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Two-step two-point hybrid methods for general second order differential equations

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Two-step two-point hybrid numerical methods for direct solution of initial value problems of general second order differential equations are proposed in this study. Chebyshev polynomials without perturbation terms are used as basic function for the development of the methods in predictor-corrector mode. The collocation and interpolation equations are generated at both grid and off-grid points. The resulting methods are zero-stable, consistent and normalized. The main predictors, having the same order with the scheme, are developed for the implementation of the methods. Accuracy of a discrete scheme from the methods is tested with linear and non-linear problems. The results show a better performance over the existing methods.

Key words: Hybrid method, chebyshev polynomials, predictor-corrector mode, off-grid points, normalized.

AMS Subject Classification: 65L05, 65L06.

INTRODUCTION

In this paper, we shall consider a direct solution of general second order problem of the form

$$y'' = f(x, y, y'),\ y(x_0) = \delta_0,\ y'(x_0) = \delta_1 \qquad (1)$$

Several literatures have shown that this type of equations is conventionally reduced to systems of first order ordinary differential equations in attempting to solve them. It is also revealed in literature that some researchers have attempted the direct solution of (1) using linear multistep methods (Lambert, 1973; Brown, 1977; Awoyemi, 2003; Adesanya et al., 2008; Kayode, 2010). These authors independently proposed methods of various order of accuracies to proffer solution to problem (1) at only grid points.

A few authors, (Kayode, 2011; Yahaya and Badmus 2009; Majid et al., 2009; Alabi et al., 2008; Ehigie et al., 2010; Kayode and Adeyeye 2011) have introduced hybrid methods to solving problem (1) but with lower order of accuracies.

In this work, Chebyshev series was used as basic function in generating the interpolation and collocation equations for the development of continuous hybrid linear multistep method (CHLMM) for the direct solution of problem (1).

MATERIALS AND METHODS

In this work, we considered using a partial sum of Chebyshev series in the form.

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$$y(x)=\sum_{n=0}^{c+i} a_n T_n(x) \qquad (2)$$

as the basic function for the development of the method, where c and i are the number of collocation and interpolation points respectively; a_j's are the determinate parameters and $T_n(x)$ is the Chebyshev polynomial of first kind.

The differential system arising from Equation (2) is given as:

$$y''(x)=\sum_{n=0}^{c+i} a_n T_n''(x) \qquad (3)$$

Interpolating the basic function (2) at grid points $x_{n+i},\ i=0,1$ and collocating the differential system (3) at grid and off-grid points $x_{n+j},\ j=0(1)2,$ x_{n+r}, and x_{n+s}, $0<r<1$ and $k-1<s<k$ respectively, gave rise to a system of equations

$$\sum_{j=0}^{c+i} a_{n,j} T_{n,j}(x)=y_{n+i},\ i=0,1,2,... \qquad (4)$$

$$\sum_{j=0}^{c+i} a_{n,j} T''_{n,j}(x)=f_{n+i},\ i=0,1,2,... \qquad (5)$$

$$\sum_{j=0}^{c+i} a_{n,j} T''_{n,j}(x)=f_{n+r},\ \ 0<r<1 \qquad (6)$$

$$\sum_{j=0}^{c+i} a_{n,j} T''_{n,j}(x)=f_{n+s},\ \ k-1<s<k \qquad (7)$$

where

$f_{n+j}=f\left(x_{n+j},y_{n+j},y'_{n+j}\right);\ y_{n+i}\approx y\left(x_{n+i}\right);\ x_{n+i}=x_n+ih$, k is the stepnumber and h is the stepsize.

Determining a_j's from Equations (4) – (7) and substituting the values into Equation (2) yields the continuous hybrid method:

$$y_k(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h^2\left(\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\tau_1(x)f_{n+r}+\tau_2(x)f_{n+s}\right) \qquad (8)$$

Taking $t=\dfrac{x-x_{n+k-1}}{h}$, $k=2$, the continuous coefficients α_j, β_j, τ_1, τ_2 and their respective first derivatives are obtained as:

$\alpha_0(t)=-t$; $\alpha_1(t)=1+t$;

$$\beta_0(t)=G_1\begin{Bmatrix}2t^4-t^3\left(3s+3r-1\right)-t^2\left(-5rs-3s-3r+6\right)\\ -t\left(15rs-7s-7r+4\right)-7r+4+15rs-7s\end{Bmatrix}$$

$$\beta_1(t)=G_2\begin{Bmatrix}2t^4-t^3\left(3s+3r-4\right)-t^2\left(-5rs+2s+2r+4\right)\\ -t\left(5rs-12s-12r+16\right)+18r+18s-25rs-14\end{Bmatrix}$$

$$\beta_2(t)=G_3\begin{Bmatrix}2t^4-t^3\left(3s+3r-7\right)-t^2\left(-5rs+7s+7r-8\right)\\ -t\left(-5sr+3s+3r-2\right)-5rs+3s+3r-2\end{Bmatrix}$$

$$\tau_1(t)=G_4\begin{Bmatrix}2t^4-t^3\left(3s-1\right)-t^2\left(-3s+6\right)\\ -t\left(-7s+4\right)-7s+4\end{Bmatrix}$$

$$\tau_2(t)=G_5\begin{Bmatrix}2t^4+t^3\left(-3r+1\right)+t^2\left(3r-6\right)\\ +t\left(7r-4\right)-7r+4\end{Bmatrix} \qquad (9)$$

where

$$G_1=\frac{h^2}{120sr}t\left(t+1\right)$$

$$G_2=-\frac{h^2}{60(s-1)(r-1)}t\left(t+1\right)$$

$$G_3=\frac{h^2}{120(r-2)(s-2)}t\left(t+1\right)$$

$$G_4=-\frac{h^2}{60r(r-2)(s-r)(r-1)}t\left(t+1\right)$$

$$G_5=\frac{h^2}{60s(s-2)(s-1)(s-r)}t\left(t+1\right)$$

$\alpha_0'(t)=-\dfrac{1}{h}$; $\alpha_1'(t)=\dfrac{1}{h}$;

$$\beta_0'(t)=H_1\begin{Bmatrix}-12t^5+t^4\left(15s-15+15r\right)+t^3\left(-20rs+20\right)\\ +t^2\left(30rs-30s-30r+30\right)+7s+7r-15rs-4\end{Bmatrix}$$

$$\beta_1'(t)=H_1\begin{Bmatrix}-12t^5+t^4\left(15s+15r-30\right)+t^3\left(-20rs+20s+20r\right)\\ +t^2\left(-30s-30r+60\right)+t\left(60+60rs-60r-60s\right)\\ +25rs-18s-18r+14\end{Bmatrix}$$

$$\beta_2'(t)=H_3\begin{Bmatrix}-12t^5+t^4\left(15s+15r-45\right)+t^3\left(40s-20rs+40r-60\right)\\ +t^2\left(-30rs+30s+30r-30t\right)-3s-3r+2+5rs)\end{Bmatrix}$$

$$\tau_1'(t)=H_4\begin{Bmatrix}-12t^5+t^4\left(15s-15\right)+20t^3\\ +t^2\left(-30s+30\right)+7s-4\end{Bmatrix}$$

$$\tau_2'(t) = H_5 \begin{Bmatrix} 12t^5 + t^4(15-15r) - 20t^3 \\ +t^2(30r-30) - 7r + 4 \end{Bmatrix} \quad (10)$$

Where

$$H_1 = -\frac{h}{120rs}$$
$$H_2 = \frac{h}{60(s-1)(r-1)}$$
$$H_3 = \frac{h}{120r(r-2)(s-2)}$$
$$H_4 = \frac{h}{60r(r-2)(s-r)(r-1)}$$
$$H_5 = \frac{h}{60s(s-2)(s-1)(s-r)}$$

Evaluating Equation (8) at the last end grid point where $t = 1$ yields the discrete scheme

$$y_{n+2} = \alpha_1 y_{n+1} + \alpha_0 y_n + (\beta_0 f_n + \tau_1 f_{n+r} + \beta_1 f_{n+1} + \tau_2 f_{n+s} + \beta_2 f_{n+2}) \quad (11)$$

Where

$$\alpha_0(t) = -1;\ \alpha_1(t) = 2;$$
$$\beta_0(t) = (5sr-3)J_1$$
$$\beta_1(t) = (25sr - 25s + 28 - 25r)J_2$$
$$\beta_2(t) = (5sr - 10r - 10s + 17)J_3$$
$$\tau_1(t) = J_4, \qquad \tau_2(t) = J_5$$

and

$$J_1 = \frac{h^2}{60sr}$$
$$J_2 = \frac{h^2}{30(s-1)(r-1)}$$
$$J_3 = \frac{h^2}{60(r-2)(s-2)}$$
$$J_4 = \frac{h^2}{10r(r-1)(s-r)(r-2)}$$
$$J_5 = -\frac{h^2}{10s(s-1)(s-r)(s-2)}$$

The first derivative of Equation (11) is

$$y'_{n+2} = \alpha'_1 y_{n+1} + \alpha'_0 y_n + (\beta'_0 f_n + \tau'_1 f_{n+r} + \beta'_1 f_{n+1} + \tau'_2 f_{n+s} + \beta'_2 f_{n+2}) \quad (12)$$

where

$$\alpha_0'(t) = -\frac{1}{h};\ \alpha_1'(t) = \frac{1}{h};$$
$$\beta_0'(t) = K_1(8r+8s+5rs-19)$$
$$\beta_1'(t) = K_2(73r+73s-65rs-92)$$
$$\beta_2'(t) = K_3(45rs-82s-82r+145)$$
$$\tau_1'(t) = K_4(8s-19)$$
$$\tau_2'(t) = K_5(8r-19)$$

and

$$K_1 = -\frac{h}{120rs}$$
$$K_2 = -\frac{h}{60(r-1)(s-1)}$$
$$K_3 = \frac{h}{120r(r-2)(s-2)}$$
$$K_4 = \frac{h}{60r(r-s)(r-1)(r-2)}$$
$$K_5 = -\frac{h}{60s(r-s)(s-1)(s-2)}$$

Implementation of the CHLMM

A sample discrete scheme is obtained for the implementation of the method by taking the values of *r* and *s* at the mid-point of the subintervals containing *r* and *s* respectively to obtain

$$y_{n+2} = 2y_{n+1} - y_n + \frac{h^2}{60}(f_n + 16f_{n+\frac{1}{2}} + 26f_{n+1} + 16f_{n+\frac{3}{2}} + f_{n+2}) \quad (13)$$

And

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{360}(3f_n + 112f_{n+\frac{1}{2}} + 126f_{n+1} + 240f_{n+\frac{3}{2}} + 59f_{n+2}) \quad (14)$$

The discrete scheme (13) is zero stable, normalized (Lambert, 1973) and its order of accuracy p is 6. The absolute error constant $C_{p+2} = 8.\ 2672\times10^{-6}$. The derivative (14) is also of order $p = 6$ and $C_{p+2} = 1.\ 9841\times10^{-4}$.

The Predictors

The major disadvantage of predictor-corrector mode has been the use of predictors of lower order to implement the scheme. In order to overcome this setback, we developed a predictor that is of the same order as the scheme. The predictor and its first derivative are developed using Chebyshev series as a basic function as discussed above to obtain

$$y_{n+2} = -16y_{n+s} + 34y_{n+1} - 16y_{n+r} - y_n + \frac{h^2}{3}(2f_{n+r} + 11f_{n+1} + 2f_{n+s}) \quad (15)$$

Table 1. Numerical results for Problem 1 $h = 0.1$

X	y-exact	y-computed
0.2	-0.22140275816	-0.22140194098
0.3	-0.34985880757	-0.34985570401
0.4	-0.49182469764	-0.49181812807
0.5	-0.64872127070	-0.64870983266
0.6	-0.82211880039	-0.82210083478
0.7	-1.01375270747	-1.01372626007
0.8	-1.22554092849	-1.22550370623
0.9	-1.45960311115	-1.45955243246
1.0	-1.71828182845	-1.71821456690

and

$$y'_{n+2} = \frac{1}{21h}\left(-2920y_{n+s} + 6245y_{n+1} - 3176y_{n+r} - 149y_n\right) + \frac{h}{630}\begin{pmatrix} -18f_n + 3394f_{n+r} \\ +19531f_{n+1} + 2818f_{n+s} \end{pmatrix} \quad (16)$$

The main predictor (14) and its derivative (15) are also of order 6 with absolute error constant $C_{p+2} = 2.3768\times10^{-5}$ and $C_{p+2} = 2.2821\times10^{-4}$ respectively. Taylor's series expansion for $y_{n+\frac{1}{2}}$ was adopted from (Yahaya and Badmus, 2009).

NUMERICAL EXPERIMENTS

Test problems

The usability of the derived schemes is confirmed with three test problems and the results are compared with the results of some existing methods.

Problem 1

$$y'' = y', y(0) = 0, y'(0) = -1$$

Exact solution: $y(x) = 1 - e^x$

Problem 2

$$y'' = \frac{(y')^2}{2y} - 2y, y\left(\frac{\pi}{6}\right) = \frac{1}{4}, y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Exact solution: $y(x) = Sin^2 x$

Problem 3

Brown (1977)

$$y_1'' = -y_2' + \cos x,\ y_1(0) = -1,\ y_1'(0) = -1,$$
$$y_2'' = y_1 + \sin x,\ y_2(0) = 1,\ y_2'(0) = 0$$

Exact solution: $y_1(x) = -\cos x - \sin x;\ \ y_2(x) = \cos x$

Problem 4

Suleiman (1989) in Majid et al. (2009)

$$y_1'' = \frac{-y_1}{r},\ y_1(0) = 1,\ y_1'(0) = 0,$$
$$y_2'' = \frac{-y_2}{r},\ y_2(0) = 0,\ y_2'(0) = 1$$
$$r = \sqrt{y_1^2 + y_2^2}$$

Exact solution: $y_1(x) = \cos x;\ \ y_2(x) = \sin x$

RESULTS

Results are explained in Tables 1-6

Conclusion

This work has produced a two-point hybrid method for the direct solution of general second order initial value non-stiff and mildly-stiff problems. Chebyshev series was used as basic function for the approximate solution to the

Table 2. Comparison of errors for Problem 1.

X	Errors in Yahaya and Badmus (2009)	Errors in Ehigie et al. (2010)	Method (13)
0.2	3.27E-04	1.16E-02	8.17176E-07
0.3	2.22E-03	3.50E-02	3.10356E-06
0.4	4.86E-03	7.18E-02	6.56957E-06
0.5	9.10E-03	1.23E-01	1.14380E-05
0.6	1.44E-02	1.91E-01	1.79656E-05
0.7	2.15E-02	2.77E-01	2.64474E-05
0.8	2.99E-02	3.84E-01	3.72222E-05
0.9	4.03E-02	5.12E-01	5.06786E-05
1.0	5.26E-02	6.65E-01	6.72615E-05

Table 3. Numerical results for Problem 2 $h = 0.01$.

X	y-exact	y-computed
0.544	0.26751586298	0.26751586348
0.554	0.27641504148	0.27641504257
0.564	0.28540365098	0.28540365300
0.574	0.29447809616	0.29447809933
0.584	0.30363474736	0.30363475191
0.594	0.31286994205	0.31286994820
0.604	0.32217998626	0.32217999423
0.614	0.33156115611	0.33156116611
0.624	0.34100969925	0.34100971149

Table 4. Comparison of errors of Problem 2.

x	Errors in Ehigie et al. (2010)	Error in new method (13)
0.544	4.70E-08	4.04E-10
0.554	1.46E-07	1.10E-09
0.564	3.09E-07	2.02E-09
0.574	5.45E-07	3.17E-09
0.584	8.65E-07	4.55E-09
0.594	1.28E-06	6.15E-09
0.604	1.79E-06	7.97E-09
0.614	2.42E-06	9.99E-09
0.624	3.17E-06	1.22E-08

Table 5. Comparison of errors for Problem 3.

TOL	Results in Majid et al (2009)				Results in New Scheme (13)			
	MTD	TS	MAXE	TIME	NMTD	TS	MAXE	TIME
10^{-2}	2P4SDIR	33	2.73003E-2	710	2PHM	33	2.768463E-10	119
10^{-4}	2P4SDIR	42	1.72828E-3	837	2PHM	55	1.275646E-13	213
10^{-6}	2P4SDIR	69	6.87609E-6	1182	2PHM	74	3.519407E-14	262
10^{-8}	2P4SDIR	84	9.64221E-7	1552	2PHM	130	7.510659E-13	447
10^{-10}	2P4SDIR	160	2.04449E-9	2485	2PHM	278	3.088640E-13	922

Table 6. Comparison of errors for Problem 4.

TOL	Results in Majid et al. (2009)				Results in New Scheme (13)			
	MTD	TS	MAXE	TIME	MTD	TS	MAXE	TIME
10^{-2}	2PFDIR	67	7.98175E-2	938	2-STEP	67	9.763298E-08	635
10^{-4}	2PFDIR	140	6.93117E-4	1472	2-STEP	140	4.170707E-10	1346
10^{-6}	2PFDIR	316	7.46033E-6	3318	2-STEP	316	2.100171E-12	2614
10^{-8}	2PFDIR	394	2.45673E-6	4181	2-STEP	394	3.214551E-15	2788
10^{-10}	2PFDIR	938	2.53897E-8	9932	2-STEP	938	2.473336E-17	5590

TOL - Tolerance, MTD - Method employed, TS - Total steps taken, MAXE - Magnitude of the maximum error of the computed solution, TIME - The execution time taken in microseconds, 2P4SDIR - Direct two point four step implicit block method of variable step size [9], NMTD - New method employed, 2PHM - 2-Point hybrid method employed with hybrid points $\frac{1}{2}$ and $\frac{3}{2}$

given problem. A discrete scheme from the derived methods was implemented to test its usability and accuracy using the main predictor of the same order of accuracy. The results, as shown in the Tables 1 to 6, revealed that the developed methods are significantly better than those of the existing methods.

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