Full Length Research Paper

Quasi-radical operation on the submodules in a module

Esra Şengelen Sevim¹* and Ünsal Tekir²

¹Department of Mathematics, Istanbul Bilgi University, Dolapdere, Istanbul, Turkey.
²Department of Mathematics, Marmara University, Göztepe-Ziverbey, Istanbul, Turkey.

Accepted 04 April, 2011

All rings are commutative with identity and all modules are unital. The purpose of this paper is to introduce interesting and useful properties of quasi-radical operation on the submodules in a module.

Key words: Prime submodules, quasi-radical operation.

INTRODUCTION

Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let $R$ be a ring and $M$ be a unital $R$-module. For any submodule $N$ of $M$, we define $(N: M) = \{ r \in R : rM \subseteq N \}$. A submodule $N$ of $M$ is called prime if $N \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in N$, $m \in N$ or $r \in (N: M)$. Let $PSpec(M)$ denote the collection of all prime submodules. Note that some modules have no prime submodules (that is, $PSpec(M) = \emptyset$). In recent years, prime submodules have attracted a good deal of attention (Lu, 1984; John, 1978; James and Patrick, 1992; Shahabaddin, 2004). An $R$-module $M$ is called a multiplication module provided for each submodule $N$ of $M$ there exists an ideal $I$ of $M$ such that $N = IM$. We say that $I$ is a presentation ideal of $N$. We say that $N$ is a presentation ideal of $M$. Let $N$ be a submodule of a multiplication $R$-module $M$ with $N = I_1M$ and $K = I_2M$ for some ideals $I_1$ and $I_2$ of $R$. The product $N$ and $K$ denoted by $NK$ is defined by $NK = I_1I_2M$. Then, the product of $N$ and $K$ is independent of presentation of $N$ and $K$ (Reza, 2003, Theorem 3.13).

In this paper, we generalize some properties of quasi-radical operation on the ideals in a ring to quasi-radical operation on the submodules in a module (Magnus, 2004).

Definition 1

Let $M$ be an $R$-module. An operation $F$ on the submodules of $M$ is a correspondence that to every submodule $N$ of $M$ associates a submodule $F(N)$ in $M$.

Definition 2

(i) Let $M$ be an $R$-module. Let $F$ be an operation on the submodules of $M$, and let $N$ be a submodule in $M$. We say that $F(N)$ is the $F$-radical of $N$.

(ii) Let $M$ be an $R$-module. We say that $N$ is $F$-radical if $F(N) = N$. A prime submodule $N$ is called $F$-prime if it is $F$-radical.

Definition 3

Let $M$ be an $R$-module and $F$ an operation on the submodules of $M$. We define $F$-prime spectrum of $M$ as:

$$Spec(M) = \{ F - prime submodules N \subseteq M \}.$$
Definition 4

Let $M$ be an $R$-module. Let $F$ be an operation on the submodules in $M$. We say that $M$ satisfies the ascending chain condition (acc) for $F$-radical submodules if for every chain $\{N_i\}_{i \in I}$ of $F$-radical submodules we have that $N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$ stabilizes.

Relations 1

Let $M$ be an $R$-module and $F$ be an operation on the submodules of $M$. It is natural to ask if $F$ satisfies the following relations for any submodules $N$, $K$ and $\{N_i\}_{i \in I}$ in $M$:

(a) $N \subseteq F(N)$,
(b) $F(F(N)) = F(N)$,
(c) $F(N \cap K) = F(N) \cap F(K)$,
(c') if $M$ is a multiplication $R$-module then $F(N \cap K) = F(N) \cap F(K) = F(NK)$,
(d) $F(\bigcup_{i \in I} N_i) = F(\bigcup_{i \in I} F(N_i))$,
(e) $\sqrt{N} \subseteq F(N)$ if $M$ is a multiplication $R$-module,
(f) $N \subseteq K$ implies $F(N) \subseteq F(K)$,
(g) $F(\bigcup_{i \in I} N_i) = F(\bigcup_{i \in I} F(N_i))$ if $\{N_i\}_{i \in I}$ is ordered family.

Proposition 1

Let $M$ be an $R$-module. Let $F$ be an operation on the submodules in $M$. The following assertions hold for (a) - (g) of Relations 1.

1. If $F$ satisfies (a), (b) and (f) then $F$ satisfies (d).
2. If $F$ satisfies (c) then $F$ satisfies (f).
3. Let $M$ be a multiplication $R$-module. If $F$ satisfies (a) and (c') then $F$ satisfies (e).
4. If $F$ satisfies (d) then $F$ satisfies (b).
5. If $F$ satisfies (a) and (d) then $F$ satisfies (f) and (g).

Proof

1. We have from (a) that $N_i \subseteq F(N_i)$ for each $i \in I$. It follows that $\bigcup_{i \in I} N_i \subseteq \bigcup_{i \in I} F(N_i)$. Consequently, we see by (f) that $F(\bigcup_{i \in I} N_i) \subseteq F(\bigcup_{i \in I} F(N_i))$. Conversely, since $N_i \subseteq \bigcup_{i \in I} N_i$ for each $i \in I$, we have by (f) that $F(\bigcup_{i \in I} N_i) \subseteq F(\bigcup_{i \in I} F(N_i))$. Thus since $F(\bigcup_{i \in I} N_i)$ is an submodule we see that $\bigcup_{i \in I} F(N_i) \subseteq F(\bigcup_{i \in I} N_i)$. This implies, again by (f), that $F(\bigcup_{i \in I} F(N_i)) \subseteq F(F(\bigcup_{i \in I} N_i))$. Now since from (b) $F(F(N)) = F(N)$ for any submodule $N$ in $M$ we get that $F(\bigcup_{i \in I} F(N_i)) \subseteq F(\bigcup_{i \in I} N_i)$. This shows that $F(\bigcup_{i \in I} N_i) = F(\bigcup_{i \in I} F(N_i))$, that is (d) holds.
2. Assume (f) is not true. There exist $N, K$ such that $N \subseteq K$ but $F(N) \not\subseteq F(K)$. This implies $F(N \cap K) = F(N) \cap F(K)$ which contradicts (c). Thus $N \subseteq K$ implies $F(N) \subseteq F(K)$ for any submodules $N, K \subseteq M$, that is (f) holds.
3. From the relation (c') we get $F(t^2) = F((t)^n) = F((t))$ for every $t \in M$. By induction on $n$, we obtain $F(t^n) = F((t))$ for all positive integer $n$. Let $N$ be a submodule of $M$ and $t \in \sqrt{N}$. Then $t^n \subseteq N$ for some positive integer $n$. We have that and from relation (a) that $t \in F((t))$. Hence $t \in F(N)$ and we have proved that $\sqrt{N} \subseteq F(N)$, that is (e) holds.
4. If $F(N) \neq F(F(N))$ then $F(\bigcup_{i \in I} N_i) \neq F(\bigcup_{i \in I} F(N_i))$ for $I = 1$ and $N_i = N$ that is we get a contradiction of (f). Thus (b) is satisfied.
5. If $N \subseteq K$ does not imply that $F(N) \subseteq F(K)$ then there exist submodules $N \subseteq K$ in $M$ such that $F(N) \not\subseteq F(K)$. Then $F(K) \nsubseteq F(N) + F(K)$ so we have by (a) that $F(N + K) = F(K) \neq F(N) + F(K)$, which contradicts (d). Thus (f) satisfied. If $\{N_i\}_{i \in I}$ is an ordered family then it is clear that $\bigcup_{i \in I} N_i = \bigcup_{i \in I} F(N_i)$ since from (f) we have that $N \subseteq K$ implies $F(N) \subseteq F(K)$; it follows that $F(\{\bigcup_{i \in I} N_i\}_{i \in I})$ is an ordered family of submodules as well. Then that $\bigcup_{i \in I} F(N_i) = \bigcup_{i \in I} F(N_i)$. Thus (d), that is $F(\bigcup_{i \in I} N_i) = F(\bigcup_{i \in I} F(N_i))$, implies $F(\bigcup_{i \in I} N_i) = F(\bigcup_{i \in I} F(N_i))$, that is (g) holds.

Lemma 1

Let $M$ be an $R$-module. Let $F$ be a prime submodule in $M$ and let $F$ be an operation on the submodules in $M$ satisfying (a) and (f) of Relations 1. The following two conditions are equivalent:

1. $F(N) = N$
2. $A \subseteq N$ implies $F(A) \subseteq N$ for each submodule $A$ in $M$.

Proof

Assume (1) does not hold, that is by (a) we have that $N \not\subseteq F(N)$ then condition (2) with $A = N$ does not hold. Thus (2) implies (1). Conversely, assume that (2) does not hold. Then there is a submodule $A$ in $M$ such that $A \subseteq N$ but $F(A) \not\subseteq N$. From (f) we get that $F(A) \subseteq F(N)$. Thus by (a) we see that $N \subseteq F(N)$ that is condition (1) does not hold. This shows that (1) implies (2).

Definition 5

Let $M$ be a multiplication $R$-module. A quasi-radical operation $F$ on the submodules in $M$ is defined as an operation on the submodules in $M$ such that for all submodules $A$ and $B$ in $M$ the following conditions hold:

(a) $A \subseteq F(A)$
(b) \( F(F(A)) = F(A) \)
(c) \( F(A \cap B) = F(A) \cap F(B) = F(AB) \)

**Remark 1**

From Proposition 1 we see that any quasi-radical operation satisfies \((a) – (g)\) of Relations 1.

**Proposition 2**

Let \( M \) be a multiplication \( R \)-module. A quasi-radical operation \( F \) on the submodules in \( M \) satisfies \( F(N) = \sqrt{F(N)} = F(\sqrt{N}) \) for any submodules \( N \subseteq M \).

**Proof**

It is clear that \( F(N) \subseteq \sqrt{F(N)} \). Conversely, let \( m \in \sqrt{F(N)} \). Then \( m^n \in F(N) \) for some positive integer \( n \). Therefore \( F(m^n) \subseteq F(F(N)) \) and so \( m \in F((m)) \subseteq F(N) \). Hence, \( \sqrt{F(N)} \subseteq F(N) \). Thus we have that \( F(N) = \sqrt{F(N)} \). Since \( F \) is quasi-radical it satisfies Relations 1 \((b), (e) \) and \((f)\). This implies that \( F(N) \subseteq F(\sqrt{N}) \subseteq F(F(N)) = F(N) \). Thus \( F(N) = F(\sqrt{N}) \) and we have proved the proposition.

**Proposition 3**

Let \( M \) be a multiplication \( R \)-module. Let \( F \) be a quasi-radical operation on the submodules in \( M \). Then for each submodule \( A \) in \( M \) the following holds:

\[
F(A) = \bigcap_{F(A) \subseteq N, \ N \text{ a prime submodule}} N
\]

**Proof**

We have that:

\[
F(A) = \sqrt{F(A)} = \bigcap_{F(A) \subseteq N, \ N \text{ a prime submodule}} N
\]

By Proposition 2, we get first equality. The second equality is clear.

**Theorem 1**

Let \( M \) be a multiplication \( R \)-module. Let \( F \) be a quasi-radical operation on the submodules in \( M \). If \( M \) satisfies the acc for \( F \)-radical submodules, then any \( F \)-radical submodule is the intersection of a finite number of \( F \)-prime submodules.

**Proof**

Let \( T \) be the set of \( F \)-radical submodules which are not intersection of a finite number of \( F \)-prime submodules. Assume that \( T \neq \emptyset \). Then \( T \) admits a maximal element \( N \), because the acc for \( F \)-radical submodules holds. Then \( N \) is \( F \)-radical and can not be prime. Take \( m \in N \) and \( r \notin (N:M) \) such that \( rm \in N \), then \( N \subset N + rm \) and \( N \subset N + rM \). Since \( N \) is maximal in \( T \) these two new modules are not in \( T \). From \((a)\) we get \( N \subset N + rm \subseteq F(N + rm) \) and \( N \subset N + rM \subseteq F(N + rM) \). Thus the submodules \( F(N + rm) \) and \( F(N + rM) \) are \( F \)-radical by \((b)\) but are not in \( T \) and therefore expressible as a finite intersection of \( F \)-prime submodules. By \((c)\) we have:

\[
N \subseteq F(N + rm) \cap F(N + rM) = F((N + rm)(N + rM)) = F(N^2 + rN + mN + rmM) \subseteq F(N) = N
\]

So, \( N = F(N + rm) \cap F(N + rM) \) and thus, a finite intersection of \( F \)-prime submodules, which contradicts the assumption that \( N \) is in \( T \). Thus \( T = \emptyset \).

**REFERENCES**