In this paper, a three-species Lotka-Volterra food-chain model with spatial diffusion and time delays is investigated. We first analyze the local stability of the steady states and the existence of Hopf bifurcation to this system under homogeneous Neumann boundary conditions. We consider the effects of impulses on the dynamics of the above food-chain model without spatial diffusion. Numerical simulations show that the system with constant periodic impulsive perturbations admits rich complex dynamics.

Key words: Hopf bifurcation, food chain, reaction diffusion, delay, stability, chaos.

INTRODUCTION

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. The classical Lotka-Volterra type systems are very important in the models of multi-species population dynamics and have been studied by many authors for example (Huang and Zou, 2002; Kuang, 1993; Liu et al., 2005; Xu et al., 2004; Yan and Chua, 2006).

Recently, the effect of spatial dispersion on population dynamics has received considerable attention. In this situation, the governing equations for the population densities are described by a system of reaction-diffusion equations for example (Cosner and Lazer, 1984; Gan et al., 2009; Pao, 2003, 2004, 2007; Tang and Zhou, 2007). On the other hand, time delays of one type or another have been incorporated into biological models by many researchers; we refer to the monographs of Gopalsamy (1992), Kuang (1993) and references cited therein for general delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate for example, (Beretta and Kuang, 1998; Busenberg and Huang, 1996; Faria, 2001; Gan et al., 2009; Song et al., 2004). Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effects of time delays. In this paper, motivated by the above discussions, we are concerned with the following three-species food chain model with spatial diffusion and time delays (Xu and Zhien, 2009; Kaifa et al., 2007):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + u(t,x)(r_1 - a_{11}u(t,x) - a_{12}v(t,x)) \\
\frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + v(t,x)(r_2 - a_{21}u(t-\tau,x) - a_{22}v(t,x) - a_{23}w(t,x)) \\
\frac{\partial w}{\partial t} &= D_3 \frac{\partial^2 w}{\partial x^2} + w(t,x)(r_3 + a_{31}v(t-\tau,x) - a_{33}w(t,x))
\end{align*}
\]

with initial conditions;

\[
u(t,x) = \rho_1(t,x), v(t,x) = \rho_2(t,x), w(t,x) = \rho_3(t,x) \quad t \in [-\tau,0], x \in \Omega
\]

In system (1)-(2), $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. The data $\rho_i(t,x)$ ($i=1,2,3$) are nonnegative and Hölder continuous and satisfy $\partial \rho_i / \partial x = 0$ in $(-\infty,0) \times \overline{\Omega}$. $u(t,x), v(t,x)$ and $w(t,x)$ represent the densities of the prey, predator and top predator population at time $t$, and location $x$ respectively. The parameters $a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{33}$,

*Corresponding author. E-mail: lixiaodong11111@sina.com.
We note that impulsive differential equations are suitable for the mathematical simulation of evolutionary process in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change (that is, jumps) in their values. Recently, equations with impulsive effect have been found in almost every domain of applied science. Numerous examples are given in Bainov and his collaborator's book (Bainov and Simeonov, 1993; Lakshmikanthan et al., 1989). Some impulsive differential equations have recently been introduced in population dynamics in relation to impulsive birth (Roberts and Kao, 1998; Tang and Chen, 2002), birth (Roberts and Kao, 1998; Tang and Chen, 2002), chemotherapeutic treatment of disease (Lakmeche and Arino, 2000; Panetta, 1996), and population ecology (Ballinger and Liu, 1997). Motivated by the work above, in this paper, we further discuss the effect of impulses on the dynamics of Equation (1). To this end, we discuss the following impulsive equations:

\[
\begin{align*}
    u(t) &= u(t,x)(t_i - a_iu(t,x) - a_iy(t,x)) \\
    v(t) &= v(t,x)(-r_1 + a_1u(t,x) - a_2v(t,x) - a_3w(t,x)) \\
    w(t) &= w(t,x)(r) + a_3v(t,x) - a_3w(t,x)) \\
    \Delta u(t) &= u(t^+ - u(t^-) = 0 \\
    \Delta v(t) &= v(t^+ - v(t^-) = 0 \\
    \Delta w(t) &= w(t^+ - w(t^-) = \xi v(t^+)
\end{align*}
\]  

(3)

LOCAL STABILITY AND HOPF BIFURCATION

In this section, we investigate the local stability of the steady states and the existence of Hopf bifurcation to Equation (1) with the initial conditions in Equation (2) and the homogeneous Neumann boundary conditions:

\[
\frac{\partial u(t,x)}{\partial x} = \frac{\partial v(t,x)}{\partial x} = \frac{\partial w(t,x)}{\partial x} = 0, \quad t \geq 0, x \in \partial \Omega
\]

where \( \partial / \partial x \) denotes the outward normal derivative on \( \partial \Omega \), the homogeneous Neumann boundary conditions imply that the populations do not move across the boundary \( \partial \Omega \).

It is easy to show that Equation (1) always has a trivial steady state \( E_0(0,0,0) \) and a semi-trivial steady state \( E_1(r_1/a_1, 0, 0) \). If \( r_1a_2 > r_2a_1 \), Equation (1) has a semi-trivial steady state \( E_2(s_1, 0, 0) \), where

\[
s_1 = -\frac{r_1a_2 + r_2a_1}{a_1a_2 + a_2a_1}, \quad s_2 = -\frac{r_1a_2 - r_2a_1}{a_1a_2 + a_2a_1}
\]

If the following holds:

\[
(H_1) \quad a_2a_3r_1 - a_1a_2r_2 - a_1a_3r_3 - a_1a_2r_3 > 0
\]

system (1) has a unique positive steady state \( E^*(k_1, k_2, k_3) \), where

\[
k_1 = \frac{a_2a_3r_1 + a_2a_3r_2 - a_1a_2r_3}{a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3}
\]

\[
k_2 = \frac{a_2a_3r_1 - a_1a_2r_2 + a_1a_3r_3}{a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3}
\]

\[
k_3 = \frac{a_2a_3r_1 + a_2a_3r_2 - a_1a_2r_3}{a_1a_2a_3 + a_1a_2a_3 + a_1a_2a_3}
\]

Let \( 0 = \mu_1 < \mu_2 < \cdots \) be the eigenvalues of the operator \( -\Delta \) on \( \text{Homogeneous Neumann boundary conditions, and } E(\mu_i) \) be the eigenspace corresponding to \( \mu_i \) in \( C^1(\Omega) \). Let \( X = [C^1(\Omega)]^\infty \), \( \{\phi_j; j = 1, \cdots, \dim E(\mu_i)\} \) be an orthonormal basis of \( E(\mu_i) \), and \( X_{ij} = \{\phi_j \in R^i\} \). Then

\[
X = \bigoplus_{i=0}^{\infty} X_i \quad \text{and} \quad X = \bigoplus_{j=0}^{\infty} X_{ij}
\]

Let

\[
D = \text{diag}(D_1, D_2, D_3)
\]

\[
LZ = \mu DZ + \zeta(\hat{E})Z, \quad \hat{E} \in \left( u, v, w \right)
\]

\[
\zeta(\hat{E})Z = \begin{pmatrix}
    r_1 - a_{11}u^0 & -a_{11}v^0 & 0 \\
    0 & -r_2 + a_{21}u^0 - a_{22}v^0 - a_{23}w^0 & -a_{23}w^0 \\
    0 & 0 & -r_3 + a_{31}u^0 - a_{32}v^0 - a_{33}w^0
\end{pmatrix}
\]

\[
X = \begin{pmatrix}
    u(t,x) & 0 & 0 & 0 \\
    v(t,x) & 0 & 0 & 0 \\
    w(t,x) & 0 & 0 & 0
\end{pmatrix}
\]

And \( \hat{E}(u^0, v^0, w^0) \) represents any uniformly stable state of Equation (1). The linearization of system (1) at \( \hat{E} \) of the form \( Z_i = LZ_i \). For each \( i \geq 1 \), \( X_i \) is invariant under the operator \( L \), and \( \lambda \) is an eigenvalue of \( L \) if and only if it is an eigenvalue of the matrix \( -\mu D + \zeta(\hat{E}) \) for some \( i \geq 1 \). In which case, there is an eigenvector in \( X_i \). The characteristic equation of \(-\mu D + \zeta(\hat{E})\) is of the form:

\[
(\lambda + \mu_1 D - r_1)(\lambda + \mu_2 D + r_2)(\lambda + \mu_3 D + r_3) = 0
\]

Clearly, for \( i = 1 \), Equation (4) always has a positive real root \( r_1 \).

Therefore, there is a characteristic root \( \lambda \), with positive real part in the spectrum of \( L \). Accordingly, the trivial uniform steady state \( E_0(0,0,0) \) is always unstable. The characteristic equation of \(-\mu D + \zeta(\hat{E})\) is of the form
\[ (\lambda + \mu_D + r_i)(\lambda + \mu_D + r_2 - a_{2i}r_i/a_{1i})(\lambda + \mu_D + r_3) = 0 \]  
\[ (\lambda + \mu_D + r_3 - a_{32}s_2)(\lambda + \mu_D + a_{11}s_1 + a_{32}s_2)(\lambda + \mu_D + a_{22}s_2 + a_{12}a_{21}s_1)e^{-\lambda \tau} = 0 \]  
\[ (\lambda + \mu_D + r_2 - a_{21}r_1/a_{11})(\lambda + \mu_D + r_3 - a_{32}s_2)(\lambda + \mu_D + a_{11}s_1 + a_{32}s_2)(\lambda + \mu_D + a_{22}s_2 + a_{12}a_{21}s_1)e^{-\lambda \tau} = 0 \]  
(5)  

Clearly, for any \( i \geq 1 \), Equation (5) always has two negative real roots \(-\mu_D_r - r_i\) and \(-\mu_D_D - r_3\). Its other root is \(-\mu_D_D - r_2 + a_{21}r_1/a_{11}\).

If \( r_{a_{21}} > r_{a_{11}} > r_2 + a_{21}r_1/a_{11} > 0 \). Hence, when \( i = 1 \), Equation (5) has a positive real root. Therefore, there is a characteristic root \( \lambda \) with positive real part in the spectrum of \( L \). Accordingly, if \( r_{a_{21}} > r_{a_{11}} > r_2 + a_{21}r_1/a_{11} > 0 \), Equation (5) has a positive real root. Therefore, all characteristic roots \( \lambda \) are negative constants in the spectrum of \( L \). Accordingly, if \( r_{a_{21}} > r_{a_{11}} > r_2 + a_{21}r_1/a_{11} > 0 \) for any \( i \geq 1 \). Therefore, all characteristic roots \( \lambda \) are negative constants in the spectrum of \( L \). Accordingly, if \( r_{a_{21}} < r_{a_{11}} > r_2 + a_{21}r_1/a_{11} > 0 \), Equation (5) has a locally asymptotically stable. The characteristic equation of \(-\mu_D + \xi(E_2)\) is of the form

\[ (\lambda + \mu_D + r_3 - a_{32}s_2)(\lambda + \mu_D + a_{11}s_1 + a_{32}s_2)(\lambda + \mu_D + a_{22}s_2 + a_{12}a_{21}s_1)e^{-\lambda \tau} = 0 \]  
(6)  

If \( (H_1) \) holds, for \( i = 1 \), \(-\mu_D_D - r_3 + a_{32}s_2 > 0 \). Equation (6) always has a positive real root. Therefore, there is a characteristic root \( \lambda \) with positive real part in the spectrum of \( L \). Accordingly, if \( (H_1) \) holds, \( E_2(s_1, s_2, 0) \) is unstable. If the following \( (H_2) \) holds:

\[ a_{23}a_{32}r_1 - a_{13}a_{32}r_2 - a_{12}a_{23}r_2 + a_{11}a_{22}r_3 < 0 \]  

For \( i \geq 1 \), Equation (6) always has a negative root \(-\mu_D_D - r_3 + a_{32}s_2\). Its other roots are determined by the following equation:

\[ \lambda^2 + a_0 \lambda + a_0 + b_0 e^{-\lambda \tau} = 0 \]  
(7)  

Where

\[ a_0 = (\mu_D_D + a_{11}s_1)(\mu_D_D + a_{22}s_2) \]  
\[ a_1 = \mu_D_D + a_{11}s_1 + \mu_D_D + a_{22}s_2 \]  
\[ b_0 = a_{12}a_{21}s_1 + a_{12}a_{21}s_1 \]  

It is easy to see that the roots of Equation (7) are negative real constants when \( \tau = 0 \), then \( E_2(s_1, s_2, 0) \) is locally asymptotically stable when \( \tau = 0 \). If \( i \sigma (\sigma > 0) \) is a solution of Equation (7), separating real and imaginary parts, we can derive that

\[ \left\{ \begin{array}{l} \sigma^2 - a_0 = b_0 \cos \sigma \\ a_0 \sigma = b_0 \sin \sigma \end{array} \right. \]  
(8)  

Squaring and adding the two equations of Equation (8), it follows that

\[ \sigma^4 + (a_0^2 - 2a_0)\sigma^2 + a_0^2 - b_0^2 = 0 \]  
(9)  

If \( a_0^2 > 2a_0 > 0 \) and \( a_0^2 - b_0^2 > 0 \), then Equation (9) have no positive roots for all \( i \geq 1 \). Therefore, all characteristic roots \( \lambda \) are negative constants in the spectrum of \( L \).

Accordingly, if \( (H_2) \) holds and \( a_{11}a_{22} > a_{12}a_{21} \), \( E_2(S_1, S_2, 0) \) is a pair of purely imaginary roots of Equation (8) with \( \tau = \tau^* \).

Then \( \tau = \tau^* \), for \( i = 1 \), Equation (7) has a pair of purely imaginary roots \( \pm \sigma_0 \) and all roots of Equation (7) have negative real parts for \( i \geq 2 \). Noting that if \( (H_2) \) holds, the positive uniform steady state \( E^*_2 \) is locally stable when \( \tau = 0 \), by the general theory on characteristic equations of delay differential equations from (Kuang, 1993) (Theorem 4.1), \( E^*_2 \) remains stable for \( \tau = \tau^* \).

We now claim that

\[ \frac{d(\text{Re} \lambda)}{d\tau} \bigg|_{\tau = \tau^*} > 0 \]

This will signify that there exists at least one eigenvalue with positive real part for \( \tau > \tau^* \). Moreover, the conditions for the existence of a Hopf bifurcation (Hale, 1977) are then satisfied yielding a periodic solution. To this end, differentiating Equation (7) with respect to \( \tau \), we obtain

\[ 2\lambda \frac{d\lambda}{d\tau} + a_1 \frac{d\lambda}{d\tau} - b_0 \lambda e^{-\lambda \tau} = b_0 \lambda e^{-\lambda \tau} \]

Hence, we derive that

\[ \left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{a_1}{\lambda \lambda^2 + a_1 \lambda + a_0} - \frac{2}{\lambda^2 + a_1 \lambda + a_0} - \frac{\tau}{\lambda} \]

It therefore follows that

\[ \text{sign} \left( \frac{d(\text{Re} \lambda)}{d\tau} \right)_{\lambda = i\sigma_0} = \text{sign} \left( \text{Re} \left( \frac{d\lambda}{d\tau} \right) \right)_{\lambda = i\sigma_0} = \text{sign} \left( \frac{2a_0^2 + a_1^2 - 2a_0}{(a_0^2 + a_1^2 - a_0)} \right) \]

(11)
Noting that \(a_i^2 - 2a_i > 0\), it follows that

\[
\frac{d(\text{Re}\, \lambda)}{d\tau} \bigg|_{\tau = \tau^*} = \sigma - \sigma_0 > 0
\]

Therefore, the transversal condition holds. From Equation (11), we know that a Hopf bifurcation occurs at \(\sigma = \sigma_0\), \(\tau = \tau^*\).

The characteristic equation of \(-\mu D + \zeta(E^*)\) is of the form

\[
\lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 + (q_1 \lambda + q_0) e^{-\lambda\tau} = 0
\]

where

\[
p_0 = (\mu D_1 + a_1 k_1)(\mu D_2 + a_2 k_2)(\mu D_3 + a_3 k_3)
\]

\[
p_1 = (\mu D_1 + a_1 k_1)(\mu D_2 + a_2 k_2)(\mu D_3 + a_3 k_3)
\]

\[
p_2 = (\mu D_1 + a_1 k_1)(\mu D_2 + a_2 k_2)(\mu D_3 + a_3 k_3)
\]

\[
q_0 = a_1, a_2, a_3, k_1, k_2, k_3
\]

\[
q_1 = a_1 a_2 k_1 k_2 k_3 + a_1 a_3 k_1 k_2 k_3 + a_2 a_3 k_1 k_2 k_3
\]

When \(\tau = 0\), Equation (12) becomes

\[
\lambda^3 + p_2 \lambda^2 + (p_1 + q_1) \lambda + p_0 + q_0 = 0
\]

It is easy to verify that \(p_2 > 0\), \(p_0 + q_0 > 0\) and \((p_1 + q_1) p_2 > p_0 + q_0\). Then it follows from Hurwitz criterions that all roots of Equation (13) have negative parts. Hence, the positive uniform steady state \(E^*\) is locally asymptotically stable when \(\tau = 0\).

If \(i \omega k(\omega > 0)\) is a solution of Equation (12), separating real and imaginary parts, we derive that:

\[
\begin{align*}
-\omega^3 + p_1 \omega k_0 \sin \omega \tau - q_1 \omega k_0 \cos \omega \tau = 0 \\
q_2 \omega^2 - p_0 k_0 = q_0 \cos \omega \tau + q_1 \omega k_0 \sin \omega \tau
\end{align*}
\]

Squaring and adding Equations (14), it follows that

\[
\omega^6 + (p_2^2 - 2p_1) \omega^4 + (p_1^2 - 2p_0 p_2 - q_1^2) \omega^2 + p_0^2 - q_0^2 = 0
\]

Let \(z = \omega^2\), Equation (15) becomes

\[
z^3 + (p_2^2 - 2p_1) z^2 + (p_1^2 - 2p_0 p_2 - q_1^2) z + p_0^2 - q_0^2 = 0
\]

In the following, we need to seek conditions under which Equation (16) has at least one positive root. Denote

\[
h(z) = z^3 + (p_2^2 - 2p_1) z^2 + (p_1^2 - 2p_0 p_2 - 2q_1^2) z + p_0^2 - q_0^2
\]

If the following holds:

\[
(H_3) \quad a_1 a_2 a_3 - a_2 a_3 a_4 - a_1 a_3 a_4 > 0
\]

It is easy to verify that \(p_2^2 - 2p_0 > 0\), \(p_0^2 - q_0^2 > 0\) and \(p_1^2 - 2p_0 p_2 - q_1^2 > 0\) for all \(i \geq 1\). Hence, Equation (15) has no positive roots in this case.

If the following holds:

\[
(H_4) \quad a_1 a_2 a_3 - a_2 a_3 a_4 - a_1 a_3 a_4 < 0
\]

\[
\text{we have } \quad p_2^2 - q_0^2 = k_1 k_2 k_3 (a_1 a_2 a_3 - a_2 a_3 a_4 - a_1 a_3 a_4) (p_0 + q_0) < 0
\]

and \(\lim_{\tau \to \infty} h(z) = \infty\) for \(i = 1\), then Equation (15) has at least one positive root. Hence, Equation (12) has a pair of purely imaginary roots \(\pm i \omega k\), and all roots of Equation (12) have negative real parts for \(i \geq 2\).

Suppose that Equation (16) has positive roots. Without loss of generality, we assume that it has three positive roots, defined by \(z_1, z_2, z_3\) respectively. Then Equation (15) has three positive roots \(\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}\) and \(\omega_3 = \sqrt{z_3}\). From (14), we have

\[
\cos \omega \tau = \frac{q_0 (p_2 \omega^2 - p_0) + q_1 (\omega^2 - p_1) \omega^2}{q_0^2 + q_1^2 \omega^2}
\]

Thus, if we denote

\[
\tau_k^{(j)} = \frac{1}{\omega_k} \left\{ \arccos \left( \frac{q_0 (p_2 \omega^2 - p_0) + q_1 (\omega^2 - p_1) \omega^2}{q_0^2 + q_1^2 \omega^2} \right) + 2j\pi \right\}
\]

(18)

Where \(k = 1, 2, 3; j = 0, 1, \cdots\), then \(\pm i \omega_k\) is a pair of pure imaginary roots of Equation (14) with \(\tau_k^{(j)}\). Define

\[
\tau_0 = \tau_k^{(0)} = \min_{k \in \{1,2,3\}} \{ \tau_k^{(0)} \}, \quad \tau < \tau_0
\]

(19)

Noting that the positive uniform steady state \(E^*\) is locally stable when \(\tau = 0\), by the general theory on characteristic equations of delay differential equations from (Theorem 4.1), \(E^*\) remains stable for \(\tau < \tau_0\). We now claim that

\[
\frac{d(\text{Re}\, \lambda)}{d\tau} \bigg|_{\tau = \tau_0} > 0
\]

Differentiating Equation (12) with respect \(\tau\), it follows that:

\[
(3\lambda^2 + 2p_1 \lambda + p_0) \frac{d\lambda}{d\tau} + q e^{-\lambda \tau} \frac{d\lambda}{d\tau} - \tau q_0 \lambda + q_0 e^{-\lambda \tau} \frac{d\lambda}{d\tau} = \lambda (q_0 \lambda + q_0) e^{-\lambda \tau}
\]

Hence, we derive that

\[
\frac{d\lambda}{d\tau} = \frac{3\lambda^2 + 2p_1 \lambda + p_0}{\lambda (q_0 \lambda + q_0) e^{-\lambda \tau}} - \frac{\tau}{\lambda}
\]
Example 1

In Equation (1), we set \( D_1 = D_2 = D_3 = 1 \), \( r_1 = 2 \), \( r_2 = r_3 = 0.5 \), \( a_{11} = a_{22} = a_{33} = 0.3 \), \( a_{12} = 3 \), \( a_{23} = 0.5 \), \( a_{21} = 0.6 \), \( a_{32} = 0.7 \). It is easy to show that Equation (1) has three steady states \( E_0(0,0,0) \), \( E_1(20/3,0,0) \) and \( E_2(10/9,5/9,0) \). By Theorem 2.1 we see that \( E_0 \) and \( E_1 \) are unstable for all \( \tau > 0 \), \( E_2 \) is asymptotically stable when \( 0 \leq \tau < \tau^* = 0.4676 \) and the bifurcation occurs when \( \tau \) crosses \( \tau^* \) to the right (\( \tau > \tau^* \)). These facts are illustrated by the numerical simulations in Figures 1 and 2.

Example 2

In Equation (1), we let \( D_1 = D_2 = D_3 = 1 \), \( r_1 = 2 \), \( r_2 = r_3 = 0.5 \), \( a_{11} = a_{22} = a_{33} = 0.3 \), \( a_{12} = a_{32} = 0.5 \), \( a_{21} = 0.6 \), \( a_{31} = 0.7 \). It is easy to show that system (1.1) has four steady states \( E_0(0,0,0) \), \( E_1(20/3,0,0) \), \( E_2(85/39,35/13,0) \) and \( E^*(415/111,65/37,90/37) \). It follows from Theorem 2.1 that \( E_0 \), \( E_1 \), \( E_2 \) are unstable for all \( \tau > 0 \), \( E^* \) is asymptotically stable when \( 0 \leq \tau < \tau_0 = 0.4749 \) and the bifurcation takes place when \( \tau \) crosses \( \tau_0 \) to the right (\( \tau > \tau_0 \)). These facts are illustrated by the numerical simulations in Figures 3 and 4.

Chaotic behavior in Equation (3)

The influences of \( \tau \) may be documented by stroboscopically sampling some of the variables over a range of \( \tau \) values.

In Equation (3), we let \( r_1 = 2 \), \( r_2 = r_3 = a_{12} = a_{32} = 0.5 \), \( a_{11} = a_{22} = a_{33} = 0.3 \), \( a_{21} = 0.6 \), \( a_{32} = 0.7 \), \( \zeta = 0.8 \), \( T = 2 \), \( 0.5 \leq \tau \leq 0.9 \).

The influences of \( \tau \) may be documented by stroboscopically sampling some of the variables over a range of \( \tau \) values. We numerically integrate Equation (3) for 500 pulsing cycles at each value of \( \tau \). For each \( \tau \), we plot the last 100 measures of prey \( u \), predator \( v \) and top predator \( w \). Since we sample at forcing period, the \( T \)-periodic solutions appear as fixed points, the \( 2T \)-periodic solutions appear as two cycles, and so forth. The resulting bifurcation diagrams (Figure 5) clearly show that: with the increasing of \( \tau \) from 0.5 to 0.9, Equation (3) experiences process of cycles \( \rightarrow \) periodic doubling cascade \( \rightarrow \) chaos (Figure 6). This periodic-doubling route to chaos is the hallmark of the logistic and Ricker maps.
Figure 1. The temporal solution found by numerical integration of problem (1.1)-(1.2) with \( D_1 = D_2 = D_3 = 1 \), \( r_1 = 2 \), \( r_2 = r_3 = a_{23} = 0.5 \), \( a_{11} = a_{22} = a_{33} = 0.3 \), \( a_{12} = 3 \), \( a_{21} = 0.6 \), \( a_{13} = 0.7 \), \( \tau = 0.2 \) and \( \rho_1(t,x) = \rho_2(t,x) = \rho_3(t,x) = 1 + e^{-20x}, \ t \in [-0.2,0] \).

Figure 2. The temporal solution found by numerical integration of problem (1.1)-(1.2) with \( D_1 = D_2 = D_3 = 1 \), \( r_1 = 2 \), \( r_2 = r_3 = a_{23} = 0.5 \), \( a_{11} = a_{22} = a_{33} = 0.3 \), \( a_{12} = 3 \), \( a_{21} = 0.6 \), \( a_{13} = 0.7 \), \( \tau = 0.6 \) and \( \rho_1(t,x) = \rho_2(t,x) = \rho_3(t,x) = 1 + e^{-20x}, \ t \in [-0.6,0] \).
Figure 3. The temporal solution found by numerical integration of problem (1.1)-(1.2) with
\[ D_1 = D_2 = D_3 = 1, \quad r_1 = 2, \quad r_2 = r_3 = a_{12} = a_{23} = 0.5, \quad a_{11} = a_{22} = a_{33} = 0.3, \quad a_{21} = 0.6, \quad a_{32} = 0.7, \quad \tau = 0.3, \quad \text{and} \quad \rho_1(t,x) = 3 + e^{-20t}, \quad \rho_2(t,x) = 1 + e^{-20t}, \quad \rho_3(t,x) = 2 + e^{-20t}, \quad t \in [-0.3,0]. \]

Figure 4. The temporal solution found by numerical integration of problem (1.1)-(1.2) with
\[ D_1 = D_2 = D_3 = 1, \quad r_1 = 2, \quad r_2 = r_3 = a_{12} = a_{23} = 0.5, \quad a_{11} = a_{22} = a_{33} = 0.3, \quad a_{21} = 0.6, \quad a_{32} = 0.7, \quad \tau = 0.6, \quad \text{and} \quad \rho_1(t,x) = 3 + e^{-20t}, \quad \rho_2(t,x) = 1 + e^{-20t}, \quad \rho_3(t,x) = 2 + e^{-20t}, \quad t \in [-0.6,0]. \]
Figure 5. Bifurcation diagrams of Equation (3) showing the effect of $\tau$ with $r_i = 2$, $r_2 = r_3 = a_{12} = a_{23} = 0.5$, $a_{11} = a_{22} = a_{33} = 0.3$, $a_{21} = 0.6$, $a_{32} = 0.7$, $\zeta = 0.8$, $T = 2$, $0.5 \leq \tau \leq 0.9$, and initial values $u = 2.5$, $v = 4$, $w = 6$.

Figure 6. Chaos of Equation (3) for $\tau = 0.89$: (a) time series of prey $u$; (b) time series of predator $v$; (c) time series of top predator $w$; (d) phase portrait.
(May, 1974; May and Oster, 1976) and has been studied extensively by Mathematicians Collet and Eeckmann, 1980; Venkatesan and Parthasarathy, 2003). For the predator-prey system, chaotic behaviors are usually obtained by continuous system with periodic forcing (Vandermeer et al., 2001).

REFERENCES


