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Contractions on Hilbert space with the smallest local unitary spectra

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Let $T$ be a contraction on a complex Hilbert space $H$, let $\sigma_T(x)$ be the local spectrum of $T$ at $x \in H$, and let $\sigma_T(x) \cap \Gamma$ be the local unitary spectrum of $T$ at $x$; $\Gamma = \{ z \in C : |z| = 1 \}$. We show that if $\sigma_T(x) \cap \Gamma$ is of Lebesgue measure zero, then $\lim_{n \to \infty} \| T^n x \|$ exists and is equal to $\inf_{n \geq 0} \| T^n x \|$. A contraction $T \in B(H)$ is said to be a $C_1$-contraction if $\inf_{n \geq 0} \| T^n x \| > 0$, for every $x \in H \setminus \{ 0 \}$. For an arbitrary $T \in B(H)$, we denote as usual by $\sigma(T)$ the spectrum of $T$ and by $R(z, T) = (zI - T)^{-1}$ the resolvent of $T$. In this paper, we will $D = \{ z \in C : |z| < 1 \}$, $\Gamma = \{ z \in C : |z| = 1 \}$, and $A(D)$ denotes for the disc-algebra. If $T \in B(H)$ is a contraction, then the spectrum of $T$ lies in $\overline{D}$. The set $\sigma_T(\Gamma) = \sigma(T) \cap \Gamma$ is called the unitary spectrum of $T$.

For an arbitrary $T \in B(H)$ and any $x \in H$, we define $\rho_T(x)$ to be the set of all $\lambda \in C$ for which there exists a neighborhood $U_\lambda$ of $\lambda$ with $u(\lambda)$ analytic on $U_\lambda$ having values in $H$, such that $(zI - T)u(\lambda) = x$ on $U_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of $T$. By definition, the local spectrum of $T$ at $x$, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$. If $T \in B(H)$ is a contraction and $x \in H$, then the set $\sigma_T(x) \cap \Gamma$ will be called the local unitary spectrum of $T$ at $x$. Consider the case where $U$ is a unitary operator on $H$. Let $E(\cdot)$ be the spectral measure of $U$.

For given $x \in H$, let $\mu_\lambda$ be the vector-measure defined on the Borel subsets of $\Gamma$ by $\mu_\lambda(\Delta) = E(\Delta)x$. One can easily see that $\sigma_T(x) = \sup \{ \mu_\lambda \}$.

Generally, the local spectrum of an operator $T \in B(H)$ may be very "small" with respect to its usual spectrum. Indeed, let $\sigma$ be a "small" part of $\sigma(T)$ such that both $\sigma$ and $\sigma(T) \setminus \sigma$ are closed sets. Let $P_\sigma$ be the spectral projection associated with $\sigma$ and let $H_\sigma = P_\sigma H$. We know that $H_\sigma$ is a (closed) $T$-invariant subspace and $\sigma(T |_{H_\sigma}) = \sigma$. Now, we can...
readily verify that \( \sigma_\tau(x) \subset \sigma \) for every \( x \in H_\sigma \).

Let \( T \in \mathcal{B}(H) \) be a contraction and let \( x \in H \). We can see that \( \xi \in \rho_\tau(x) \cap \Gamma \) if and only if \( R(z,T)x (|z| > 1) \)
admits an analytic extension to some neighborhood of \( \xi \).
It follows that if \( \xi \in \rho_\tau(x) \cap \Gamma \) for every \( x \in H \), then \( \xi \in \rho(T) \). Hence, we have
\[
\sigma_\tau(T) = \bigcup_{x \in H} (\sigma_T(x) \cap \Gamma).
\]

Note that there exist a contraction \( T \in \mathcal{B}(H) \) and \( x \in H \)
such that \( \sigma_\tau(x) \cap \Gamma = \emptyset \), but \( \sigma_\tau(T) = \Gamma \). Indeed, let \( H^2(K) \)
be the Hardy space of \( K \)-valued analytic functions on \( D \) and let \( S \) be the unilateral shift operator
on \( H^2(K) \); \( (S_\lambda^*_K f)(\lambda) = \lambda f(\lambda) \). Its adjoint, the backward shift, is given by:
\[
(S^* f)(\lambda) = \frac{f(\lambda) - f(0)}{\lambda}, \quad f \in H^2(K).
\]

It is easy to verify that for every \( f \in H^2(K) \) and \( z \in C \)
with \(|z| > 1\),
\[
(z I - S^*) f(\lambda) = \frac{z^{-1} f(z^{-1}) - \lambda f(\lambda)}{1 - \lambda z}.
\]

Hence \( \sigma_{S^*} f(\lambda) \cap \Gamma \) consists of all \( \xi \in \Gamma \) such that \( f \)
has no analytic extension to a neighborhood of \( \xi \). It
follows that if \( f \) admits an analytic extension across the
unit circle, then \( \sigma_{S^*} f(\lambda) \cap \Gamma = \emptyset \). However, \( \sigma_{S^*} (S_\lambda^* K) = \Gamma \).

Note also that for every nonzero \( f \in H^2(K) \),
\( \sigma_{S^*} f(\lambda) = \overline{D} \).

Recall that a contraction \( T \in \mathcal{B}(H) \) is said to be
completely non-unitary if it has no proper reducing
subspace on which it acts as a unitary operator. As is
well known (Nikolski, 1986), if \( T \in \mathcal{B}(H) \) is a contraction,
then there exists a canonical decomposition (with respect
to \( T \)) of the space \( H \) into two \( T \)-invariant subspaces:
\( H = K \oplus L \) such that: i) \( K \) and \( L \) reduce \( T \); ii) \( S := T|_K \) is a completely non-unitary contraction; iii) \( Z \)
\( U := T|_L \) is a unitary operator, where the subspace \( L \)
is defined by:

\[
L = \{ x \in H : \| T^n x \| = \| x \| \quad n \in \mathbb{R} \}.
\]

The operator \( S \) (respectively \( U \)) will be called
completely non-unitary (unitary) part of \( T \). According to
this decomposition, every \( x \in H \) can be written as
\( x = x^u_T + x^u_T \). The vector \( x^u_T \) (respectively \( x^u_T \)) will be called
completely non-unitary (unitary) part of \( x \).

It can be seen that if \( T \in \mathcal{B}(H) \), \( \lim_{n \to \infty} \| T^n x \| = 0 \) if
and only if \( \sigma_{S^*} f(\lambda) = \emptyset \). Generally, the asymptotic
behavior of the sequence \( \{ T^n \} \) is frequently related to
unitary spectrum of the underlying operator. This is well
illustrated by the following classical result of Nagy-Foias
(Nagy and Foias, 1966). If the unitary spectrum of a
completely non-unitary contraction \( T \in \mathcal{B}(H) \) has
Lebesgue measure zero, then \( \lim_{n \to \infty} \| T^n x \| = 0 \) for all
\( x \in H \) (the proof based on unitary dilation arguments). In
this paper, we address the problem whether local and
quantitative versions of the Nagy-Foias Theorem hold.
For related results see (Allan and Ransford, 1989; Batty
et al., 1998; Mustafayev, 2010).

RESULTS

The following theorem is the main result of this paper.

Theorem 1

Let \( T \in \mathcal{B}(H) \) be a contraction and let \( x \in H \) be such
that \( \sigma_\tau(x) \cap \Gamma \) is of Lebesgue measure zero. Then, we have:
\[
\lim_{n \to \infty} \| T^n x \| = \| x^u_T \|,
\]
where \( x^u_T \) is the unitary part of \( x \) in the canonical
decomposition of the space \( H \) with respect to \( T \).

For the proof, we need the following lemmas.

Lemma 1

Let \( T \in \mathcal{B}(H) \) be a contraction, let \( E \) be a \( T \)-invariant
subspace, and let \( \pi : H \to H/E \) be the canonical
mapping. Then, the following assertions hold:

a) \( \sigma_{T|_E}(x) \cap \Gamma = \sigma_{T}(x) \cap \Gamma \quad \) for every \( x \in E \);
b) \( \sigma_T(x^c_T) \cap \Gamma \subset \sigma_T(x) \cap \Gamma \), where \( x^c_T \) is the completely non-unitary part of \( x \in H \) in the canonical decomposition of \( H \).

c) \( \sigma_T(\pi x) \subset \sigma_T(x) \) for every \( x \in H \), where \( \tilde{T} \) is the induced mapping; \( \tilde{T} \circ \pi = \pi \circ T \).

**Proof**

a) Let \( x \in E \). It is easy to see that \( \sigma_T(x) \subset \sigma_{T \mid E}(x) \), and so

\[
\sigma_T(x) \cap \Gamma \subset \sigma_{T \mid E}(x) \cap \Gamma .
\]

For the reverse inclusion, let an arbitrary \( \xi \in \rho_T(x) \cap \Gamma \) be given. Then, there exists a neighborhood \( U_\xi \) of \( \xi \) with \( u(z) \) analytic on \( U_\xi \) having values in \( H \), such that \( (zI-T)u(z) = x \) on \( U_\xi \). Since

\[
u(z) = R(z; T)x = \sum_{n=0}^{\infty}z^{-n-1}T^n x \in E,
\]

for all \( z \in U_\xi \) with \( |z| > 1 \), we have \( \nu(z) = 0 \) for all \( z \in U_\xi \) with \( |z| > 1 \). By uniqueness theorem, \( \nu(z) = 0 \) for all \( z \in U_\xi \), so that \( u(z) \in E \). Thus, we have \( (zI-T)u(z) = x \) on \( U_\xi \). This shows that \( \xi \in \rho_T(x) \cap \Gamma \).

b) Let \( H = K \Theta L \) be the canonical decomposition of \( H \) and let \( S = T \mid K \). It follows from a) that

\[
\sigma_T(x^c_T) \cap \Gamma = \sigma_S(x^c_T) \cap \Gamma .
\]

It remains to show that \( \sigma_T(x^c_T) \subset \sigma_T(x) \). If \( \lambda \in \rho_T(x) \), then there exists a neighborhood \( U_{\lambda_0} \) of \( \lambda \) with \( u(z) \) analytic on \( U_{\lambda_0} \), having values in \( H \), such that \( (zI-T)u(z) = x \) on \( U_{\lambda_0} \). Let \( P \) be the orthogonal projection from \( H \) onto \( K \). Then, we have \( (zP-P\tilde{T})u(z) = x^c_T \). Since \( PT = TP = SP \), we obtain \( (zI-S)Pu(z) = x^c_T \). This shows that \( \lambda \in \rho_S(x^c_T) \).

c) If \( \lambda \in \rho_T(x) \), then there exists a neighborhood \( U_{\lambda_0} \) of \( \lambda \) with \( u(z) \) analytic on \( U_{\lambda_0} \), having values in \( H \), such that \( (zI-T)u(z) = x \) on \( U_{\lambda_0} \). It follows that \( (zI-T)u(z) = x \) on \( U_{\lambda_0} \). Consequently, we have \( (zI-T)u(z) = \pi x \) on \( U_{\lambda_0} \). This shows that \( \lambda \in \rho_T(\pi x) \).

Recall that \( V \in B(H) \) is called an isometry if \( \|Vz\| = \|z\| \) for all \( z \in H \). It is well known that if \( V \) is a non-unitary isometry, then \( \sigma(V) = \overline{D} \). Recall also that \( x \in H \) is a cyclic vector of \( T \in B(H) \), if the set \( \{e^{\imath \theta} : n = 0,1,2,...\} \) spans the whole space \( H \).

**Lemma 2**

If \( V \in B(H) \) is an isometry and \( x \in H \) is a cyclic vector of \( V \), then

\[
\sigma_{\|V\|}(V) = \sigma_{\|V\|}(x) \cap \Gamma .
\]

**Proof**

Assume that \( VH = H \), that is, \( V \) is a unitary operator. We must show that \( \sigma(V) = \sigma_{\|V\|}(x) \). By Spectral Theorem, there exists a positive measure \( \mu \) on \( \Gamma \) such that the operator \( M \) on \( L^2(\Gamma, \mu) \) defined by \( Mf = e^{\imath \theta}f \) is unitary equivalent to \( V \). Let \( \chi_\Delta \) denotes the characteristic function of any Borel subset \( \Delta \) of \( \Gamma \) and let 1 be the constant one function on \( \Gamma \). Then, we have \( \sigma(V) = \text{supp}(\mu) \) and \( \sigma_{\|V\|}(x) = \text{supp}(\nu) \), where \( \nu \) is a vector measure on \( \Gamma \) that is defined by \( \nu(\Delta) = \chi_\Delta 1 \).

Since \( \|\nu(\Delta)\| = \|\nu(\Delta)\| \), we have \( \text{supp}(\mu) = \text{supp}(\nu) \) and so, \( \sigma(V) = \sigma_{\|V\|}(x) \).

Assume that \( VH \neq H \). In this case \( \sigma(V) = \overline{D} \), so that \( \sigma_{\|V\|}(x) = \overline{D} \). Let \( K = H \Theta VH \). By Wold's Decomposition Theorem (Nagy and Foias, 1966), there exists a decomposition \( H = H_0 \oplus H_1 \) such that \( H_0 \) and \( H_1 \) reduce \( V \). \( V_0 = V \mid_{H_0} \) is unitary and \( V_1 = V \mid_{H_1} \) is unitary equivalent to the unilateral shift operator \( S \) on \( H_1^2 \). Let \( x = x_0 + x_1 \), where \( x_0 \in H_0 \) and \( x_1 \in H_1 \). Since \( x_1 \) is a cyclic vector of \( V_1 \), \( x_1 \neq 0 \), so that \( \sigma_{\|V_1\|}(x_1) = \overline{D} \). It remains to show that \( \sigma_{\|V_1\|}(x_1) \subset \sigma_{\|V\|}(x) \). If \( \xi \in \rho_{\|V\|}(x) \),
then there exists a neighborhood \( U_{\xi} \) of \( \xi \) with \( u(z) \) analytic on \( U_{\xi} \) having values in \( H \), such that
\[
(zI - V)u(z) = x
\]
on \( U_{\xi} \). Let \( P \) be the orthogonal projection \( H \) onto \( H_1 \). Then, we have \( (zP - PV)u(z) = x \). Since \( PV = VP \), we obtain \( (zI - V)P\) \( u(z) = x \). This shows that \( \xi \in \rho_{\chi}(x) \).

**Lemma 3**

Let \( T \in B(H) \) be a \( C_1 \)-contraction and let \( x \in H \), if 
\[
f \in A(D)
\]
vanishes on \( \sigma_f(x) \cap \Gamma \), then \( f(T)x = 0 \).

**Proof**

By Nagy-Foias Theorem (Nagy and Foias, 1966), there exist an isometry \( V \) and a quasi-affinity \( X \) on \( H \) intertwining \( T \) and \( V ; XT = VX \). First, we claim that
\[
\sigma_f(x) \subset \sigma_f(x).
\]

(1)

If \( \lambda \in \rho_f(x) \), then there exists a neighborhood \( U_{\lambda} \) of \( \lambda \) with \( u(z) \) analytic on \( U \lambda \) having values in \( H \), such that \( (zI - T)u(z) = x \) on \( U \lambda \). It follows that
\[
(zX - XT)u(z) = Xx \quad (z \in U \lambda)
\]
Consequently, we have \( (zI - V)Xu(z) = Xx \) on \( U \lambda \). This shows that \( \lambda \in \rho_f(Xx) \).

Set
\[
K = \text{span}\{V^n Xx : n = 0, 1, 2, \ldots\},
\]
and
\[
L = \text{span}\{T^n x : n = 0, 1, 2, \ldots\}.
\]

Since \( V^n Xx = XT^n x \ (n \in \mathbb{N}) \), the operator \( X|_k \) is a quasi-affinity from \( L \) to \( K \) and
\[
(V|_k)x = (X|_k)f|_x.
\]

Also, since \( Xx \) is a cyclic vector of \( V|_k \), by Lemma 2
\[
\sigma_f(V|_k) = \sigma_f(x) \cap \Gamma.
\]

On the other hand, taking into account Lemma 1 a) and (1), we can write
\[
\sigma_f(x) \cap \Gamma = \sigma_f(x) \cap \Gamma \subset \sigma_f(x) \cap \Gamma.
\]

Hence, we have
\[
\sigma_f(V|_k) \subset \sigma_f(x) \cap \Gamma.
\]

(3)

We see that under the hypotheses of the Lemma, the Lebesgue measure of \( \sigma_f(x) \cap \Gamma \) is necessarily zero. It follows from (3) that \( \sigma_f(V|_k) \) has Lebesgue measure zero and therefore, \( V|_k \) is a unitary operator. Since \( f \in A(D) \) vanishes on \( \sigma_f(x) \cap \Gamma \), it follows that \( f \) vanishes on \( \sigma(V|_k) \), and so \( f(V)\) \( K = \{0\} \). Using now the identity (2), we can write \( Xf(T)K = \{0\} \). In particular, we have \( Xf(T)x = 0 \). Since \( X \) has zero kernel, we obtain that \( f(T)x = 0 \).

**Lemma 4**

Let \( T \in B(H) \) be a \( C_1 \)-contraction and let \( x \in H \) such that \( \sigma_f(x) \cap \Gamma \) is of Lebesgue measure zero, then
\[
\|T^n x\| = \|T^{*n} x\| = \|x\| \quad \text{for all } n \in \mathbb{R}.
\]

**Proof**

Set \( M = \sigma_f(x) \cap \Gamma \). Let us define a mapping \( h : C(M) \rightarrow H \) as a following way: Take a function \( f \in C(M) \). By Rudin-Carleson Theorem (Beauzamy, 1988), there exists a function \( \overline{f} \in A(D) \) such that \( \overline{f}(\xi) = f(\xi) \) for all \( \xi \in M \), and
\[
\|\overline{f}\|_{L^\infty} = \sup_{\xi \in M} |f(\xi)|.
\]

(4)

Set \( h(f) = \overline{f}(T)x \). By Lemma 3, \( h \) is a well-defined linear mapping. On the other hand, it follows from von Neumann inequality and the identity (4), the mapping \( h \) is bounded. Note also that if \( f, g \in C(M) \), then
\[
h(fg) = \overline{f}(T)\overline{g}(T)x. \]
and \( \tilde{f}_1 \) on \( M \) is defined by \( f_{-1}(\xi) = \xi^{-1} \), \( f_0(\xi) = 1 \) and \( f_1(\xi) = \xi \). Then, we have
\[
x = h(f_0) = h(f_{-1} f_1) = \tilde{f}_{-1}(T) \tilde{f}_1(T)x.
\]

Set \( S = \tilde{f}_{-1}(T) \). Then, \( S \) is a contraction on \( H \) which commutes with \( T \). Since \( f_1(T) = T \), we have \( STx = x \) so that \( ST^n x = T^{-n} x \) for all \( n \in \mathbb{R} \). It follows that
\[
||T^{-n} x|| = ||ST^n x|| \leq ||T^n x|| \leq ||T^{-n} x||.
\]

Thus, \( ||T^n x|| = ||x|| \) for all \( n \in \mathbb{R} \). We know (Nagy and Foias, 1966) that if \( T \) is an arbitrary contraction and \( \xi \) is an eigenvector of \( T \) for the eigenvalue \( \lambda = 1 \), then \( \xi \) is also an eigenvector of \( T^* \) for the eigenvalue \( \lambda = 1 \). Since \( ST \) is a contraction and \( STx = x \), we have \( S T^* x = x \). It follows that \( ||T^* x|| = ||x|| \) for all \( n \in \mathbb{R} \).

We are now able to prove the Theorem 1

**Proof of Theorem 1**

Let \( H = K \oplus L \) be the canonical decomposition of \( H \) and let \( S = T | _K \) be the completely non-unitary part of \( T \).

Let \( x = x_T^v + x_T^u \), where \( x_T^v \) is the completely non-unitary and \( x_T^u \) is the unitary part of \( x \). Let us show that
\[
\lim_{n \to \infty} ||T^n x_T^v|| = 0.
\]

For this reason, set
\[
K_0 = \{ x \in K : \lim_{n \to \infty} ||S^n x|| = 0 \}.
\]

Let \( \pi : K \to K/K_0 \) be the canonical mapping and let \( \tilde{S} : K/K_0 \to K/K_0 \) be the induced mapping; \( \tilde{S} \circ \pi = \pi \circ S \). First, we claim that \( \tilde{S} \) is a \( C_1 \)-contraction. For this, it is enough to show that for every \( x \in K \),
\[
\lim_{n \to \infty} ||\tilde{S}^n \pi x|| = ||S^n x||.
\]

Indeed, let
\[
\alpha = \lim_{n \to \infty} ||\tilde{S}^n \pi x|| = \lim_{n \to \infty} ||S^n x + K_0||.
\]

Then, we have \( \alpha \leq \lim_{n \to \infty} ||S^n x|| \). On the other hand, for an arbitrary \( \varepsilon > 0 \), there exist \( k \in \mathbb{R} \) and \( y \in K_0 \), such that \( ||S^n x - y|| \leq \alpha + \varepsilon \), which implies
\[
||S^n k x - S^n y|| \leq \alpha + \varepsilon,
\]
for all \( n \in \mathbb{R} \). It follows that
\[
||S^{n+k} x|| \leq ||S^{n+k} x - S^n y|| + ||S^n y|| \leq \alpha + \varepsilon + ||S^n y||.
\]

As \( n \to \infty \), we obtain \( \lim_{n \to \infty} ||S^n x|| \leq \alpha + \varepsilon \), so that
\[
\lim_{n \to \infty} ||S^n x|| = \alpha.
\]

Further, it follows from the identity \( \hat{S}^* \hat{S} = S^* S \) that \( \hat{S} \) is a completely non-unitary contraction. Using Lemma 1 c), a), and b), respectively; we have
\[
\sigma_{\hat{S}}(\pi \xi_T^v) \cap \Gamma \subseteq \sigma_{\hat{S}}(\pi \xi_T^u) \cap \Gamma = \sigma_{\hat{S}}(\pi \xi_T^v) \cap \Gamma \subseteq \sigma_{\hat{S}}(\pi \xi_T^v) \cap \Gamma.
\]

It follows that \( \sigma_{\hat{S}}(\pi \xi_T^v) \cap \Gamma \) has Lebesgue measure zero. Since \( \hat{S} \) is a completely non-unitary \( C_1 \)-contraction, by Lemma 4, \( \pi \xi_T^v = 0 \), and so
\[
\lim_{n \to \infty} ||T^n x_T^v|| = \lim_{n \to \infty} ||S^n x_T^v|| = 0.
\]

Also, since \( ||T^n x_T^u|| = ||x_T^u|| \) for all \( n \in \mathbb{R} \), we have that
\[
\lim_{n \to \infty} ||T^n x|| = \lim_{n \to \infty} ||T^n x_T^v + T^n x_T^u|| = \lim_{n \to \infty} ||T^n x_T^u|| = ||x_T^u||.
\]

**CONCLUSION**

It is easy to verify that if \( T \in B(H) \), then
\[
\lim_{n \to \infty} ||T^n x|| = 0 \quad \text{if and only if} \quad \sigma_{\pi}(T) = \emptyset.
\]
In general, the asymptotic behavior of the sequence \( \{ T^n \}_{n=1}^{\infty} \) is frequently related to unitary spectrum of the underlying operator. This is well illustrated by the classical result of Nagy-Foias (Nagy and Foias, 1966). If the unitary spectrum of a completely non-unitary contraction \( T \in B(H) \) has Lebesgue measure zero, then
\[
\lim_{n \to \infty} ||T^n x|| = 0 \quad \text{for all} \quad x \in H.
\]

In this note we show that if \( \sigma_{\pi}(x) \cap \Gamma \) of Lebesgue measure zero, then
\[
\lim_{n \to \infty} ||T^n x|| = ||x_T^v||.
\]

Consequently, local and quantitative version of the well known Nagy-Foias Theorem is proved.
REFERENCES


