Full Length Research Paper

Some affine connexions in a generalized structure manifold

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In this paper we have studied some affine connexions in a generalised structure manifold. Certain theorems are have also been proved, which are of great geometrical importance.

Key words: $C^\infty$-manifold, generalised structure, generalised metric structure, $F$-structure, $\pi$-structure, AMS Subject Classification Number: 53.

INTRODUCTION

We consider a differentiable manifold $V_n$ of differentiability class $C^\infty$ and of dimension n. Let there exist in $V_n$ a tensor field $F$ of the type $(1, 1)$, s linearly independent vector fields $U_i, i = 1, 2\ldots s$ and s linearly independent 1-forms $u^i$ such that for any arbitrary vector field $X$, we have

\[ \overline{X} = b^2 X + cu^i(X)U_i, \]

Where

\[ F(X) = \overline{X} \text{ and } b^2, c \text{ are constants} \]

Then the structure \{F, $u^i, U_i, p^i_j; i, j=1,2,\ldots, s$\} will be known as generalised structure and $V_n$ will be known as generalised structure manifold of order $s$ where $s < n$.

Agreement 1.1

All the equations which follow hold for arbitrary vector fields $X, Y, Z, \ldots$, etc.

Now replacing $X$ by $\overline{X}$ in (1.1), we get

\[ \overline{\overline{X}} = b^2 \overline{X} + cu^i(\overline{X})U_i \]

Operating $F$ in (1.1), we get

\[ \overline{\overline{X}} = b^2 \overline{X} + cu^i(X)U_i \]

Using (1.2) in above, we get

\[ \overline{\overline{X}} = b^2 \overline{X} + cu^i(X)p^i_jU_j \]

From (1.3) and (1.4), we have

\[ u^i(\overline{X}) = u^i(X)p^i_j \]

Further, operating $F$ in (1.2) and using (1.1) and (1.2), we get

\[ p^i_j = b^2\delta^i_j + cu^j(U_i) \]

Where

\[ p^i_j = (r-1)p^i_j \]

On generalised structure manifold $V_n$, let us introduce a metric tensor $g$ such that $\overline{\overline{F}}$ defined by

\[ \overline{\overline{F}}(X, Y) = g(\overline{X}, Y) \]

is skew-symmetric, then $V_n$ is
called generalised metric structure manifold.
We have on a generalised metric structure manifold
\[ g(\overline{X}, Y) + g(X, \overline{Y}) = 0. \]
Replacing \( Y \) by \( \overline{Y} \) in above equation and using (1.1), we obtain
\[ g(\overline{X}, \overline{Y}) + b^2 g(X, Y) + cu(X)u(Y) = 0 \quad (1.7) \]
Where
\[ u(X) = g(U, X) \quad (1.8) \]
Then \( V_n \) satisfying (1.7), (1.8) is called generalised metric structure manifold (Mishra, 1984).

Agreement 1.2: The generalised metric structure manifold will always be denoted by \( V_n \).

Definitions: (Boothby, 1975; Kobayasi and Nomizu, 1996)

Almost tangent metric manifold: A differentiable manifold \( M_n \) on which there exists a tensor field \( F \) of the type (1, 1) such that
\[ F^2 = 0 \quad (1.9) \]
is called an almost tangent manifold and \( \{F\} \) is called an almost tangent structure on \( M_n \).

On almost tangent manifold, let us introduce a metric \( g \) such that \( F \) defined by \( F(X, Y) = g(X, Y) \) is alternating. Then \( M_n \) is called an almost tangent metric manifold and structure \( \{F, g\} \) is called an almost tangent metric structure.

Almost Hermite manifold: A differentiable manifold \( M_n \) on which there exists a tensor field \( F \) of the type (1, 1) such that
\[ F^2 = -I_n \quad (1.10) \]
is called an almost complex manifold and \( \{F\} \) is called an almost complex structure.

An almost complex manifold endowed with an almost complex structure and a metric \( g \) such that
\[ g(\overline{X}, \overline{Y}) = g(X, Y) \quad (1.11) \]
is called an almost Hermite manifold and structure \( \{F, g\} \) is called an almost Hermite structure.

Metric \( \pi \)-structure manifold: (Mishra and Singh, 1975; Duggal, 1969). A differentiable manifold \( M_n \) on which there exists a tensor field \( F \) of the type (1, 1) such that
\[ F^2 = -\lambda^2 I_n \quad (1.12) \]
where \( \lambda \) is a non zero complex constant. Then \( \{F\} \) is called a \( \pi \)-structure or G-F structure and \( M_n \) is called \( \pi \)-structure manifold or G-F structure manifold.

On almost tangent manifold, let us introduce a metric \( g \) such that \( F \) defined by \( F(X, Y) = g(\overline{X}, Y) \) is alternating. Then \( \{F, g\} \) is called metric \( \pi \)-structure or H-structure and \( M_n \) is called metric \( \pi \)-structure manifold or H-structure manifold.

F-Structure Manifold: (Yano, 1963). Let \( M_n \) be an n dimensional differentiable manifold of class \( C^\infty \) and let there be a tensor field of the type (1, 1) and rank \( r \) \( (1 \leq r \leq n) \) everywhere such that
\[ F^3 + F = 0 \quad (1.13) \]
Then \( \{F\} \) is called an F-structure and \( M_n \) is called F-structure manifold.

Almost Grayan manifold: (Sasaki, 1960): If on an differentiable manifold \( M_n \) \( (n = 2m+1) \) of differentiability class \( C^{r+1} \), there exist a tensor field \( F \) of the type (1, 1), a 1-form \( u \) and a vector field \( U \), satisfying
\[ F^2 = -I_n + u \otimes U \quad (1.14) \]
and
\[ \overline{U} = 0 \quad (1.15) \]
Then \( M_n \) is called an almost contact manifold and the structure \( \{F, U, u\} \) is said to give an almost contact structure to \( M_n \).

On an almost contact manifold, let us introduce a metric \( g \) such that \( F \) defined by \( F(X, Y) = g(\overline{X}, Y) \) is skew symmetric. Then \( M_n \) is called an almost Grayan manifold and the structure \( \{F, g, U, u\} \) is called an almost Grayan structure. In this manifold it can be easily calculated
\[ g(\overline{X}, \overline{Y}) = g(X, Y) - u(X)u(Y) = 0 \quad (1.16) \]
Torsion tensor: A vector valued, skew-symmetric, bilinear function $S$ defined by

$$S(X,Y) \overset{\text{def}}{=} D_X Y - D_Y X - [X,Y]$$ (1.17)

is called torsion tensor of a connexion $D$ in a $C^\infty$ manifold $V_n$.

For the symmetric or torsion free connexion $D$, the torsion tensor vanishes.

Curvature tensor: The tensor $K$ of the type $(1,3)$ defined by

$$K(X,Y,Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$ (1.18)

is called the curvature tensor of the connexion $D$.

Remark 1.1
It may be noted that $V_n$ gives an almost tangent metric manifold, an almost Hermite manifold, a metric $\pi$-structure manifold, an almost Grayan manifold and $\{F, g^1, u^1, u^2, U_1, U_2\}$ structure manifold according as $(b^2 = 0, c = 0); (b^2 = -1, c = 0); (b^2 = \lambda^2, c = 0); (b^2 = -1, p^j = 0); (b^2 = -1, c = 1, p^j = 0)$ and $(b^2 = -1, c = 1, p^i + p^j = 0 ; i, j = 1, 2)$ respectively.

Affine Connexion $D$: Let us consider in $V_n$ an affine connexion $D$ satisfying (Duggal, 1971; Mishra, 1984)

$$(D_X F) Y = 0$$ (2.1a)

and we call it as F-connexion.

(2.1a) is equivalent to

$$D_X \bar{Y} = D_X Y$$ (2.1b)

Replacing $Y$ by $\bar{Y}$ and using (1.1), (2.1a) in above, we get

$$c_1 (U_i) (D_X U_i) + (D_X u^i)(Y) U_i = 0$$ (2.1c)

Theorem (2.1): In $V_n$, we have

$$cu^i (D_X U_i) U_j = -(b^2 \delta^i_j - (2) p^i_j) (D_X U_j)$$ (2.2)

Proof
Operating $u^i$ in (2.1)c and using (1.6), we get (2.2).

Putting $U_i$ for $Y$ in (2.1)c and using (1.6), we obtain (2.3).

Theorem 2.2
In $V_n$, we have

$$S(X,Y) + b^2 S(X,Y) + cu^i (S(X,Y)) U_i - S(X,Y) - S(X,Y) = -[X,Y] - b^i [X,Y] - cu^i ([X,Y]) U_i + [X,Y] + [X,Y]$$ (2.5)

Proof
From (2.1)b, we get

$$D_X \bar{Y} = D_X Y, \quad D_Y \bar{X} = D_Y X, \quad D_X \bar{Y} = D_X Y,$$

$$D_Y \bar{X} = D_Y X$$ (2.6)

Now in view of (1.1), we have

$$S(X,Y) + b^2 S(X,Y) + cu^i (S(X,Y)) U_i - S(X,Y) - S(X,Y) = S(X,Y) + S(X,Y) - S(X,Y) - S(X,Y)$$

Using (1.17) and (2.6) in right hand side of above, we get (2.5).

Now, we consider in $V_n$ a scalar valued bilinear function $\mu$, vector valued linear function $v$ and a 1-form $\sigma$ given by,

$$\mu(X,Y) \overset{\text{def}}{=} (D_i u^i)(\bar{X}) - (D_X u^i)(\bar{Y}) + (D_Y u^i)(X) - (D_X u^i)(Y)$$ (2.7)

$$v(X) \overset{\text{def}}{=} (D_U, F)(X) - (D_X F)(U_i) - D_X U_i$$ (2.8)

and

$$\sigma(X) \overset{\text{def}}{=} (D_X u^i)(U_i) - (D_U, u^i)(X)$$ (2.9)

$i, j = 1, 2, \ldots, s.$

Theorem (2.3)
In $V_n$, we have
\[(b^2 \delta^j - (2)p^j)\mu (X, Y) = c[u'(X)u'(D_x U_i) - u'(Y)u'(D_x U_i) - u'(\overline{X})(D_x u^j)U_i + u'(\overline{Y})(D_x u^j)U_i] \]

(2.10a)

\[(b^2 \delta^j - (2)p^j)\mu (X, Y) = -c[u'(X)u'(\sigma Y) - u'(Y)u'(\sigma Y) + u'(\overline{X})(D_x u^j)U_i - u'(\overline{Y})(D_x u^j)U_i] \]

(2.10b)

and

\[(b^2 \delta^j - (2)p^j)\mu (X, Y) = -(u'(X)\sigma(X) + (D_x u^j)\overline{X}) - u'(Y)\sigma(Y) + (D_x u^j)\overline{Y} - u'(X)(D_x u^j)U_i + u'(Y)(D_x u^j)U_i \]

(2.10c)

**Proof**

Replacing \(Y\) by \(\overline{Y}\) in (2.2), we get

\[cu'(\overline{Y})u'(D_x U_i) = -(b^2 \delta^j - (2)p^j)(D_x u^j)(Y) \]

(2.11)

Replacing \(X\) by \(\overline{X}\) in (2.2), we get

\[cu'(Y)u'(D_x U_i) = -(b^2 \delta^j - (2)p^j)(D_x u^j)(Y) \]

(2.12)

Further by using (2.11), (2.12) in (2.7), we get (2.10a). Using (2.1)a in (2.8), we get

\[u'(X) = -(D_x U_i) \]

(2.13)

Using (2.13) in (2.10)a, we get (2.10)b. Replacing \(X\) by \(\overline{X}\) in (2.9), we get

\[-u'(D_x U_i) = \sigma(\overline{X}) + (D_{U_i} u^j)(\overline{X}) \]

(2.14)

Using (2.14) in (2.10)a, we get(2.10)c.

**Theorem 2.4**

In \(V_n\), we have

\[K(X, Y, \overline{Z}) = b^2 K(X, Y, Z) + cu'(K(X, Y, Z)U_i) \]

(2.16a)

\[p^j u'(K(X, Y, \overline{Z})) = b^2 u'(K(X, Y, Z)) + (b^2 \delta^j - (2)p^j)u'(K(X, Y, Z)) \]

(2.16b)

and

\[b[K(\overline{X}, \overline{Y}, Z) + (X, Y, Z) + K(\overline{Z}, \overline{X}, Y)] \]

(2.16c)

**Proof**

Replacing \(Z\) by \(\overline{Z}\) in (1.18) and using (2.1)b, we get

\[K(X, Y, \overline{Z}) = \overline{K(X, Y, Z)} \]

(2.17)

Operating \(F\) in (2.17) and using (1.1), we obtain (2.16)a.

Operating \(u^j\) on both sides of (2.16)a and using (1.5) and (1.6), we get(2.16)b. Bianchi’s first identity of symmetric connexion \(D\) is given by

\[K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0 \]

(2.18)

Operating \(F\) in (2.18), we get

\[K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0 \]

(2.19)

Using (2.17) in (2.19), we get

\[K(X, Y, \overline{Z}) + K(Y, Z, X) + K(Z, X, \overline{Y}) = 0 \]

(2.20)

Replacing \(X\) by \(\overline{X}\), \(Y\) by \(\overline{Y}\) & \(Z\) by \(\overline{Z}\) in (2.20) and using (1.1), we get (2.16)c.

**Affine connexion \(\tilde{D}\):** Let us consider in \(V_n\) an affine connexion \(\tilde{D}\) satisfying

\[u'(Y)(\tilde{D}_X U_i) + (\tilde{D}_X u^j)(Y)U_i = 0 \]

(3.1)

**Theorem 3.1**

In \(V_n\), we have

\[u'(Y)[b^2(\tilde{D}_X U_i) + cu'(\tilde{D}_X U_i)U_i] + (\tilde{D}_X u^j)(Y)p^j u^i U_i = 0 \]

(3.2a)

\[\left(\overline{2}p^j - b^2 \delta^j\right) \text{ div } U_j = cu'(\tilde{D}_{U_i} U_i) \]

(3.2b)

Where

\[\text{div } X = (C_i^i \nabla X) \]

(3.3) and

\[\nabla X Y = (D_Y X) \]

(3.4)
Proof

Operating $F^2$ in (3.1) and using (1.1) and (1.2), we get (3.2)a. Now contracting (3.1) with respect to $X$ and using (3.3) and (3.4), we get

\[ u'(Y)d\text{iv}U_i + (D_{U_i} u')(Y) = 0 \quad (3.5) \]

Replacing $i$ by $j$, then $Y$ by $U_i$ in (3.3) and using (1.6), we get (3.2)b.

Theorem 3.2

In $V_n$, we have

\[ cu'(Y)u'(D_X U_i) + (2p' - b^2 \delta)'(D_X u')(Y) = 0 \quad (3.6)a \]

\[ (2p' - b^2 \delta)'(D_X u')(Y)u'(D_U U_j) = cu'(Y)u'(D_U U_j)(D_X u')(U_j) \quad (3.6)b \]

Proof

By operating $u'$ on (3.1) and using (1.6), we obtain (3.6)a.

Multiplying (3.2)c with $u'(D_U U_j)$, we get (3.6)d.

Affine connexion $D$

Let us consider in $V_n$ an affine connexion $D$ satisfying

\[ u'(Y)(D_X U_i) + (D_X u')(Y)U_i = 0 \quad (4.1)a \]

And

\[ (D_X F)(Y) + (D_Y F)(X) = 0 \quad (4.1)b \]

It may be noted that all the results of the section above hold for $D$. In addition we have the following results:

Theorem 4.1

In $V_n$, we have

\[ D_{U_j} Y + (D_{U_j} F)Y = b^2 (D_{U_j} Y) + cu'(D_{U_j} Y)U_i \quad (5.2)a \]

Proof

(5.1)b is equivalent to

\[ D_{U_i} Y + (D_{U_i} F)Y = b^2 (D_{U_i} Y) + cu'(D_{U_i} Y)U_i \quad (5.2)b \]
Using (1.1) in (5.3), we get (5.2)a. Replacing $X$ by $U_i$ in (5.3), we get

$$D_x Y + D_x Y = D_x Y + D_x Y \quad (5.3)$$

Replacing $X$ by $U_i$ in (5.2)b, we get

$$\ast (D_{U_i} Y - D_{U_i} Y) + p^i_j [D_{U_j} (b^2 Y + c u^i(Y)U_i)] = p^i_j (D_{U_j} Y) \quad (5.4)$$

From (5.4) and (5.5), we get

$$p^i_j (D_{U_i} F Y) + p^i_j [D_{U_i} (b^2 Y + c u^i(Y)U_i)] = p^i_j (D_{U_i} Y) \quad (5.6)$$

Using (1.1)a in (5.6), we get (5.1)b.

**Theorem 5.2**

In $V_n$, we have

$$\hat{D} Y - b^2 (D_x Y) + b^i (D_x Y) =$$

$$\hat{D} Y - b^2 (D_x Y) - b^i (D_x Y) U_i + c u^i (X) [b^j (D_V Y) + u^i (D_V Y) U_i] + (D_V Y) \quad (5.7)$$

**Proof**

Replacing $X$ by $\hat{X}$ in (5.3) and using (1.1), (5.1), we get (5.7).

**REFERENCES**


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