefits. On the other hand, if managers target an extremely low rating and makes overlevered choices, tax benefits from a rating-linked coupon debt are smaller than the base case since the firm paying the rating-linked coupon may experience a downward jump in the coupon payment. Thus this yields a larger optimal ratio of debt-to-total capital.

To clarify why a firm in the practice issues a rating-linked coupon debt, the study co-focuses on the market value of debt (numbers in Columns 4 and 9) and the changes in per-share price due to leverage (numbers in Columns 5 and 10) reported by Table 4. It is observed that the debt with rating-linked coupon scheme is more valuable than the standard debt under the same par value, except for the case of target initial rating CCC. Similarly, the levered benefits to shareholders derived from issuing a rating-linked coupon debt are usually greater than those from issuing a constant coupon debt. The intuition behind these facts implies that unless the firm is overlevered and holds too low of a credit rating, rating-linked coupon scheme cannot only allow debt-holders to earn more coupons but also benefit shareholders via additional tax shields. Loosely speaking, as long as the underlying credit quality at the time of issuance is not too bad, using the rating-linked coupon debt in general is truly more beneficial to the firm.

**DEBT WITH TARGET MINIMUM RATING POLICY**

The study starts from the point to credit rating effect on the allowance of investing in corporate debts. Regulations based on ratings, in practice, determine whether certain investor groups can purchase risky bonds (Cantor and Packer, 1994; SEC, 2003). For example, banks have already been restricted from holding the bonds with speculative-grade ratings since 1936. In 1989, savings and loans were prohibited from investing in junk bonds, too. Due to these facts, Kisgen (2006b) argues that, in reality, maintaining a particular rating level provides benefits to firms. Thus managers are motivated to target a minimum rating at which the regulations affect the investments in a firm’s bonds, such as targeting the investment-grade level.
Table 4. Comparison between model outputs with rating-linked/constant coupon scheme where objective is to maximize share value.

<table>
<thead>
<tr>
<th>TICR</th>
<th>Debt/total capital (%)</th>
<th>Tax benefits ($)</th>
<th>Bankruptcy costs ($)</th>
<th>Value of debt ($)</th>
<th>Change in share price ($)</th>
<th>Debt/total capital (%)</th>
<th>Tax benefits ($)</th>
<th>Bankruptcy costs ($)</th>
<th>Value of debt ($)</th>
<th>Change in share price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.5158</td>
<td>0.3476</td>
<td>0.0003</td>
<td>3.6165</td>
<td>0.0035</td>
<td>3.5162</td>
<td>0.3371</td>
<td>0.0003</td>
<td>3.5902</td>
<td>0.0034</td>
</tr>
<tr>
<td>AA</td>
<td>4.5775</td>
<td>0.4673</td>
<td>0.0011</td>
<td>4.7997</td>
<td>0.0049</td>
<td>4.5792</td>
<td>0.4517</td>
<td>0.0011</td>
<td>4.7186</td>
<td>0.0045</td>
</tr>
<tr>
<td>A</td>
<td>8.4513</td>
<td>1.0652</td>
<td>0.0208</td>
<td>9.3040</td>
<td>0.0104</td>
<td>8.4721</td>
<td>0.9367</td>
<td>0.0208</td>
<td>8.9825</td>
<td>0.0092</td>
</tr>
<tr>
<td>BBB</td>
<td>17.7075</td>
<td>2.5089</td>
<td>0.3863</td>
<td>20.0687</td>
<td>0.0212</td>
<td>17.7566</td>
<td>2.2264</td>
<td>0.3863</td>
<td>19.3599</td>
<td>0.0184</td>
</tr>
<tr>
<td>BB</td>
<td>22.1800</td>
<td>3.5653</td>
<td>0.8300</td>
<td>26.0836</td>
<td>0.0274</td>
<td>22.2560</td>
<td>3.2132</td>
<td>0.8300</td>
<td>25.1963</td>
<td>0.0238</td>
</tr>
<tr>
<td>B</td>
<td>25.7608</td>
<td>4.4500</td>
<td>1.3331</td>
<td>30.9361</td>
<td>0.0312</td>
<td>24.4334</td>
<td>4.0905</td>
<td>1.3331</td>
<td>29.0596</td>
<td>0.0296</td>
</tr>
<tr>
<td>CCC</td>
<td>37.6325</td>
<td>6.4182</td>
<td>3.8009</td>
<td>43.8945</td>
<td>0.0397</td>
<td>37.6206</td>
<td>6.4509</td>
<td>3.8009</td>
<td>43.9792</td>
<td>0.0265</td>
</tr>
<tr>
<td>No target</td>
<td>25.7608</td>
<td>4.4500</td>
<td>1.3331</td>
<td>30.9361</td>
<td>0.0312</td>
<td>24.4334</td>
<td>4.0905</td>
<td>1.3331</td>
<td>29.0596</td>
<td>0.0296</td>
</tr>
</tbody>
</table>

TICR denotes target initial credit rating.

Table 5. Comparison between the structures of present value of rating-linked/constant coupon payment.

<table>
<thead>
<tr>
<th>TICR</th>
<th>Value of rating-linked coupon ($)</th>
<th>Value of constant coupon ($)</th>
<th>Value of rating-linked coupon ($)</th>
<th>Value of constant coupon ($)</th>
<th>Value of rating-linked coupon ($)</th>
<th>Value of constant coupon ($)</th>
<th>Value of rating-linked coupon ($)</th>
<th>Value of constant coupon ($)</th>
<th>Value of rating-linked coupon ($)</th>
<th>Value of constant coupon ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.4566E-01</td>
<td>3.4566E-01</td>
<td>4.2622E-01</td>
<td>4.1470E-01</td>
<td>7.8865E-02</td>
<td>6.8689E-02</td>
<td>1.6653E-02</td>
<td>1.2665E-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BBB</td>
<td>1.5382E-07</td>
<td>2.0225E-07</td>
<td>5.6677E-07</td>
<td>7.2505E-07</td>
<td>3.5236E-04</td>
<td>4.0351E-04</td>
<td>2.3291E+00</td>
<td>2.3291E+00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BB</td>
<td>9.8163E-09</td>
<td>1.5088E-08</td>
<td>5.0225E-08</td>
<td>7.5111E-08</td>
<td>2.9631E-05</td>
<td>3.9667E-05</td>
<td>4.4749E-03</td>
<td>5.2312E-03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>4.4068E-07</td>
<td>1.0537E-06</td>
<td>1.2898E-05</td>
<td>2.8844E-05</td>
<td>7.2314E-04</td>
<td>1.3789E-03</td>
<td>9.0634E-03</td>
<td>1.4551E-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CCC</td>
<td>9.7054E-08</td>
<td>2.0130E-07</td>
<td>7.6006E-07</td>
<td>1.5338E-06</td>
<td>1.1407E-04</td>
<td>2.0607E-04</td>
<td>1.2116E-03</td>
<td>1.9113E-03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No target</td>
<td>25.7608</td>
<td>4.4500</td>
<td>1.3331</td>
<td>30.9361</td>
<td>0.0312</td>
<td>24.4334</td>
<td>4.0905</td>
<td>1.3331</td>
<td>29.0596</td>
<td>0.0296</td>
</tr>
</tbody>
</table>

TICR denotes target initial credit rating.
Figure 1. Probabilities of firm’s credit remaining in the ratings lower than initial target as a function of time.
or minimum B rating. Following rating downgrades, firms attempt reducing capital market activity and leverage ratio to regain their previous better rating. To study the impact of such interesting behavior, an extended model that considers a rating-dependent callable option on the debt was offered. In particular, this option is effective only when the firm’s credit has already fallen below the target minimum rating.

The model
Following the economic setup given previously, replace subscript \( L \) with \( R \) to refer to quantities of this model.
Table 6. Outputs of model with/without minimum investment-grade policy where the objective is to maximize share value.

<table>
<thead>
<tr>
<th>TICR</th>
<th>Debt/total capital (%)</th>
<th>Repurchase ratio</th>
<th>Regained rating via repurchase</th>
<th>Tax benefits ($)</th>
<th>Bankruptcy costs ($)</th>
<th>Change in share price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.516156</td>
<td>N/A</td>
<td>N/A</td>
<td>0.337116</td>
<td>2.6646E-04</td>
<td>3.3685E-03</td>
</tr>
<tr>
<td>AA</td>
<td>4.579165</td>
<td>N/A</td>
<td>N/A</td>
<td>0.451711</td>
<td>1.0943E-03</td>
<td>4.5062E-03</td>
</tr>
<tr>
<td>A</td>
<td>8.472111</td>
<td>N/A</td>
<td>N/A</td>
<td>0.936665</td>
<td>2.0848E-02</td>
<td>9.1582E-03</td>
</tr>
<tr>
<td>BBB</td>
<td>17.756551</td>
<td>N/A</td>
<td>N/A</td>
<td>2.226419</td>
<td>3.8626E-01</td>
<td>1.8402E-02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TICR</th>
<th>Debt/total capital (%)</th>
<th>Repurchase ratio</th>
<th>Regained rating via repurchase</th>
<th>Tax benefits ($)</th>
<th>Bankruptcy costs ($)</th>
<th>Change in share price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.516155</td>
<td>13.4165%</td>
<td>BBB</td>
<td>0.337016</td>
<td>1.1906E-04</td>
<td>3.3690E-03</td>
</tr>
<tr>
<td>AA</td>
<td>4.579164</td>
<td>4.5829%</td>
<td>BBB</td>
<td>0.451498</td>
<td>8.5810E-04</td>
<td>4.5064E-03</td>
</tr>
<tr>
<td>A</td>
<td>8.472111</td>
<td>0.0000%</td>
<td>N/A</td>
<td>0.936665</td>
<td>2.0848E-02</td>
<td>9.1582E-03</td>
</tr>
<tr>
<td>BBB</td>
<td>17.756551</td>
<td>&gt;0.0000%*</td>
<td>BBB</td>
<td>2.226419</td>
<td>3.8626E-01</td>
<td>1.8402E-02</td>
</tr>
</tbody>
</table>

TICR denotes target initial credit rating. *, When a firm initially targets rating at BBB, there is no distance between initial and minimum target for credit rating. Since our system is operated in continuous time, managers can immediately regain BBB by the exercise of callable option following a downgrade, even if repurchase ratio is fairly small.

Consider now the case where the reference firm has a policy of jointly targeting the initial rating at $k$, consistent with long-term target, and a minimum rating at $m$ where $m \leq k$. Under this policy, managers are endowed with a right to restructure the firm’s capital, namely, one-time callable option on the debt.\(^{15}\) The option is rating-dependent and can be exercised only if the firm’s rating fails to remain in the target range $[m, n]$ during the life of debt.

To manage the credit rating, managers can undertake a debt repurchase activity as soon as the debt’s rating first enters into a state lower than the target minimum rating (for example, state $m-1$):

$$
\psi_m = \inf (\eta^s(s) = m-1 \mid n \geq k \geq m > 2, s \in [0, T])
$$

$$
= \inf (V(s) \leq D^k_m \mid n \geq k \geq m > 2, s \in [0, T]).
$$

(14)

In making a repurchase activity, managers will retire the outstanding debt at par. The repurchase size, $\rho \in [0,1]$, is predetermined based on the goal of regaining a rating consistent with their initial target.\(^{16}\) After the repurchase, the firm’s default-triggered threshold drops from $F^k_R$ to $(1-\rho)F^k_R$, equaling the par value of remaining debt. The corresponding rating transition boundaries fall by the same factor to become $(1-\rho)^kD^k_1, \ldots, (1-\rho)^kD^k_n$. Since $(1-\rho)^kF^k_R = (1-\rho)^kD^k_\rho < D^k_m$, firm’s default time should never be earlier than the repurchase time, and hence has following expression:

$$
\tau^k_R = \inf (\psi^s(s) = 1 \mid n \geq k \geq m > 2, s \in (\psi_m^s, T])
$$

$$
= \inf (V(s) \leq (1-\rho)^kF^k_R \mid n \geq k \geq m > 2, s \in [0, T]).
$$

(15)

To examine the random time, Equations 14 and 15, Proposition 1 can be simplified as below.

**Corollary 1.** Conditioned on the initial credit rating $k$, the total probability of the repurchase and of the default from time zero to $s$ respectively is, for $0 \leq s \leq T$,

$$
Q(\psi_m^s \leq s \mid F_m) = 1 - q^k_m(0, s) = 1 + (D^k_m / V(0))^s N(\theta_1^0) - N(\theta_m^0)
$$

(16)

and

$$
Q(\tau^k_R \leq s \mid F_m) = 1 - q^k_m(0, s) = 1 + [(1-\rho)^kF^k_R / V(0)]^s N(\theta_1^0) - N(\theta_2^0)
$$

(17)

Where:

$$
\theta_1 = \left[ \frac{\ln (D^k_m / V(0)) + \lambda s}{\sigma^s} \right]^{-1}
$$

$$
\theta_2 = \left[ \frac{\ln (V(0) / D^k_m) + \lambda s}{\sigma^s} \right]^{-1}
$$

\(^{15}\)Since only one chance is left for managers to adjust the capital structure during the trading period, it is assumed that the costs of capital restructuring and debt repurchase are ignored in this study.

\(^{16}\)Making such an assumption helps us capture an important idea from empirical evidence that firms in reality will have a “strict” target debt ratio and will appear to return their leverage to this target in the long-run (Graham and Harvey, 2001; Fama and French, 2002; and Leary and Roberts, 2005).
\[ \theta_{11} = \left[ \ln \frac{(1 - \rho)F^k_R}{V(0)} + \lambda s \right] \sigma s^{\kappa s} \Sigma^{-1} ; \]

\[ \theta_{12} = \left\{ \ln \left( \frac{V(0)}{[(1 - \rho)F^k_R]} \right) + \lambda s \right\} \sigma s^{\kappa s} \Sigma^{-1} . \]

Now incorporate rating-dependent callable option in to the pricing of contingent claims associated with this model. Firstly, the current value of debt is shown as:

\[ D^R_{k,m}(0) = E_0^Q \left( \int_0^T \left[ C_R^{k,1} \left( \tau_m^{k,m} > s, \psi^k_m > s \right) + (1 - \rho) C_R^{k,1} \left( \tau_m^{k,m} > s, \psi^k_m > \psi^k_m \right) \right] e^{-rT} ds \right. \]

\[ + \rho F^k_R e^{-r\psi^k_m} \left( \psi^k_m \leq T \right) \left[ (1 - \rho) F^k_R \left( V(0) \right)^{-1} (1 - \alpha) V(\tau^{k,m}_R) e^{-r\tau^{k,m}_R} \right] \left( \tau_m^{k,m} \leq T \right) \]

The implication behind the payoff structure of Equation 18 is sketched as follows. Due to the repurchase effect on the debt's market value, the total contribution from the coupon payment consists of two parts: the first is the full coupon conditioned on the callable option not being exercised yet, and the other is the remaining coupon after repurchase process. The principal is repaid to debt-holders under three different conditions: (i) if the credit rating remains in the target range during the whole period, debt-holders will receive full par \( F_R^k \) at debt's maturity; (ii) during the repurchase process, managers raise fund \( \rho F^k_R \) to repay debt-holders; (iii) if the callable option has been exercised but bankruptcy does not occur until to maturity, the par value of the remaining debt \( (1 - \rho) F_R^k \) will be refunded at the expiring date. Lastly, upon bankruptcy, the amount \( TV^{k}(0) \left( V(0) \right)^{-1} (1 - \alpha) V(\tau^{k,m}_R) \) represents the contribution from recovery payment to debt-holders. Making the use of Proposition 1 and Corollary 1, the study has the explicit solution of Equation 18:

\[ D^{k,m}_{R}(0) = \int_0^T \left[ \left( 1 - \rho \right) q^{k,m}_t(0, s) + \rho q^{k,m}_t(0, s) \right] C_R^{k,1} + \rho F^{k,m}_R \int_{\psi^k_m} \left[ \psi^k_m \leq T \right] e^{-rT} \]

\[ + \int_0^T TV^{k}(0) \left( V(0) \right)^{-1} (1 - \alpha) \left( V(\tau^{k,m}_R) e^{-r\tau^{k,m}_R} \right] \left( \tau_m^{k,m} \leq T \right) \]

Where \( f_{\psi^k_m}(0, s) = -\partial q^{k,m}_t(0, s) / \partial s \) and \( f_{\tau^{k,m}_R}(0, s) = -\partial q^{k,m}_t(0, s) / \partial s \) respectively denotes the probability density of the occurrence of repurchase and default at time-s.

Considering the value of bankruptcy costs and tax benefits from the implication of Equation 15, it is obvious that the repurchase activity is always prior to the firm's default. Hence, after the debt repurchase, the bankruptcy -triggered threshold shrinks to equal the par of remaining debt, meaning that \( V(\tau^{k,m}_R) = (1 - \rho) F_R^k \). Similar to Equation 7, the present value of bankruptcy costs is:

\[ BC^{k,m}_{R} = \int_0^T \alpha \left( 1 - \rho \right) F_R^k e^{-rT} f_{\tau^{k,m}_R}(0, s) ds \]

On the other hand, when computing the value of tax benefits, it should be stressed that the coupon paid by the firm before/after a debt repurchase is different since part of the payment will be cut after the repurchase activity. In this model, the initial total value of tax shields contains two parts: the part accrues to the firm before debt repurchases, and the other part accrues after the repurchase. That means,

\[ TB^{k,m}_{R} = \int_0^T \beta \left( 1 - \rho \right) q^{k,m}_t(0, s) + \rho q^{k,m}_t(0, s) \right] C_R^{k,1} e^{-rT} ds \]

Similar to Equation 9, the equity's value reflects four terms: the firm's asset value, plus the value of tax benefits, less the value of bankruptcy costs, and less the value of debt:

\[ E^{k,m}_{R} = V(0) + TB^{k,m}_{R} - BC^{k,m}_{R} - D^{k,m}_{R} \]

The impact of target minimum rating policy on optimal capital structure

Here, this study focuses on investigating the impact of the behavior of targeting a minimum rating on optimal capital
structure decision. For analytical simplicity, let the reference firm target minimum rating at BBB\(^1\) (that is minimum investment-grade policy), and also suppose that the given combination of model parameters is identical with that in the base case. From the model setting, it is known that the debt repurchase induced by the exercise of rating-dependent callable option determines the impact of target minimum rating policy. To have a closer inspection to the impact, the study compiles the outputs of the model with/without minimum investment-grade policy as Table 6.\(^2\)

The numbers of repurchase ratio are shown in Column 2. The repurchase ratios both are increasing with the distance between the initial rating and the minimum rating targeted by the firm in Panel B and C. Interestingly, the intuition behind them is different. In Panel B (repurchase ratios are determined on the objective of maximizing share value), the result makes sense that, as the size of the departure from the firm’s target minimum rating to the initial rating is bigger, the additional tradeoff benefits (that is, the value equaling tax benefits minus bankruptcy costs) from restructuring the capital become greater. Managers, hence, are motivated to undertake a larger scale repurchase activity. The repurchase ratios, however, are determined on the objective of regaining the target initial rating in Panel C. The longer the distance between the target initial rating and the minimum rating, the more the improvement in the firm’s credit is required via repurchase. Since the size of repurchase is positively related to the rating level regained by firm, the pattern of repurchase ratio is clarified in Panel C.

Numbers in Column 6 of Table 6 report, the leverage effect on the per-share price for varying with the distance between the target initial rating and the minimum rating. Relative to the base case (Panel A), numbers with repurchase ratios derived from the traditional tradeoff theory (Panel B) seem slightly larger, while those with repurchase ratios derived from the target initial rating policy (Panel C) are smaller. To further explore this point, a depiction of tradeoff theory combined with the repurchase effect is exhibited in Figure 3.

The peak of each line in this figure implicates the choice of the repurchase ratio that maximizes the percentage changes in tradeoff benefits (that is, maximizes share value). Hence, the optimal repurchase ratios here are consistent with the numbers shown in Column 2 of Panel B, equaling 13.4165% in Panel A, 4.5829% in Panel B, and 0.0000% in Panel C and D.

It is important to note that, the improvements in the firm’s credit quality via repurchase are usually very slight in case where the ratios of repurchase are chosen for maximizing share value (Column 3 of Panel B). This suggests that, if the firm’s policy requires more improvements in the credit quality, such as regaining the target initial rating, managers necessarily make over-repurchase choices (see the numbers in Column 2 of Panel C). In the light of Figure 4, however, firm undertaking large-scale repurchase activities will experience a fractional loss in tradeoff benefits (that is why the numbers in Column 6 of Panel C are smaller than those in Panels A and B).

The outcome is not surprising. Recall the argument by CR-CS that when making the capital decision, managers should balance the benefits associated with higher rating levels against the loss in forgoing traditional tradeoff benefits. Obviously, if benefits from regaining previous target ratings dominate, managers have an incentive to make over-repurchase choices in adjusting leverage and credit rating, despite sacrificing a portion of the tradeoff benefits. The fact of making over-repurchase choices still makes a response to Kisgen (2006a) that, in the reality, possessing better credit may outweigh deriving more levered benefits when determining the capital structure.

Now consider the leverage dynamics behind the behavior of targeting minimum rating. Table 7 reports the distance between the initial target debt ratio and adjusted /unadjusted debt ratio in the case where the firm’s credit has already fallen below target minimum rate BBB. The sizes of the number shown in Column 4 measure the degree of moving current leverage ratio toward initial target. It is notable that the numbers in Panel B almost approximate to zero and are significantly smaller than those in Panel A.

More exactly, if the firm’s policy requires the credit to regain initial target rating after the repurchase, the leverage ratio will be adjusted back to a level so close to initial target.\(^3\) On the other hand, if improvements in the credit rating are slight under the policy of share value maximizing, the movement in the firm leverage becomes small due to a little-scale repurchase activity. Behind these results, the study has two-fold economical implication: (i) the degree of moving the leverage ratio toward initial target is strongly positive-correlated to the required improvements in the firm’s credit under target minimum rating policy; (ii) the leverage dynamics implied by the behavior of targeting the minimum rating follows a mean-reverting process.\(^4\) Now

\(^{15}\)such movement in leverage is corresponding with the empirical finding by Graham and Harvey (2001), Fama and French (2002), and Leary and Roberts (2003), implying that firms in the reality may adjust their current rating back to long-term target (as like initial target rating in our model) for managing credit rating.

\(^{16}\)Mean-reversion in leverage dynamics has been found by number of empirical works, including Jalilvand and Harris (1984), Fama and French (2002), and Leary and Roberts (2005). A theoretic work by Collin-Dufresne and Goldstein (2001) argues that the adjustments of outstanding debt level in response to changes in firm value will cause mean-reverting leverage ratio. Via debt repurchase activities induced by the intention of adjusting credit ratings; our results make an interesting link between Kisgen’s contention and those existing literatures.
Figure 3. Percentage change in tradeoff benefits as a function of repurchase ratio when the debt has minimum investment-grade policy.

Table 7. Distance between initial target debt ratio and adjusted/unadjusted debt ratio in case where firm’s credit has fallen below BBB.

<table>
<thead>
<tr>
<th>ITCR</th>
<th>Adj. debt ratio Via Rep.- (1) (%)</th>
<th>Un-adj. debt ratio - (2) (%)</th>
<th>Initial target debt ratio - (3) (%)</th>
<th>Distance between (1) and (3) (%)</th>
<th>Distance between (2) and (3) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>31.349567</td>
<td>36.207487</td>
<td>3.516155</td>
<td>27.833412</td>
<td>32.691332</td>
</tr>
<tr>
<td>A</td>
<td>16.250780</td>
<td>16.250780</td>
<td>8.472111</td>
<td>7.778669</td>
<td>7.778669</td>
</tr>
</tbody>
</table>

Panel A: The case where repurchase ratio is determined on the objective of maximizing share value

Panel B: The case where repurchase ratio is determined on the objective of regaining ITCR

ITCR denotes initial target credit rating.

The study puts the focus on clarifying the leverage dynamics. Due to the requirement by credit rating policy, it is known that the firm targets the initial rating at a level consistent with long-term target. The initial optimal leverage ratio, thus, can be treated as a long-term mean in dynamics. Once the firm’s credit falls below the target minimum rating during the life of debt, managers instantly exercise the callable option on the debt to adjust credit rating. No matter what the scale of repurchase is, the leverage ratio can always be moved toward initial target. If so, the numbers shown in Column 4 of Table 7 have demonstrated that the behavior of intending to regain previous better ratings account for the mean-
reversion in leverage dynamic. This will also hold in a more
generalized case with unlimited numbers of adjusting credit
rating, which gives the conclusion that mean-reverting
behavior empirically exhibited by the firms’ leverage can be
imputed to the target minimum rating policy.

CONCLUSION
By building the system of rating transition multi-boundaries,
the study proposes a new firm-value-based framework for
credit migration and rebuild the contingent claim model for
capital structure. To explore the central issue of this study,
the models we develop capture two types of empirical
behavior regarding credit rating management; the behavior
of linking the firm’s credit to the debt’s coupons and the
behavior of targeting a minimum rating.

The results elucidate the impact of credit rating
management behavior on optimal capital structure decision.
First, as long as the firm’s rating at the time of debt
issuance is not too low, the behavior of linking credit to the
debt’s interest cannot only allow the debt-holders for
earning more coupons but also benefit shareholders with
additional tax shields. When making a capital decision,
managers in general may prefer issuing rating-linked
coupon debt relative to standard debt, and hence, the
corresponded optimal leverage ratio is higher also. Further,
the study shows that levered benefits to shareholders in the
case of the firm applying rating-linked coupon scheme are
commonly greater than those in the case of constant
coupon scheme, which clarifies the motivation behind the
use of rating-linked coupon debts in practice (Lando and
Mortensen, 2005).

The behavior of targeting a minimum rating is verified to
be the cause to mean reversion in leverage dynamics. This
helps us link the argument in Kisgen (2006b) to several
interesting literatures that find leverage mean -reverting
behavior, including Jalilvand and Harris (1984), Fama and
French (2002), and Leary and Roberts (2005). Beyond the
linkage above, the analysis indicates that debt repurchase
activities induced by the intention of regaining the initial
target rating can generate another leverage ratio so close to
the initial optimal level (as the long-term mean in the model),
following a downgrade from target minimum rating. To
adjust their current rating back to the initial target,
managers appear to make over-repurchase choices,
despite fractional loss in the tradeoff levered-benefits.

This paper can be applied in several further dimensions.
The problem of jointly making optimal capital structure and
credit rating decisions can be studied. Other interesting
extensions include dynamic repurchasing or a stochastic
term structure of interest rate. The study expects that more
following studies will be devoted to this subject.

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The proof of proposition 1

The study firstly introduces some useful lemmas that describe the probability law of first-passage-time.

Lemma A.1 Let \( \tau \) stand for the first passage time to zero by the process \( Y \), that is, \( \tau := \inf \{ t \in [0, T] : Y(t) = 0 \} \). The process \( Y \) follows one-dimensional standard Brownian motion with the volatility \( \sigma > 0 \) and the drift \( \nu \in \mathbb{R} \), specifically:

\[
dY(t) = \nu dt + \sigma dB_Q(t), \quad \text{for } Y(0) = y(0).
\]

Then the random variable \( \tau \) has an inverse Gaussian probability distribution under \( Q \),

\[
Q(\tau \leq t) = N[h_1(t)] + e^{-2
u \sigma^2 t(0)}N[h_2(t)],
\]

where, \( h_1(t) = \frac{y(0) - \nu t}{(\sigma \sqrt{t})} \), and \( h_2(t) = \frac{-y(0) + \nu t}{(\sigma \sqrt{t})} \).

Proof of Lemma A.1: Note that, for any \( 0 \leq t \leq T \),

\[
Q(\tau > t) = Q(\inf_{0 \leq s \leq t} Y(s) \geq 0) = Q(\inf_{0 \leq s \leq t} X(s) \geq -y(0)), \tag{A1}
\]

where, \( X(s) := \nu s + \sigma B_Q(s) \). Recall that for every \( x < 0 \) we have (see Corollary B.3.4 in Musiela and Rutkowski, 1997):

\[
Q(\inf_{0 \leq s \leq t} X(s) \geq x) = N((-x + \nu t)/(\sigma t^{0.5})) - e^{2\nu \sigma^2 x}N((x + \nu t)/(\sigma t^{0.5})).
\]

When combined with (A1), this yields the Lemma A.1.

Lemma A.2 Denote by \( m(t) \) the minimum of standard Brownian motion \( Y \) on \([0, t]\). The joint distribution of \( (m(t), Y(t)) \) has the density, for any \( x \geq z \geq 0 \),

\[
f_{y,m}(x, z | F_0) = -\partial^2 Q(\inf_{0 \leq s \leq t} Y(u) \leq z, Y(t) \geq x | F_0) / \partial x \partial z = -\partial^2 \left[ \exp\left((2z-x-y(0)+\nu t)/(\sigma \sqrt{t})\right) x \right] / \partial x \partial z \tag{A2}
\]

\[
= \exp((2z-x-y(0)+\nu t)/(\sigma \sqrt{t})-z-x+[y(0)-2z]2\nu \sigma^2)
\]

\[
\times(2\pi^{-1})^{0.5} [x+y(0)-2z]t^{-15} \sigma^{-3}
\]


Through combining expressions (2) and (3) with the probability law under risk-neutral measure, rating transition probabilities of going from the current rating \( k \) to \( j \) at time-\( s \) \( q_j^n(0, s) \equiv Q(\eta^n(s) = j | \eta^n(0) = k) \) has the explicit form. For \( j = n \),

\[
q_j^n(0, s) \equiv Q(\inf_{0 \leq u \leq s} V(u) \geq D_j^k, V(s) \geq D_j^k | F_0);
\]

for \( j = n-1 \cdots 2 \),

\[
q_j^n(0, s) \equiv Q(\inf_{0 \leq u \leq s} V(u) \geq D_{j+1}^k, D_j^k \geq V(s) \geq D_j^k | F_0)
\]

\[
= Q(\inf_{0 \leq u \leq s} V(u) \geq D_j^k, V(s) \geq D_j^k | F_0) - Q(\inf_{0 \leq u \leq s} V(u) \geq D_j^k, V(s) \geq D_{j+1}^k | F_0) \tag{A4}
\]
and for \( j = 1 \),

\[
q^j_1(0, s) = Q\left( \inf_{0 \leq u \leq s} V(u) \leq D^j_s \bigg| \mathcal{F}_s \right) = 1 - \sum_{i=2}^{\infty} q^j_i(0, s) \tag{A5}
\]

From the joint view of (A3) and (A4), it is observable that the probability term \( Q\left( \inf_{0 \leq u \leq s} V(u) \geq D^j_s, V(s) \geq D^j_s \bigg| \mathcal{F}_s \right) \) serves as the building block of main derivation. Thus we now restrict the attention on deriving (A3). Recall the formula in Lemma A.2 that:

\[
Q\left( \inf_{0 \leq u \leq s} Y(u) \leq z, Y(t) \geq x \bigg| \mathcal{F}_u \right) = N\left[ \frac{(2z - x - y(0) + \nu t)}{\sigma \sqrt{t}} \right] \exp\left[ \frac{-(z - y(0))^2}{2\sigma^2} \right] \tag{A6}
\]

Because the process \( Y \) can be treated as the log-price of firm’s assets with \( V = \lambda \) and \( y(0) = \ln V(0) \), using Ito’s formula and (A6) with letting \( t = s, x = \ln D^j_s, \) and \( z = \ln D^j_s \) has the following derivation:

\[
q^j_1(0, s) = Q\left( V(s) \geq D^j_s \bigg| \mathcal{F}_s \right) - Q\left( \inf_{0 \leq u \leq s} V(u) \leq D^j_s, V(s) \geq D^j_s \bigg| \mathcal{F}_s \right).
\]

\[
= N\left[ \left( \ln V(0) - \ln D^j_s + \lambda s \right) / \left( \sigma \sqrt{s} \right) \right] - \left[ D^j_s / V(0) \right]^{2\lambda \sigma^{-2}} N\left( (2 \ln D^j_s - \ln V(0) - \ln D^j_s + \lambda s) / \left( \sigma \sqrt{s} \right) \right)
\]

Due to the connection between (A3) and (A4), the latter can be easily derived with same doing. Finally, in solving (A5), using Lemma A.1 and choosing \( V = \lambda, t = s \), and \( y(0) = \ln V(0) - \ln D^j_s \) yields:

\[
q^j_1(0, s) = 1 - N\left[ \left( \ln V(0) - \ln D^j_s + \lambda s \right) / \left( \sigma \sqrt{s} \right) \right] + \left[ D^j_s / V(0) \right]^{2\lambda \sigma^{-2}} N\left( (\ln D^j_s - \ln V(0) + \lambda s) / \left( \sigma \sqrt{s} \right) \right).
\]

Thus this completes the proof. \( \square \)