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Review

Review of applications of partial differential equations for image enhancement

Yuanfeng Jin[1], Tinghuai Ma[2], Donghai Guan[3]*, Weiwei Yuan[4] and Chengmin Hou[1]

[1]Department of Mathematics, Yanbian University, China.
[2]School of Computer and Software, Nanjing University of Information Science and Technology, China.
[3]Department of Computer Engineering, Kyung Hee University, Korea.
[4]College of Computer Science and Technology, Harbin Engineering University, China.

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Image restoration and enhancement are important parts of digital image processing, belonging to the early visual image processing problems. Image pre-processing is the necessary preliminary work of image analysis, such as filtering to reduce image noise and to enhance the image edges. The image enhancement technique plays an important role in improving image quality and is good for image post-processing e.g. image segmentation and image tracking. Image restoration and enhancement have been widely used in military, medical, industrial production and other fields. Partial differential equation (PDE) as a sophisticated method of image analysis and processing is of great values of research and application, which needs a deep study. As both the variational model and the anisotropic diffusion model have a complete theoretical framework, a variety of models and sophisticated numerical schemes, introduction of which to the fields of digital image processing and computer vision provides a powerful tool to solve problems undoubtedly . This paper concerns about the applications of the PDE in image restoration and image enhancement. We mainly assay traditional methods of image analysis, study applications of the variational method and diffusion equations in image restoration, as well as their improved algorithm for image enhancement.

Key words: Image restoration, image enhancement, forward and backward diffusion filtering.

INTRODUCTION

Image enhancement means a processing method to highlight some information in an image according to the specific needs, meanwhile weaken or remove the information unwanted. Its main purpose is to make the processed image more suitable for a particular application than the original image. Therefore, such treatments improve image quality for some application purpose. There are two kinds of image enhancement methods: the spatial domain method and the frequency domain method. The spatial domain method is mainly used to do direct operation on the pixel gray values in the space domain, such as the grayscale transformation, histogram modification, spatial domain smoothing, image sharpening and pseudo-color processing.

The frequency domain method means calculating the image transformation value in a certain image transform domain, such as the Fourier transform first, then the image frequency domain filtering, finally performing inverse transformation of the filtered image transformation value to the spatial domain, to obtain the enhanced image. This is an indirect approach, the principle and process of which are shown in Figure 1 (Lysaker et al., 2004; Ruan, 2007; Xia and Li, 2005). In Figure 1, $F(u,v), G(u,v)$ respectively represent the Fourier transforms of $f(u,v)$, $g(u,v)$ before and after process of the image, $H(u,v)$ corresponds to the filter transfer function, F^{-1} is the inverse Fourier transform operator. The process is:

$$G(u,v) = H(u,v)F(u,v) \quad (1)$$

*Corresponding author. E-mail: donghai@oslab.khu.ac.kr.

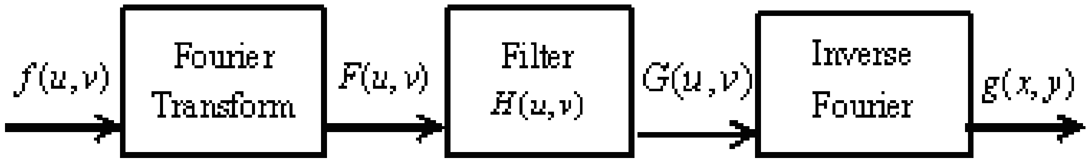


Figure 1. The principle and process of the frequency domain method.

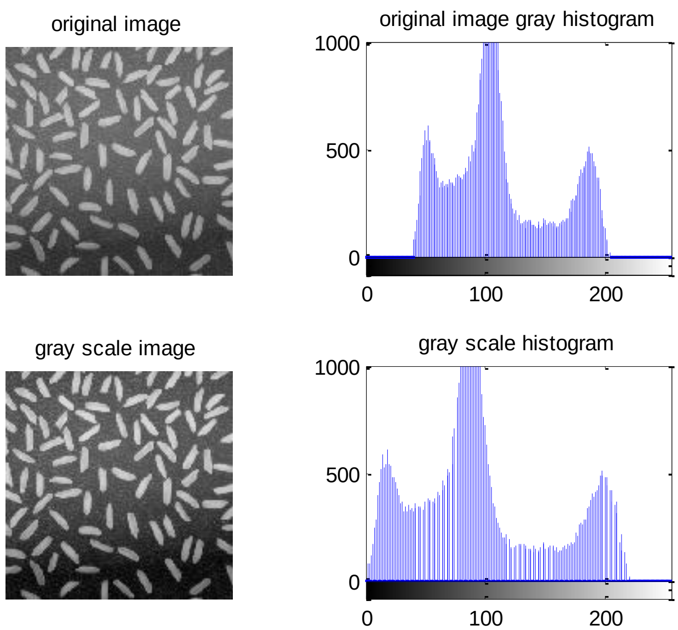


Figure 2. Grayscale image contrast enhancement.

$$g(u,v) = F^{-1}G(u,v) \tag{2}$$

GRAYSCALE TRANSFORMATION, IMAGE SHARPENING AND EVALUATION

Grayscale transformation

The grayscale modification is a method to enhance an image in the spatial domain, simple and of remarkable effect. Different modification methods can be adopted depending on different degraded performances. There are two common methods:

1. The grayscale transformation is used against a whole underexposed image or just a part of it. It aims at increasing the contrast of the image grayscale. The gray value of the original image shown in Figure 2 focuses in the middle gray area, after the contrast enhancement, the interval range of gray value in the original image gets a linear expansion, visual effects improve.

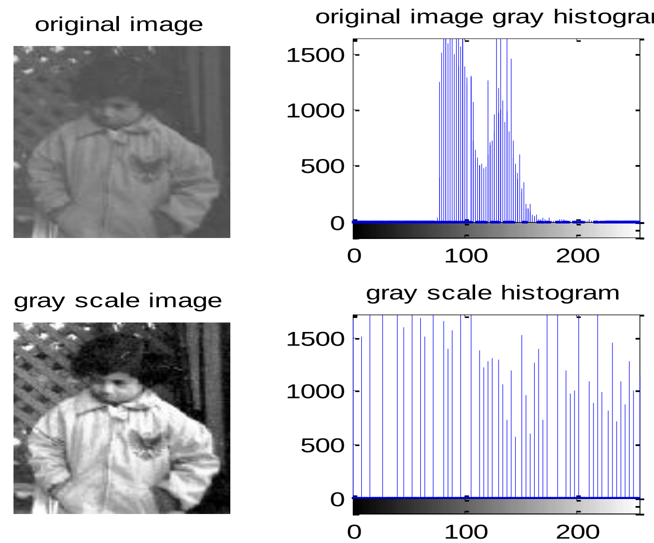


Figure 3. Grayscale histogram equalization.

2. Histogram modification. The method can make an image possess a grayscale distribution as wanted, and then highlight the desired image features selectively (Wang, 2004). The gray value distribution of the original image shown in Figure 3 is of non-linear expansion, the low gray value changes more, the overall vision of the image gets brighter, and some details are more prominent.

Image sharpening

Image sharpening is mainly used to enhance the edge of an image and the hopping part of the grayscale. It has two methods including spatial and frequency domain processing. The most common spatial domain method is to do image processing with the second-order Laplacian, which is similar to the differential process. The result of differential will make the edge of the image prominent. Therefore, the differential is one of the image sharpening processing methods. Take the image sharpening filter based on second-order differential for example, the simplest isotropic second-order differential operator is Laplacian, a Laplace transform of an image can be defined as:

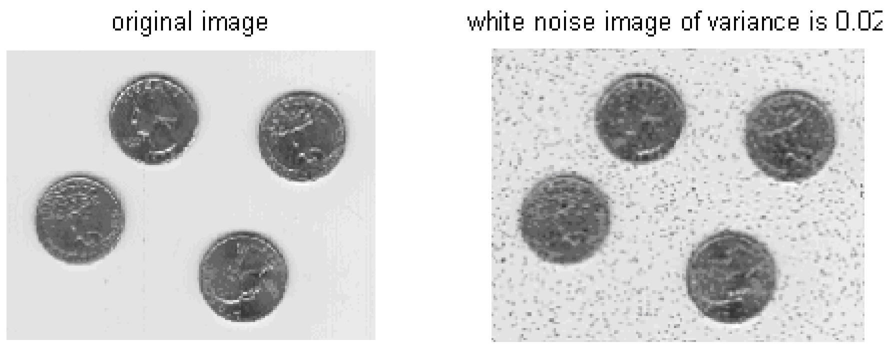


Figure 4. Filtered image with a 3*3 Laplacian second-order differential operator.

0	1	0
1	-4	1
0	1	1

Matrix 1. 3x3 Laplacian second-order differential operator.

$$\nabla^2 u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} \quad (3)$$

When in discrete format, we replace differential by difference, so the first-order differential about x_1 is

$$\frac{\partial u}{\partial x_1} = u(x_1 + 1, x_2) - u(x_1, x_2). \quad (4)$$

The second-order differential about x_1 is

$$\frac{\partial^2 u}{\partial {x_2}^2} = u(x_1 + 1, x_2) + u(x_1 - 1, x_2) - 2u(x_1, x_2). \quad (5)$$

Similarly, the first-order differential about x_2 is

$$\frac{\partial^2 u}{\partial {x_1}^2} = u(x_1, x_2 + 1) + u(x_1, x_2 - 1) - 2u(x_1, x_2) \quad (6)$$

So,

$$\nabla^2 u = [u(x_1 + 1, x_2) + u(x_1 - 1, x_2) + u(x_1, x_2 + 1) + u(x_1, x_2 - 1) - 4u(x_1, x_2 \quad (7)$$

This formula can be achieved with the filter corresponding to Matrix 1. So sharpening achieved by Laplacian can also be regarded as a filtering process. Filter results are shown in Figure 4.

Evaluation of image enhancement results

Evaluation methods of image processing quality

The research of evaluation of image processing quality is one of the basic studies in image information science. The image is the mainstay of information for image processing or image communication morphology, while the quality of an image is the important indicator when measuring the system. Image enhancement is to improve the subjective visual display quality. While image restoration is used to compensate for image degradation, so that the quality of restored image can keep the same as the original one as possible. All these call for an appropriate evaluation method of image processing quality.

Image processing quality encompasses two aspects: one is the fidelity of an image, that is the degree of the deviation between evaluated image and the original standard image; the other one is the intelligibility of an image, which means the ability of the image providing information for mankind or machines. The ideal situation is to find a quantitative discription method of fidelity and intelligibility of an image that can be seen as the basis of the image evaluation and the image system design. However, due to the lack of full understanding of both the characteristics of human vision system and quantitative description methods of human psychological factors, the most frequently used and most authorative method is the so-called subjective evaluation method.

Subjective evaluation of images

Subjective evaluation of images is to observe the images through people, who judge the qualities of images subjectively. We can get the final evaluation results by collecting and averaging their assessments. The evaluated qualities of images are associated with the observers' characters and the observational conditions.

Objective evaluation of images

Although the subjective evaluation of images is the most authoritative way, on some researchful occasions or owing to the restriction of experimental conditions, the quantitative discription of image qualities is preferred. The fidelity measurement is in common use. Fidelity can be defined as a normalized mean square error (NMSE). Another method is peak mean squar error (PMSE).

As for digital images, assume $f(j,k)$ as the original reference image, $\overline{f}(j,k)$ as its degraded image, then:

$$NMSE = \frac{\sum_{j=0}^{N-1}\sum_{k=0}^{M-1}[f(j,k)-\overline{f}(j,k)]^2}{\sum_{j=0}^{N-1}\sum_{k=0}^{M-1} f^2(j,k)} \tag{8}$$

$$PMSE = \frac{\sum_{j=0}^{N-1}\sum_{k=0}^{M-1}[f(j,k)-\overline{f}(j,k)]^2}{M \times N \times A^2} \tag{9}$$

Where $A = 255$, M, N are the sizes of the image.

Other methods

On some specific occasions, there are some other evaluation methods. For example, when developing the MPEG.4 standard, ISO proposed two ways to evaluate the quality of a video image, one is called the quality evaluation based on feelings; the other one is called the quality evaluation based on tasks (Zhu et al., 2002).

Features of traditional image processing

Traditional algorithms of image enhancement are comparatively easiser and faster, but have limited effects. The results can not improve the signal to noise ratio (SNR),which can only make some characters more obvious to recognize. Nevertheless, when an image is polluted by noise, traditional enhencement algorithms are more likely to failure. When a polluted original image is sharpened by Laplace second-order differential, the outputting image no longer has the specific borderline as the original one, but becoming a blurred image with noise. However, in the real world, the images captured by photographic equipments are usually with noise. For the images with noise, denoising must be done before using traditional image enhancement algorithms. While some partial differential models can do the image denoising and the edge enhancement simultaneously, moreover, they are faster and more accurate.

OVERVIEW ON IMAGE RESTORATION AND ENHANCEMENT BASED ON THE PARTIAL DIFFERENTIAL EQUATIONS

Applications of partial differential equations in image processing

In the past 20 years, because human vision has high-level pursuit of image restoration and enhancement, doing image processing by mathematical methods has became an important project. Among those methods, partial differential equations and functional analysis are preferred. Thus a series of mathematical methods are posed, including partial differential equation model used in image processing and solving partial differential equations with computer. The idea of doing image processing with partial differential equation could date back to Gabor and Jain (Gabor, 1965), but this method was not established until Koenderind (1984) and Witkin (1983). They introduced the concept of Scale Space, which represented a group of images in different scales simultaneously. Their distribution constituted the basis of doing image processing with partial differential equations to a great degree. The scales of an image are obtained by Gaussian smoothing, using the classical heat conduction equation for the image evolution can also bring the Scale Space.

In the late 1980s, Humme (1989) proposed that the heat conduction equation is not the only one for Scale Space constitution and suggested some principles instead. The anisotropic diffusion model proposed by Perona and Malik (1990) is the most influential one in this field. They suggested replacing the Gaussian diffusion with a diffusion which can keep the selectivity of edge. This incurred large amounts of researches on theoretical and practical problems. Under the same framework, the Shock filter proposed by Osher and Rudin (1990) and the method of total variation (TV) reduction proposed by Rudin et al. (1992) further stressed the importance of the partial differential equations in image processing. In the fields of image processing and computer vision, some other partial differential equations are based on the curve and surface of curvilinear motion. Osher and Sethian (1998) developed the level set numerical algorithm. They tried to describe the deformation of curves, surfaces or images with a higher dimensional and hypersurface level set. This technique not only made the numerical results more accurate, but also solved the topology problem difficult to solve previously. Partial differential equations can also be used in image segmentation. The model proposed by Mumford and Shah (1989) combined a variety of image segmentation algorithms, incurring many new theoretical and practical problems. Proposed the image segmentation algorithm based on moving boundary also had a great influence, and later many scholars expanded their work with the geometric partial differential equations.

Partial differential equations can also be used in image in painting (Chan and Shcn, 2001; Mumford and Shah, 1989). It is synonymous with the image interpolation, originally coming down from the artists who restored broken works of art by hand in museums. Currently the digital image in painting technique is widely applied to the image processing, visual analysis and digital technology, for example, image restoration, image enlargement, image super resolution analysis and error concealment in the wireless image transmission, etc. As the models for image processing emerging, multiple mathematical methods that solve the continual models represented with partial differential equations also appeared.

In terms of speeding up the calculating speed, as describe, complexity calculation should be taken into account when calculating the continual integration with models based on partial differential equations. There are two kinds of methods to cope with. One kind of methods is to find a time integration algorithm without solving the anisotropic diffusion equation, the calculation speed of which is faster than solving the anisotropic diffusion equation. Meanwhile regularization can eliminate required demands of algorithms like of the sensibility to noise, instability and consistency of calculation, which means getting a parallel result to that of the anisotropic diffusion method by less calculated amount. Common methods are finite difference method, finite element, finite volume or spectral method Foyer and Zou, 2006; Caselles et al. 1998). The second kind of methods is to reduce the frequency of continuous integration equations that anisotropic diffusion equation needs by the adaptive grid algorithm, researches about the imaging of human retina to geometry structures indicate that it can result in enhanced image of large size by utilizing variable grid integrating algorithm in relatively less time.

It shows that the basic equations for image processing are shock filter, total variation, anisotropic diffusion equation, moving boundary and so on. They can realize the recovery, enhancement and segmentation of images. Algorithms we studied in this paper are based on shock filter, total variation, and anisotropic diffusion. They are important parts of many image processing methods based on partial differential equation and the improvement and numerical calculation in these models have positive meanings.

Main methods for recovery and enhancement of images based on anisotropic diffusion equation

Recovery and enhancement of images are both technologies for improving images' quality. In the actual process of imaging, the original clear image may become fuzzy for various reasons. It will encounter in many actual applications. Image recovery is to recovery fuzzy images to original clear images. In recovery process, degradation model is firstly built, then against the quality decline process. We adopt a method to recovery and rebuild original images. According to the difference of mathematical modeling, we can classify algorithms of noise removal and enhancement for common images of anisotropic diffusion equation as follows.

Regularization methods of inverse problems

Image recovery can be viewed as inverse problems of image degradation model $f = Au$ or $f = Au + boat$. That is getting original images from degradation images. In the above model, A is (usually linear case is considered) and assuming image functions u and f are uniformly bounded functions of $\Omega \in R^2 \to R$. According to actual situation of numerical image process, if problem is solved by changing it into linear equation, the spectral value of coefficient matrix will be smaller. Linear equations are always weak conditions and the solution in this condition is always ill-posed. So academia has imposed many regularization methods to solve ill-posed problems. These methods focus on maintaining stability as well as information of the solution, and use methods of statistics, iteration, and variation to get regularization solution of inverse problems. For example, by solving extreme value, among them, $\nabla f(x)$ represents the gradient of function $f(x)$. $\|Au - f\|$ is used to maintain information of the solution. $\int \phi[|\nabla f(x)|]dx$ used to maintain stability of the solution. λ used to balance effect of the two. It is dainty in selection of norms $\|\cdot\|$ and λ.

Methods based on geometric character

The diffusion methods are based on anisotropic diffusion equation. It can be divided into three kinds: linear isotropic diffusion filter, nonlinear isotropic diffusion filter and linear anisotropic diffusion filter. Linear isotropic diffusion filter is the simplest of image smoothing algorithms based on anisotropic diffusion equation which has distinct physical meaning. In the case of no production and extinction of substance, it can be viewed as a process of balance internal concentration difference. It can be represented by a mathematical formula:

$$j = -D \cdot \nabla u \tag{10}$$

The formula shows gradient of concentration ∇u has produced flow j, and it will be compensated for gradient. The relationship between ∇u and j is described by diffusion tensor D. It is a positive symmetric matrix. When ∇u and j are parallel, the diffusion methods are isotropic. In this condition, it can be represented by a diffusion coefficient g with positive scales. Usually when it is anisotropic, ∇u and j is not parallel. In image process, the diffusion rate of linear isotropic diffusion filter is constant; the diffusion rate of nonlinear isotropic diffusion filter corresponds to the local structure of images; the diffusion tensor of nonlinear anisotropic diffusion filter corresponds to the local structure of images.

In the anisotropic equation, conductivity coefficient in every point is a matrix. Usually, it makes diffusion effect relevant to weak and strong direction by setting every coefficient of thermal conductivity matrix. Along other direction, diffusion should be stronger, so that it can achieve smoothing and removing noise. The design of thermal conductivity matrix depends on the gradient in current point of the image. So diffusion tensor is changed by spatial position.

The diffusion process of conservation of matter above can be described by the following continuity equation:

$$\partial_t u = -div\ \ j \tag{11}$$

Among it t represents time. By substitution of formula (10) into formula (11), we have:

$$\partial_t u = -div\ (D \cdot \nabla u) \tag{12}$$

The equation is present to many physical transmission processes and is called heat equation in the case of heat transfer. The level set method takes images as a set consisting of isodense outline. Considering noise is the main reason of degeneration of images, we take most noise as small targets with bigger curvature of isodense outline. We make isodense outline with big curvature in noise segment shrink to a point, even disappearing with the time evolution. Images with small curvature will be preserved by evolving into the outline of images. The general model of level set method is $F|\nabla u| = 0$ or $\partial_t u(x,t) = F|\nabla u|$, where F is speed function. On this base, average curvature flow model and max/min curvature flow model all get well development.

The method based on image transformation

It takes process of image enhancement as a process of image transmission. That is making operator Z acting on image function f. To achieve image processing, operator Z needs to meet some conditions. Alvarez Guichard Lions and Morel has generalized three kinds of conditions.

Structure conditions

1. Recursion: for any $\|T_t(f+hg)-(T_t(f)+gh)\|_\infty \le Cht$ and function $f(x)$ defined in R^n, $T_0(f) = f$, $T_s o T_t(f) = T_s(T_t(f) = T_{s+t}(f) f$. character o represents the combination of operator.
2. Causality: for any $0 \le s,\ t < \infty$, $T_t = T_{t,0},\ T_{t+s} = T_{t,s} o T_s$. will be tenable with conversion operator cluster $\{T_{s,t}\}$.
Regularization: for any $0 \le h,\ t \le 1$, and smoothing functions f and g, $\|T_t(f+hg)-(T_t(f)+gh)\|_\infty \le Cht$ will be tenable.
3. Limitation: for any smoothing functions f and $|T_t(f)-T_t(g)|(x) = o(t),\quad D^\alpha f(x) = D^\alpha g(x),$ is tenable with any $|\alpha| \ge 0$ and $x \in R^n$. So, when $t \to 0^+$, there is $T_t(0) = 0,$ $T_t(f+C) = T_t(f) + C.$

Stability condition

Comparison principle: for any functions f and g ,if $f \le g$, then for any $t \ge 0$, $T_t(f) \le T_t(g)$ is always tenable.

Morphology condition

1. Gray translation invariance: for any functions f and constant C, $T_t(0) = 0,\quad T_t(f+C) = T_t(f) + C$ is tenable.
2. Gray scale invariance: assuming h is a non decreasing real function, for any functions f and any $t \ge 0,\ T_t(h(f)) = h(T_t(f))$ are always tenable.
3. Scale invariance: for any T_t and t, there will be a t' making $D_\lambda T_t = T_{t'} D_\lambda$ tenable.
4. Transmission invariance: for any $h \in R^n, t \ge 0,$ assuming $(\tau_h o f)(x) = f(x+h)$ is tenable, $T_t(\tau o f) = \tau o(T_t f)$ will be tenable too.
5. Isometry invariance: assuming R is $(Rof)(x) = f(Rx)$, where R is an orthogonal transformation in R^n. So for any f, $t \ge 0$, $T_t(Rof) = RoT_t(f)$ is tenable.
6. Projection invariance: for any operator A and t, there will be a t' relevant to A and t making $AoT_{t'} = T_t\, oA$ tenable. t' satisfies to all above conditions can do image process.

The method based on variation

Based on preserving original information of image as much as possible, it makes images smoothing so that optimization to some extent is achieved. Then a corresponding variation model is proposed. It is a mass of optimization:

$$\begin{cases} \min J(u) \\ s.t. \int_\Omega Au = \int_\Omega f, \ \int_\Omega |Au - f|^2 = \sigma^2 \end{cases} \quad (13)$$

Among them, $\begin{cases} \min J(u) \\ s.t. \int_\Omega Au = \int_\Omega f, \ \int_\Omega |Au - f|^2 = \sigma^2 \end{cases}$ is aimed at preserving image information and $J(u)$ is functional to measure degree of irregularity of images. There are several styles. In some condition, optimization problems above can be changed into extreme value problem $J(u) + \frac{\lambda}{2}|Au - f|^2$. When λ has value of 0 it can be viewed as a Gaussian Filter: $\min(J(u)) = \int_\Omega |\nabla u|^2 dx$. This is a traditional L^2 norm method based on gradient which causes solving problem of linear equation.

In the aforementioned contents, four basic methods of image process based on partial differential equation were described. They have different senses. The regularization Method of Inverse Problems takes the process of image recovery as an inverse process of image degeneration; Level set method takes images as a set of isodense outline. And the curvature of isodense outline of noise is partly bigger; the method based on image transformation focuses on conditions which image transformation need to meet; variation method focuses on not only decreasing degree of irregularity of image but also preserving original image information when processing images. It makes a feature of the image optimal.

The basic theory related to image enhancement algorithm

The image enhancement algorithms that based on partial differential equation mainly are diffusion method and variation method. Compared to other methods, there are some advantages in calculation. Firstly, they can process important geometric characters directly which affects visual effect. Such as gradient, angle, curvature and so on. The diffusion method can do simulation effectively on linear and nonlinear diffusion. Secondly, the diffusion method and variation method get well development on calculation of current relative mathematics analysis theory and partial differential equation. Next we will introduce some basic theory related to algorithms.

The representation model of images in the method of partial differential equation

For processing images efficiently, we firstly should know how to understand and represent images in term of mathematics. The image model and its representative method largely determine the process model of images. There are common three kinds of models: random field model, wavelet model and regular space model.

1. Random field model takes the image as a sampling result of random field model. The image can be simulated by some Gibbs or Markov random field model. The statistical characteristics of random field model often can be established by filter techniques and learning theory. The random field model is the most ideal in term of describing nature images (such as trees and hills) with relatively rich texture.
2. Wavelet model is based on time and frequency analysis theory. Every transient component of images maps to time. The location of frequency plat corresponds to component main frequency, occurrence time and amplitude. So the space the image located in is three-dimensional which can be viewed as a lamination. There are three techniques related to wavelet which are filter group theory, multi-resolution or time field scale analysis (especially pyramid representation) and sub-band coding. The successful compression of new JEPG2000 protocol and FBI fingerprint database are the two most influential applications.
3. In the traditional linear filter theory of image process, an image is viewed as an element in the $H^1(\Omega)$ in the Sobolev space. For the function in Sobolev space is continuous, Sobolev model is good in processing some flat field in an image. But it is not good as a model of a whole image for it blurs important visual information-edges. Currently there are two famous models represented by partial differential equation can process edge problems. One is Mumford and 'object-edge' from Shah. The other is an image model BV (bounded difference) from Rudin, Osher and Fatemi. "object, Edge" model is based on the assumption that image U is consisted of different flat blocks $[u_k, \Omega_k]$ $(u_k \in H^1(\Omega_k))$ and regular boundary $\partial\Omega_k$. BV model assumes that the total variation $\int_\Omega |Du|$ of the image is bounded.

From the point of specific styles of partial differential equation, here our study involves two kinds. The most representative is oval Laplace equation $\frac{\partial^2 I}{\partial^2 x} + \frac{\partial^2 I}{\partial^2 y} = 0$ and parabolic heat conduction equation $\frac{\partial^2 I}{\partial^2 x} + \frac{\partial^2 I}{\partial^2 y} = \frac{\partial I}{\partial t}$.

For determining the solution of partial differential equation completely, proper conditions of definite solution should be given. The definite solution involved in this paper contains initial condition and edge condition. Usually initial condition is an image with noise which is going to be processed. Boundary should meet condition Neumann. That is on the boundary of the image, the value of exterior normal derivative is 0. Both differential equation and definite condition consist of the definite solution problem.

Gaussian smoothing

Convolution is a very important method in typical image processing techniques. And in convolution method the most common convolution kernel is Gaussian convolution. The convolution on a gray image f and Gaussian filter can be represented as follow:

$$\left(K_\sigma * f\right)(x) = \int_R K_\sigma(x-y) f(y) dy \tag{14}$$

Where K_σ represents two-dimension Gaussian kernel of variance σ^2:

$$K_\sigma(x) = \frac{1}{2\pi\sigma^2} \cdot \exp[-\frac{|x|^2}{2\sigma^2}] \tag{15}$$

The frequency of convolution of time field can be represented as follow:

$$F[K_\sigma * f](\omega) = F[K_\sigma](\omega) \cdot F[f](\omega) \tag{16}$$

Among it,

$$F[K_\sigma](\omega) = \exp[-\frac{|\omega|^2}{2\sigma^2}],\ F[f](\omega) = \int_{R^2} f(x)\exp(-iw \cdot x)dx.$$

The equivalence relation between linear diffusion process and Gaussian filter

For image f, the linear diffusion process is as follows:

$$\partial_t u = \Delta u \tag{17}$$

$$u(x,0) = f(x) \tag{18}$$

It has only one solution

$$u(x,t) = \begin{cases} f(x) & (t=0) \\ (K_{\sqrt{2t}} * f)(x) & (t>0) \end{cases} \tag{19}$$

Assuming $u(x,t)$ meets the condition: $|u(x,t)| \le M \cdot \exp(a|x|^2)\ (M, a > 0),$ image f meets max and min theorem in $R^2 \times [0,\infty)$: $\inf_{R^2} f \le u(x,t) \le \sup_{R^2} f.$ In the theory of partial differential equation, the Gaussian convolution for initial signal is just solution to heat conduction equation. Gaussian kernel is the basic solution to heat conduction equation. The variance of Gaussian kernel is relative to the time of basic solution. We can see from formula (3.3) that time t corresponds to the variance of time field of Gaussian kernel $\sqrt{2t}$. The smoothing structure of degree σ should be stopped on the time $T = \frac{1}{2}\sigma^2$. According to this observation, people proposed the first partial differential equation- heat transmission equation used as image process:

$$\frac{\partial I(x,y,t)}{\partial t} = \Delta I(x,y,t) = \frac{\partial^2 I(x,y,t)}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2} \tag{20}$$

Among it, $I(x,y,t)$ represents the image on time t. Initial image $I(x,y,0)$ is initial condition. Through the Fourier analysis, we can see the high frequency component in initial image is gradually removed as the time to getting the solution of the equation increases. So the equation has the effect of low pass filter. Heat transmission equation is a linear equation with the diffusion feature of isotropic. Heat transmission coefficient is always equal to 1 at any place. This is equivalent to apparent 'heat' (gray value) will be spread. At last we will get an image with consistent gray value. It is equivalent to the average of initial 'heat' (gray value). We can conclude that the linear diffusion filter is similar to the mean filter. In terms of filter effect, Gaussian filter and average filter is equivalent. At the same time of removing noise, the boundary of the image is often blurred too (Figure 5).

Gaussian function

In order to know the structure of the image, one should analyze the deviation of gray value in the neighborhood of every pixel. That is calculating the gradient information of the image. However, affected by noise and so on, a small disturbance of the initial image can cause arbitrarily large deviation of derivative. So the regularizing method is needed. One available regularizing method is making convolution between the image and Gaussian kernel firstly. We can find that all images seeking derivative experience the same Gaussian smoothing process which is equivalent to the convolution between an image and a Gaussian function from equation $\partial_{x_1}^n \partial_{x_2}^m (K_\sigma * f) = K_\sigma * (\partial_{x_1}^n \partial_{x_2}^m K_\sigma) * f$. Gaussian derivative can also show differential invariance after rotation transmission. Such as $|\nabla K_\sigma * u|$ and $\Delta K_\sigma * u$. It is very useful in checking the boundary and other structures of the image.

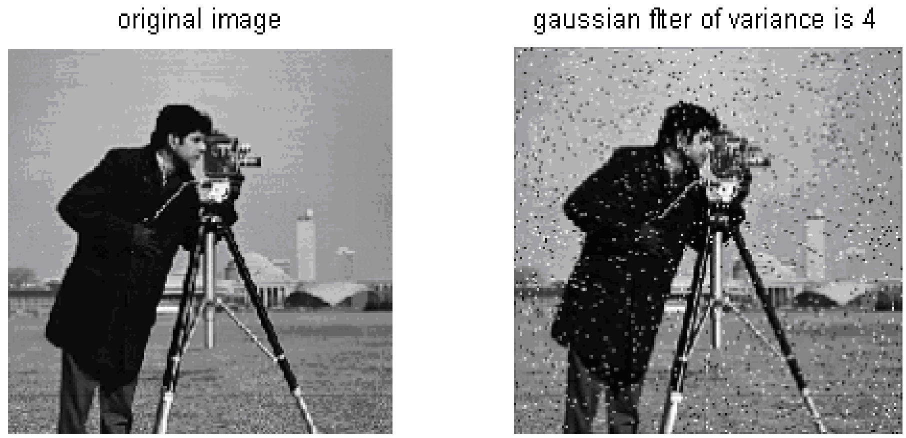


Figure 5. The image after Gaussian lowpass filtering.

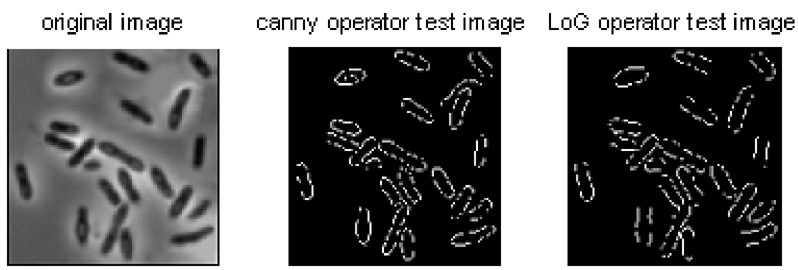


Figure 6. Cannyhe and LoG operator boundary detector.

The boundary detector Canny often used is from first derivative of the image after Gaussian smoothing. The location with bigger gradient amplitude is set as boundary as shown in Figure 6. The method is the best linear boundary detector. It is almost a criterion of boundary detector. When doing specific calculation, Gaussian function can be decomposed to simplify calculation process. Another interesting boundary detector is operator Marr-Hildreth. It is a convolution kernel used in Gaussian Laplace (LoG) ΔK_{σ} as shown in the right of Figure 6.

The boundary of the image f is defined by zero crossings $\Delta K_{\sigma} \cdot f$. This does not need further post-processing and can always get a close boundary. An interesting phenomenon can be viewed if we study time field evolution of zero crossings of the image after linear diffusion filter. When increasing smoothing scale σ, it will not appear new zero crossings with more precise scale. The evolution feature is closely related to the max-min principle of determining parabolic operator. To get actually stable result, additional information should be added when rebuilding original image from time field evolution of zero crossing of Laplace operator. However, the evolution feature of zero crossings is important thought in scale space theory.

Scale space

Scale space is the representation of the image in continuous scale space. We assume that the image have the smooth structure with Invariance, the image f consists of its gradually simplified version family. When the image's scale changes from fine to rough, this smooth transformation should never generate the artifact, this is so called the process of information reducing. In Oshe and Sethian (1998), many people like Alvarez study the relation between the space theory and partial differential equation (PDE). The image after filtering must become the viscous solutions of second order parabolic partial differential equations using the following filtering axiom: Comparison theorem can keep characteristics when we make sure power exponent (the time of continuously differential), but it do not have enhancement. Axiom is the generalization of nonnegative smooth nuclear in the nonlinear cases.

Assuming $T_t : C_b^{\infty}(R^2) \to C_b^{\infty}(R^2)$ is the bounded function space having any number of derivatives. We use X to express $C_b^{\infty}(R^2)$ below.

(A1) Recursiveness:
$T_0(u) = u, T_s \circ T_t = T_{s+t}(u), \forall s, t \ge 0, \forall u \in X.$

(A2) Consistency:

$\left|T_t(u+hv)-(T_t(u)+hv)\right| \le cht, \forall u,v \in X, c$ rely on u and v.

(3) Directionality:

$(T_t(u)-T_t(v))(x)=o(t), t\to 0^+$, for all the $\forall u,v\in X$ and all the $\alpha \ge 0$ satisfy $\nabla^\alpha u(x)=\nabla^\alpha v(x)$

(A4) Comparison theorem (limit theorem):

For all the $t \ge 0$ and $u,v\in X$, if $T_t(u)\le T_t(v)$ we have $u \le v$.

(I1) The invariance of gray translation:

For all the $u,v\in X$, and all the constants c, if $T_t(0)=0$ then we have $T_t(u+c)=T_t(u)+c.$

(I2) The invariance of transference:

For all the h, in the place which satisfies $(\tau_k u)(x)=u(x+h),$, we have $T_t(\tau_k u)=T_k(\tau_t u)$ in linear cases, many demands above can be satisfied, but they may not be met in nonlinear cases.

We join linear conditions in the axioms above, for example $T_t(au+bv)=aT_t(u)+bT_t(v), \quad \forall t\ge 0, \forall a,b\in R,$ and we join the Isometric unchanged characteristics, for instance, for all the $f, t\ge 0$ and all the orthogonal transformation R, we have $T_t(Rf)=RT_t(f).$ in reference [12], we can obtain the linear scale space (Gaussian scale space). Gaussian scale space is gained from the convolution of itself and Gaussian functions. Gaussian functions have increased width which is equal to linear diffusion filter.

The estimation criterion which variational method adopts

The estimation criterion which variational method removing noise adopts is MAP criterion. Its core ideology is creating regular functions according the MAP criterion and geometric knowledge. We assume that for the unknown digital image u, its prior probability is $p(u)$, its MAP estimator can be expressed as $\bar{u}=\arg\max_u\left\{\log p(u_0|u)+\log p(u)\right\}$, $p(u_0|u)$ is the conditional probability for u to get u_0. The normal mode of prior probability is MRF, its characteristic is Gibbs distribution. $p(u)=\frac{1}{Z}\exp\left\{-\frac{F(u)}{\lambda}\right\}$. Z is the partition function, λ is a constant, F is the energy function, reference [38] prove that Gibbs distribution is equal to MER. For the image which is polluted by Gaussian noise, the formula $p(u_0|u)=K\exp\left\{-\frac{|u-u_0|^2}{2\sigma^2}\right\}$ is workable, then MAP estimator can be expressed as $\bar{u}=\arg\min_u\left\{F(u)+\frac{\lambda}{2}|u-u_0|^2\right\}$.

The Dirichlet integral function and total variational integral function in image denoising can be separately defined as $D(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx$ and $TV(u)=\int_\Omega|\nabla u|dx.$ Then the minimization problem total variational method of corresponding can be expressed as $\bar{u}=\arg\min_{u\in X}\int_\Omega\left\{F(|\nabla u|)+\frac{\lambda}{2}|u-u_0|^2\right\}dx,$ x is the appropriate image space which has smooth functions like $C^1(\bar{\Omega})$ or image function space $BV(\Omega)$ which have bounded variation or Sobolev space $H^1(\Omega)=W^{1,2}(\Omega)$. $BV(\Omega)$ is the Banach space which have norm $\|u\|^2_{H^1(\Omega)}=\|u\|^2+\|\nabla u\|^2$.

The discrete method of partial differential equation

In generally, the partial differential models used to image denosing and image enhancement are continuous, it is tough to gain analytical solutions, in generally, and we often obtain the numerical approximation solution (approximate solution). In practical problems, if the image has the value in fixed and equidistant grid (or pixel), it needs to be discretized into the continuous partial differential functions. We adopt the equivalent discretization to approximate convolution process and the diffusion equation when we solve the linear diffusion equations and handle the Gaussian convolution nuclear. Like images and Gaussian kernel for airspace convolution, we use the image in the limited domain and Gaussian kernel separately to do FFT. It can reduce calculated amount if we multiply the results and do IFFT. According to the calculated amount of FDCT and IDCT, this process is very effective for the large kernels. However, when we convolve in space region, we often need to truncate the Gaussian sampling kernel, the shortcoming of this method is that it cannot keep semi-attributes of continuous Gaussian scale space.

Lindeberg (1990) put forward linear space theory specific to semi-discrete conditions. He prove that the discretization of Gaussian kernel can be expressed as rectifying Bessel function, because this scale space is obtained from the semi-discrete form of diffusion equations. He made draw a conclusion that the approximation of diffusion equations should priori to discrete the convolution integral. In many methods used to approximate linear diffusion equations, FD is constantly used. Among those methods, there are few implicit methods, in other words, explicit methods are the main methods. When we need to effectively approximate Gaussian scale space, we adopt the multigrid method. Gaussian pyramid (Butt and Adelson, 1983), presents multilevel representations of scale space which consist of different image resolution. This method uses the display

(a) (b)

Figure 7. Filtering effect of P.M model on gray noise image. (a)gray noise image, (b) the image after the P-M filter.

representations of diffusion equations to do image filtering.

According to the results limited in a coarse grid, we can gain the image representation in a thicker grid in next level. As this method is very simple and effective, pyramid decomposition is integrated into commercial hardwares. In most applications of nonlinear diffusion filters, FD is preferentially used, because it is simple to handle each point's pixels of the image for the real value image having been discretized in the fixed grid. Explicit methods are simple and popular used because it has good local qualities, which is suitable for parallelism. However, we need to set a small step length in computing to make the algorithm stable. The semi-implicit algorithm has better stability properties.

The discrete method of differential operator

The simplest discrete method of the derivative of 1-dimensional function use two-point method to become difference function.

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f^{n}(\xi) \quad (21)$$

h is the sampling interval, when h is very small, we can use the first difference to approximate $f'(x_0)$, the error of this discrete method is $o(h)$, if $h > 0$,then we call it forward difference, if $h < 0$, then we call it backward difference. 2-dimensional differential operator use small regional template convolution to compute approximately. The gradient of 2- dimensional functions is defined as

$$\nabla I(x,y) = [G_x, G_y]^T = [\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}]^T \quad (22)$$

G_x and G_y use the same template. According to the size of the template, the element values are different, now people have put forward many kinds of different operators. Some kinds of Common gradients' templates include Roberts cross operator, Prewitt operator and Sobel operator. Figure 7 is the template of the Laplace operator of 2- dimensional derivatives, when we compute operator, we adopt the way which is similar to convolution. Let the template move to 11, we compute the gradient value of the central pixel in each place. When we have the discretization, we have to ensure the correctness of discretization, also we hope to do the following points:

1. The results of discretization need to be as simple as possible. We need to use the iterative method when we solve the partial differential equations; the complicated discretization will make analysis and calculation tough.
2. The symmetry of discrete operator. It means that the discrete operator of a point is center symmetric with itself and it can ensure the invariance of direction of the operator compared to the image. The results of transformations of the operator compared to the image are the same after rotating the image 90° and rotating the image 180°. So when we compute the first derivation, we often use the central differencing scheme. We also can find that the central differencing scheme has higher precision compared with the forward differencing scheme and the backward differencing scheme.

The typical differencing scheme of diffusion equations

Taking the one-dimensional constant coefficient self-defocusing equations $\frac{\partial I}{\partial t} = \alpha \frac{\partial^2 I}{\partial x^2}$ for example, $x \in R, t > 0, a$ is positive integer. The step length of time t is Δt, the step length of space x is h. We assume that $I(x,t) = I(jh, k\Delta t)$. There are three kinds of differencing schemes often used.

1. The forward differencing scheme (explicit formulation)

$$\frac{I_j^{k+1}-I_j^k}{\Delta t}-\alpha\frac{I_{j+1}^k-2I_j^k+I_{j-1}^k}{h^2}=0 \quad (23)$$

The truncation error of the forward differencing scheme is $o(\Delta t+h^2)$ by Taylor series expansion, we can have the growth factor $G(\Delta t,\sigma)=1-4ar\sin^2\frac{\sigma h}{2}$, $r=\frac{\Delta t}{h^2}$, if $ar\le\frac{1}{2}$, then $|G(\Delta t,\sigma)|\le 1$. So Von Neumann is met, the stability condition of this forward differencing scheme is $ar\le\frac{1}{2}$.

2. The backward differencing scheme (implicit formulation)

$$\frac{I_j^{k}-I_j^{k-1}}{\Delta t}-\alpha\frac{I_{j+1}^k-2I_j^k+I_{j-1}^k}{h^2}=0 \quad (24)$$

The truncation error is $o(\Delta t+h^2)$, the growth factor is $G(\Delta t,\sigma)=\frac{1}{1+4ar\sin^2\frac{\sigma h}{2}}$, because $a>0$, for all the grid ratios r, $G(\Delta t,\sigma)\le 1$. This scheme is stable without any conditions. The forward differencing scheme and the backward differencing scheme are the scheme that time is a one order precision and space is second order precision.

3. The weighted implicit formulation combined the explicit formulation and the implicit formulation

$$\frac{I_j^{k}-I_j^{k-1}}{\Delta t}=\alpha\theta\frac{I_{j+1}^k-2I_j^k+I_{j-1}^k}{h^2}+a(1-\theta)\frac{I_{j+1}^k-2I_j^k+I_{j-1}^k}{h^2} \quad (25)$$

$0\le\theta\le 1$ is the weighting efficient . We assume that $I(x,t)$ is the fully smooth solution of the equation (22). Let equation (25) do the Taylor series expansion in (x_j,t_k) and dissolve, then we can have the truncation error $E=a(\frac{1}{2}-\theta)\Delta t[\frac{\partial^3 I}{\partial x^2\partial t}]_j^k+o(\Delta t^2+h^2)$, if $\theta\ne\frac{1}{2}$, the truncation error is $o(\Delta t^2+h^2)$; if $\theta=\frac{1}{2}$, the truncation error is $\theta=\frac{1}{2}o(\Delta t^2+h^2)$. The differencing scheme which $\theta=\frac{1}{2}$ is called Crank-Nicolson scheme or CN scheme, this is a scheme with two order precision.

The growth factor of the weighted implicit formulation is $G(\Delta t,\sigma)=\frac{1-4(1-\theta)ar\sin^2\frac{\sigma h}{2}}{1+4\theta ar\sin^2\frac{\sigma h}{2}}$, the stability condition is obtained according to $|G(\Delta t,\sigma)|\le 1$, if $0\le\theta\le\frac{1}{2}$, $r\le\frac{1}{2a(1-2\theta)}$; if $\frac{1}{2}\le\theta\le 1$, the scheme is stable without any conditions. Expected three schemes; there are three layer explicit format, three layer implicit format, stagger scheme, PMECME and Asymmetric format (Zhang, 2006). What this article adopts is the explicit format, though it is conditional stability, its computation speed is the fastest.

The characteristics and research meaning of partial differential equations

Broadly speaking, the image processing technology based on partial differential equations has the following characteristics: Let partial differential equations and the field of curvature used to image analysis, the image can be expressed as a continuous signal, partial differential equations can be considered the iteration with the Locally Filter having infinite dimension neighborhood. The discretization of local nonlinear filters is easier to be understood. The property not relying on grids and the isotropy simplify the calculation formula (Kass et al., 1988). Because Mathematics has a long-term further research about the numerical approach of partial differential equations, we can gain the numerical solution of high accuracy and stability. When we consider the image processing and the numerical solution, we will suffer the derivation problem of the non-smooth signal inevitably.

The viscous solution of computing math provides the strict mathematical theory to handle such problems. There is the good mathematics foundation of the method of partial differential equations, therefore this method can provide deep theoretical results, and the algorithm has good stability. We lead the partial differential equations theory into image processing and computer vision, because the theory not only has ready-made mature algorithms, but also provides rich theoretical results, like the provement of the existence, stability and uniqueness of results.

We deal with the image from the perspective of partial differential equations and we find new methods. These methods include more invariants compared with typical methods, like the filtering keeping the structure, linear enhancement and so on. At the same time, partial differential equations make the synthesis of image processing method very nature. For example, we give two different image processing methods: $\frac{\partial I}{\partial t}=F_1(I(x,t))$

and $\frac{\partial I}{\partial t} = F_2(I(x,t))$, they can be composited as $\frac{\partial I}{\partial t} = \alpha F_1 + F_2, \quad \alpha \in R^+$, if F_1 and F_2 are two Euler-Lagrange operators having the minimum energy E_1 and E_2, then the minimum energy flow of the composition plan is $E_1 + \alpha E_2$. The synthesis of models has the practical uses: the image smoothing and boundary keeping, this application can make the program of removing noises and keep boundary at the same time possible (Shah, 1996; Chen and Bose, 2001; Wu and Ruan, 2006; Peng et al., 2006).

Some classical methods like Gaussian filtration, median filtration, corrosion swelling and so on get a brand-new explanation in the unity framework of PDE. Taking removing noises for example, we have proved that wavelet technology is the optimal regularization problem of Sobolev space, on the other hand, by Gibbs formula statistically. Bayes procedure can be associated with the variational method based on regularization. Partial differential equations have widely used in image processing, so the study of the subject is of great significance. Firstly, it is good for the image processing research based on partial differential equations. On one hand, it makes the development in this field expand in its application field, so we can work out more various problems; on the other hand, with the development of this subject, people more and more deeply mine image and the essence of the image processing, they try to take advantage of strict mathematical theory to reform the existing image processing methods, which make the results of the image processing more ideal. Secondly, the image processing based on partial differencing equations promotes the development of the theory of partial differential equations at the same time which in the use of partial differential equations theory, it also injects new contents into the theory of partial differential equations. The study at it also has certain stimulative effect on other image processing methods and has a big effect on the final results of the image processing.

DISCUSSION AND ANALYSIS OF THE IMAGE ENHANCEMENT TECHNIQUES BASED ON THE PARTIAL DIFFERENTIAL EQUATION

The diffusion model

Many Smooth or fuzzy processes can be described with the partial differential equation. In the 1960s, Gabor noticed that the difference between the image f and the image f after the fuzzy process is proportional to the Laplacian operation of the image f; Witkin (1983) proved that, the linear diffusion of the image is equal to do the convolution with the image using Gaussian filter in the traditional image processing, while, the reverse diffusion is the same as the removing convolution process. To achieve the purpose of removing noises and keep the edge information of the image at the same time, Perona and Malik (1990) put forward the famous P-M equation, which smooth the noise with the nonlinear opposite diffusion. the P-M equation model is as follow:

$$\begin{cases} \frac{\partial I(x,y,t)}{\partial t} = div[g(|\nabla I|)\nabla I] \\ I(x,y,0) = I_0(x,y) \end{cases} \quad (26)$$

P-M equation uses the Gaussian function based on the Lorentzian norm:

$$g(s) = \exp[-(\frac{s}{K})^2] \quad (27)$$

or the Cauchy function

$$g(s) = \frac{1}{1+(\frac{s}{K})^2} \quad (28)$$

As the diffusion coefficient function, K is the gradient threshold. so, the P-M equation is the anisotropic diffusion process, its diffusion coefficient $g(|\nabla I|)$ is relying on the local features of the image. in the flat field of the image, the gradient amplitude is lower, which fulfils $|\nabla I| > K$, so the diffusion coefficient $g(|\nabla I|)$ is higher, at this time the P-M model plays a role in smoothing the noise; in the area near the edge, the gradient amplitude is higher, which fulfils $|\nabla I| > K$, so the diffusion coefficient $g(|\nabla I|)$ is lower to keep the edge. The filtering effect of the P-M model can be seen in Figure 8.

What we can know from the Figure 8 is that, the P-M model can filter out most of the noise in the image, and can keep most of the detail information of the image. However, the P-M model has the disadvantages itself, on the one hand, in the noisy point, the gradient of image $|\nabla I|$ may be very higher, the P-M model is more likely to keep the noise as the edge; on the other hand, the anisotropic diffusion equation in (26 to 28) is a sick equation in mathematics, which cannot ensure the uniqueness and stability of the solution. Catte et al. (1992), etc. prove that the diffusion equation itself has the pathological properties they put forward the regularization P-M equation.

$$\begin{cases} \frac{\partial I(x,y,t)}{\partial t} = div[g(|\nabla I_\sigma|)\nabla I] \\ I(x,y,0) = I_0(x,y) \end{cases} \quad (29)$$

Where, $I_\sigma(x,y,t) = G_\sigma * I(x,y,t)\sigma$, which show the

(c) (d)

Figure 8. Filtering effect of P-M Model c) RGB noise image (d) the image (c) after the P-M filter.

noise image after the convolution filter with the Gaussian function whose variance is σ. Because the regularization model can control the Gaussian noise, it solves the problem that the model cannot distinguish the edge and edge to a certain extent, and guarantee the existence and uniqueness of the solution of equation. However, the regularization model is easy to dim the information of the edge and texture in the image, the parameters of Gaussian kernel is hard to determine beforehand.

The diffusion coefficient in the P-M model usually convergences more slowly, it leads to the faster diffusion speed. The diffusion coefficient still keeps the feature of keeping subtle diffusion in the area where the gradient is high, so it may remove the important detail information which is not too obvious and weakens high contrast area (In Figure 8(d) the texture information of the hat is obviously blurred). According to the relation between the robust estimation model and anisotropic diffusion coefficients, Black et al. (1998), conduct the robustness anisotropic diffusion (RAD) model, it uses the diffusion coefficient based on the Turkey Error norm and the robust estimation operator:

$$g(x,k)=\begin{cases}[1-(\frac{x}{K})^2]^2 & if\ |x|\le K\\ 0 & otherwise\end{cases} \quad (30)$$

Where, K is the robustness threshold value scale. compared with the Lorentzian error norm in the P-M model, the Turkey Error norm can 'estimate' more quality image information of the edge and the detail, it can also see that how the error model 'end' the diffusion behavior. but the robustness anisotropic diffusion model still can not remove the effect of the salt and pepper noise. in allusion to the problem of the small target, low contrast, blur edge and testing difficultly, Wang Yanhua put forward the improved anisotropic diffusion algorithm to enhancing the small target, its expression is as followed:

$$\frac{\partial I}{\partial t}=div[(g|\nabla I|-v(|\nabla I|))\nabla I] \quad (31)$$

Where $g(|\nabla I|)$ means the diffusion coefficient, $v(|\nabla I|)$ is sharpen coefficient, w is the smooth and sharpen weighted factor. w is bigger, and the sharpen effect of the algorithm is larger; w is smaller, and the effect of smoothing the noise of the algorithm is larger. this algorithm couples the regional smoothing and the edge sharpening, whose essence is taking the coupling norm. this algorithm can primely combine the noise smoothing with the edge sharpening in the condition of low noise environment, which can be seen in Figure 9. Figure 9(a) is a degraded image due to noise; Figure 9(b) is the result of filtering Figure 9(a) with the Wang method. In Figure 9(b), the noise is filtered basically, and the rear window of the white minibus is sharpened, thus the image quality is lowered more.

TV model

As the quadratic norm L^2 especially emphasizes the 'punishment' to great grads, which is inimical to inherent characteristics of images, while with respect to the quadratic norm L^2, L^1 has stronger power to protect the edges so that it gives people clearer subjective feeling. Rudin et al. (1992), proposed the total variation (TV) Model based on linear norm, the energy functional of which is:

$$E(I)=\alpha\iint_{\Omega}|\nabla I|\,d\Omega+\iint_{\Omega}(h_\sigma * I-I_0)^2\,d\Omega \quad (32)$$

(a) (b)

Figure 9. The designsketch of enhancing image with the Wang model (a) noise image, (b) the filter image with the Wang method.

(a) (b)

Figure 10. Filtering effective figures of TV model (a) image with noise, (b) filtered image with TV algorithm.

The second term in the right of formula (32) is fidelity term, used to descript the appropinquity degree of filtered image and the initial one. TV model has only solution, the corresponding grads katabatic drainage is:

$$\frac{\partial I}{\partial t} = h_\sigma * (f - h_\sigma * I) + \alpha div(\frac{\nabla I}{|\nabla I|}) \tag{33}$$

Where,

$div(\frac{\nabla I}{|\nabla I|}) = \frac{I_{xx}I_{yy}^2 - 2I_xI_yI_{xy} + I_{yy}I_{xx}^2}{|\nabla I|^3}$ 。Filtering effective images of TV model are shown in Figure 10. It can be seen that TV model is a kind of image restoration models. The ‘fragmentation constant’ effect is always in the stable solution of TV model, which does not completely suit to the morphology theory of image processing.

Shock wave filter

Because of the usual diffusion model is a kind of algorithm approximately to keep edge smooth, it can’t de-noising and enhance image at the same time. So Osher and Rudin presented a Shock Wave Filter, mathematical model of which is:

$$\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = -|\nabla I| F(\Delta I) \tag{34}$$

When

$\mathrm{F}(s) = sign(s)$, $\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = -|\nabla I| sign(\Delta I)$.

The enhancement action of image signals of shock wave filter has the following characteristics:

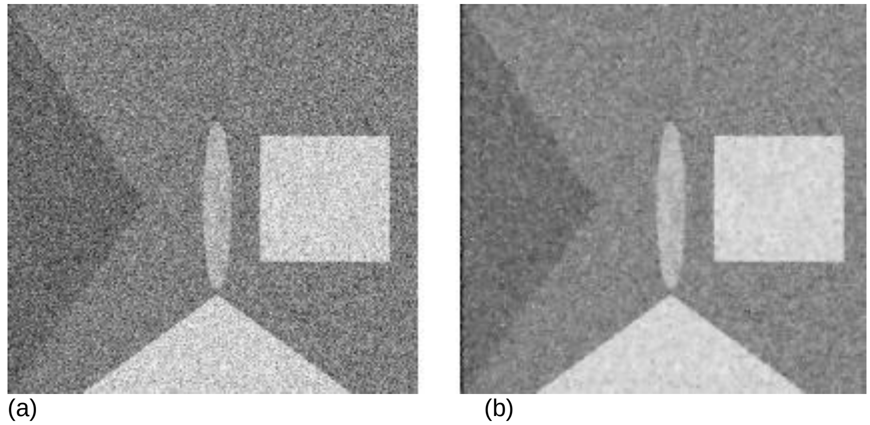


Figure 11. Filtering effect figures of Sum method (a) image with noise (b) restored image of Sum method.

1. The enhancement of signals produces on the zero-crossing point of second derivative, in another word, in the edge of the image.
2. Weak solution of Shock Wave Filter is fragmentation constant, discontinuous in the inflection point;
3. Enhancement process of Shock Wave Filter approximates the de-convolution process.

The shortage of Shock Wave Filter is that it is very sensitive to noise. In theory, in the continuous domain, any white noise joining in signals may join countless inflection point, damaging signal enhancement thoroughly.

Improved P-M model

In Sum and Cheung (2007), Anthony K. W. Sum thought fidelity could be used to strengthen the robustness in the iterative process of TV model, and when used in P-M equations, it was able to reduce the loss of image detail information affected by iteration. He has improved the P-M model:

$$\frac{\partial I}{\partial t} = div[g(|\nabla I|)|\nabla I|] + \frac{1}{2}|I - I_0| \qquad (35)$$

In the above formula, we notice that at any given time t, the second term on the right of equation is nonnegative constant, making the whole gray value of the image increase, and be easy to produce a saturation point (Figure 11), then cannot guarantee the robust in the restoration process very well, and can't sharpen image edges.

So, Zhou and Liu, (2011), gave the corresponding improvement. Firstly on account of the discomfort of original P-M model, he did regularization processing, that is: $I_\sigma = G_\sigma * I(x, y, t)$. Then, minimize the following energy functional:

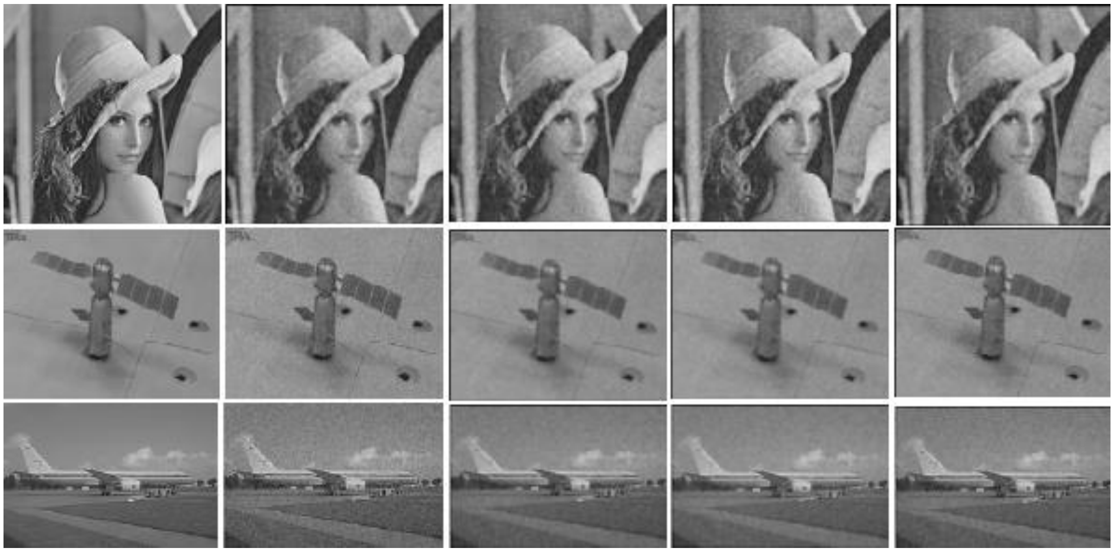
$$E(I) = \int_\Omega \rho(|\nabla I_\sigma|)dxdy + \lambda \int_\Omega [I - I_0]dxdy \qquad (36)$$

Where $\rho(|\nabla I_\sigma|) = g(|\nabla I_\sigma|)|\nabla I_\sigma|$. The first term on the right of equation (36) is used to do anisotropic diffusion to smooth image, the second term is fidelity, to ensure that image detail information in spread and restoration processes will not be lost. So, the grads drop flow is:

$$\frac{\partial I}{\partial t} = div[g(|\nabla I|)|\nabla I|] - \lambda(I - I_0) \qquad (37)$$

Comparison of various algorithms

In order to verify the performance of each algorithm, the experiment about a set of benchmark images was taken. First, by adding a certain degree of Gaussian noise to noise-free image, then the original PM model, Anthony KWSum improve the PM model (Sum method) and Zhou methods were used to filter separately, and we also compared the filtering performance. To enhance the comparability, different iterative algorithms use the same number of iterations (80 times). Figure 12 is the filtering results of the algorithms. Figure 12 (a) and Figure 12 (b) are the original image and the noise image, Figure 12 (c), Figure 12 (d), Figure 12 (e) are the results of Figure 12 (b) using the PM model, Sum method and Zhou methods. Lena image, the Shenzhou images and plane image plus noise variance were 50, 25, and 30 respectively. Peak signal-to-noise ratio (PSNR) is a criterion to measure the performance of different methods of filtering. PSNR results of each image are shown in Table 1. It can be seen from Table 1, the PM model increased fidelity avoid



(a) (b) (c) (d) (e)

Figure 12. Comparison of all the methods (a) initial image (b) noise image (c)P-M method (d)Sum method (e)Zhou method.

Table 1. PSNR comparison of different methods.

Image name (bmp)	Origin image noise	P-M model	Sum model	Improved P-M model
Lena	17.38189	29.40671	27.93841	29.56522
Shenzhou	23.03554	35.54492	34.16889	35.99998
Plane	21.47833	30.42956	28.26197	30.92465

the shortage of the original PM model take strong noise as the edge and filters, so the filtering performance has improved to some extent.

CONCLUSION

In this work, we review the applications of partial differential equations for image enhancement. The motivation of this review is two-folds: firstly, image enhancement is an important preprocessing technique in image processing area. It is worthy to do research for it. Secondly, partial differential equations are relatively new techniques which are superior to traditional methods. At the end of this review, a comprehensive discussion is presented. We believe our review can contribute to the researchers who are interested to apply partial differential equations for image enhancement.

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