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# Common fixed point theorems for some generalized non-expansive mappings and non-spreading mappings in CAT(0) spaces

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**We study the existence of a common fixed point for some generalized non-expansive mappings and non-spreading mappings in CAT(0) spaces. We also study an iterative method for approximating common fixed point of a pair of those mappings.**

**Key words:** CAT(0) Space, nonspreading mapping, common fixed point.

## INTRODUCTION

Let $E$ be a nonempty subset of a Banach space $X$. A mapping $T : E \longrightarrow E$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in E$. We denote by F(T) the set of fixed points of T, that is, $F(T) = \{x \in E : Tx = x\}$. A mapping $T : E \longrightarrow E$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\| Tx - z \| \leq \| x - z \|$ for all $x \in E$ and $z \in F(T)$.

In 2008, Suzuki (2008) introduced a condition on mappings, called condition (C), which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Moreover, he obtained some interesting fixed point theorems and convergence theorems for such mappings. In 2008, Dhompongsa et al. (2009) proved a fixed point theorem for mappings with condition (C) on a Banach space such that its asymptotic center in a bounded closed and convex subset of each bounded sequence is nonempty and compact. Nanjaras et al. (2010) extended Suzuki results on fixed point theorems and convergence theorems to a special kind of metric spaces, namely CAT(0) spaces. Kohsaka and Takahashi (2008) introduced a nonspreading mapping on Banach spaces. Let $E$ be a nonempty closed convex subset of a Banach space $X$. A mapping $T : E \longrightarrow E$ is said to be a nonspreading mapping if $2\| Tx - Ty \|^2 \leq \| Tx - y \|^2 + \| Ty - x \|^2$ for all $x, y \in E$. (For detail, one can also refer to Lemoto and Takahashi (2009).

In 2011, Lin et al. (2011) introduced generalized nonspreading mappings on CAT(0) spaces, called generalized hybrid mappings, let $E$ be a nonempty closed convex subset of a CAT(0) space $X$. We say $T : E \longrightarrow X$ is a generalized hybrid mapping if there exist $a_1 : E \longrightarrow 0,1\, a_2, a_3 : E \longrightarrow [0,1)$ such that

(P1) $d^2 \;\; Tx,Ty \;\; \leq a_1 \;\; x \;\; d^2 \;\; x,y \;\; + a_2 \;\; x \;\; d^2 \;\; Tx,y$
$+ a_3(x) d^2(x, Ty) + k_1(x) d^2(Tx, x) + k_2(x) d^2(Ty, y)$
for all $x, y \in E$;

(P2) $a_1 \;\; x \;\; + a_2 \;\; x \;\; + a_3 \;\; x \;\; \leq 1$ for all $x, y \in E$;

(P3) $2k_1 \;\; x \;\; < 1 - a_2 \;\; x$ and $k_2 \;\; x \;\; < 1 - a_3 \;\; x$
for all $x, y \in E$.

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They also gave the definition of nonspreading mappings on CAT(0) spaces. Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$. A mapping $T:E\longrightarrow E$ is said to be a nonspreading mapping if

$$2d^2(Tx,Ty)\leq d^2(Tx,y)+d^2(Ty,x) \text{ for all } x,y\in E.$$

The following iterative scheme is introduced by Dhompongsa et al. (2011). Let $E$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S:E\longrightarrow E$ be a nonspreading mapping and let $T:E\longrightarrow E$ be a mapping satisfying condition (C) such that $F(S)\cap F(T)\neq\varnothing$. They consider,

$$(A')\begin{cases} x_1=x\in E,\\ x_{n+1}=\alpha_n S\{\beta_n Tx_n+(1-\beta_n)x_n\}+(1-\alpha_n)x_n,\end{cases}$$

$$(B')\begin{cases} z_1=z\in E,\\ z_{n+1}=\alpha_n T\{\beta_n Sz_n+(1-\beta_n)z_n\}+(1-\alpha_n)z_n,\end{cases}$$

for all $n\in N$, where $\{\alpha_n\}\subset(0,1]$ and $\{\beta_n\}\subset[0,1]$.

In this paper, we extend this iterative scheme to CAT(0) spaces. Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$. Let $S:E\longrightarrow E$ be a nonspreading mapping and let $T:E\longrightarrow E$ be a mapping satisfying condition (C) such that $F(S)\cap F(T)\neq\varnothing$. We consider,

$$(A)\begin{cases} x_1=x\in E,\\ x_{n+1}=\alpha_n S\{\beta_n Tx_n\oplus(1-\beta_n)x_n\}\oplus(1-\alpha_n)x_n,\end{cases}$$

$$(B)\begin{cases} z_1=z\in E,\\ z_{n+1}=\alpha_n T\{\beta_n Sz_n\oplus(1-\beta_n)z_n\}\oplus(1-\alpha_n)z_n,\end{cases}$$

for all $n\in N$, where $\{\alpha_n\}\subset(0,1]$ and $\{\beta_n\}\subset[0,1]$.

## PRELIMINARIES

Let $X$ be a complete CAT(0) space, let $\{x_n\}$ be a bounded sequence in $X$ and for $x\in X$ set $r(x,\{x_n\})=\limsup_{n\to\infty} d(x,x_n)$.

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\})=\inf\{r(x,\{x_n\}):x\in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}.$$

It is known from Proposition 7 of Dhompongsa et al. (2006) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

**Definition 1:** Let $E$ be a nonempty subset of a complete CAT(0) space $X$. Then $T:E\longrightarrow E$ is said to satisfy condition (C) if $\frac{1}{2}d(x,Tx)\leq d(x,y)$ implies $d(Tx,Ty)\leq d(x,y)$ for all $x,y\in E$. (Nanjaras et al., 2010; Suzuki, 2008).

We see that every nonexpansive mapping satisfies condition (C) but the converse is not true (Nanjaras et al., 2010). It is easy to see that if a mapping $T$ satisfies condition (C) and has a fixed point, then $T$ is a quasi - nonexpansive mapping (Nanjaras et al., 2010). We now give the definition of $\Delta$-convergence.

**Definition 2:** A sequence $\{x_n\}$ in a complete CAT(0) space $X$ is said to $\Delta$*-converges* to $x\in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_{n\to\infty} x_n=x$ and call $x$ the D- *limit* of $\{x_n\}$. (Kirk and Panyanak, 2008; Lim, 1976).

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

**Lemma 1:** Every bounded sequence in a complete CAT(0) space always has a $\Delta$-*convergent subsequence* (Kirk and Panyanak, 2008).

**Lemma 2:** If $E$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $E$, then the asymptotic center of $\{x_n\}$ is in $E$. (Dhompongsa et al., 2007).

**Lemma 3:** Let $E$ be a nonempty closed convex subset of a CAT(0) space $X$. Let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\})=\{x\}$, and let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$. Suppose that $\lim_{n\to\infty} d(x_n,u)$ exists. Then $x=u$. (Gromov, 1999).

**Lemma 4:** Let $X$ be a CAT(0) space (Dhompongsa and Panyanak, 2008).

(i) For $x,y\in X$ and $t\in[0,1]$, there exists a unique point $z\in[x,y]$ such that $d(x,z)=td(x,y)$ and

$$d(y,z)=(1-t)d(x,y). \tag{1}$$

We use the notation $(1-t)x\oplus ty$ for the unique point $z$ satisfying (1).

(ii) For $x,y,z\in X$ and $t\in[0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

(iii) For $x, y, z \in X$ and $t \in [0,1]$, we have

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y).$$

**Lemma 5:** Let $X$ be a CAT(0) space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in $X$ with $\lim_{n\to\infty} d(y_n, x_n) = 0.$ (Lin et al., 2011).
If $\Delta - \lim_{n\to\infty} x_n = x,$ then $\Delta - \lim_{n\to\infty} y_n = x.$

**Lemma 6:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and suppose that $T: E \longrightarrow E$ satisfies condition (C) (Nanjaras et al., 2010). If $\{x_n\}$ is a sequence in $E$ such that $d(Tx_n, x_n) \longrightarrow 0$ and $\Delta - \lim_{n\to\infty} x_n = z$ for some $z \in X,$ then $z \in E$ and $z = Tz.$

**Lemma 7:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow X$ be a generalized hybrid mapping (Lin et al., 2011). Let $\{x_n\}$ be a bounded sequence in $E$ with $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $x Î E$ and $Tx = x$.

**Corollary 1:** Let $E$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$. (Nanjaras et al., 2010). Suppose that $T: E \longrightarrow E$ satisfies condition (C). Then $F(T)$ is nonempty closed, convex, and hence contractible.

**Corollary 2:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow E$ be any one of nonspreading mapping, TJ–1 mapping, TJ–2 mapping, hybrid mapping, and nonexpansive mapping. Then $\{T^n x\}$ is bounded for some $x Î E$ if and only if $F(T) \neq \varnothing.$ (Lin et al., 2011).

**Lemma 8:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow X$ be a generalized hybrid mapping (Lin et al., 2011). If $\{x_n\}$ is a bounded sequence in $E$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T),$ then $\omega_w(\{x_n\}) \subset F(T),$ where $\omega_w(\{x_n\}) := \cup A(\{u_n\})$ and $\{u_n\}$ and $\{u_n\}$ is any subsequence of $\{x_n\}$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.

## EXISTENCE THEOREM

**Theorem 1:** Let $E$ be a nonempty bounded closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow E$ satisfy condition (C) and $S: E \longrightarrow E$ be a nonspreading mapping. Let $T$ and $S$ are commuting mappings on $E$. Then $T$ and $S$ have a common fixed point.

**Proof:** By Corollary 1, we have $F(T) \neq \varnothing.$ Since $T$ and $S$ are commuting mappings on $E$, we have $Sx = S(Tx) = T(Sx),$ and hence $Sx \in F(T)$ for all $x \in F(T).$ So $S: F(T) \longrightarrow F(T).$ By Corollary 2, we have $F(S) \neq \varnothing.$ Hence there exists $y \in F(S)$ such that $y = Sy \in F(T).$ So $y \in F(T) \cap F(S).$

## $\Delta$- CONVERGENCE THEOREMS

We need the following lemmas for completing the proof of main results.

**Lemma 9:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow X$ satisfy condition (C). If $\{x_n\}$ is a bounded sequence in $E$ such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\{d(x_n, v)\}$ converges for all $\{d(x_n, v)\}$ then $\omega_w(\{x_n\}) \subset F(T),$ where $\omega_w(\{x_n\}) := \cup A(\{u_n\})$ and $\{u_n\}$ is any subsequence of $\{x_n\}$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.

**Proof.** By the assumption $\{x_n\}$ is a bounded sequence in $E$ such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0.$ Let $u \in \omega_w(\{x_n\}),$ then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}.$ By Lemmas 1 and 2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} u_n = v \in E.$ Since $\lim_{n\to\infty} d(Tv_n, v_n) = 0,$ we have $v \in F(T)$ by Lemma 6. By the assumption $\{d(x_n, v)\}$ converges for all $v \in F(T)$ then $u = v \in F(T)$ by Lemma 3. This shows that $\omega_w(\{x_n\}) \subset F(T).$ Next, we show that $\omega_w(\{x_n\})$ consists of exactly one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be a subsequence of $\{u_n\}$ with $A(\{u_n\}) = \{u\}.$ Since $u \in \omega_w(\{x_n\}) \subset F(T),$ we have seen that $u = v \in F(T)$ then $\{d(x_n, u)\}$ converges. By Lemma 3, $x = u$.

**Lemma 10:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T: E \longrightarrow E$ satisfy condition (C) and $S: E \longrightarrow E$ be a nonspreading mapping such that $F(T)\cap F(S)\neq\emptyset$. Let $\{x_n\}$ be a sequence defined as (A). Then $\lim_{n\to\infty} d(x_n,w)$ exists for all $w\in F(T)\cap F(S)$

**Proof:** Let $\{x_n\}$ be a sequences defined by (A) and $w\in F(T)\cap F(S)$. Then $d(Tx,w)\leq d(x,w)$ and $d(Sy,w)\leq d(y,w)$ for all $x,y\in E$. By Lemma 4(iii), we have

$$\begin{aligned} d^2(y_n,w) &= d^2(\beta_n Tx_n \oplus (1-\beta_n)x_n,w)\\ &\leq \beta_n d^2(Tx_n,w)+(1-\beta_n)d^2(x_n,w)-\beta_n(1-\beta_n)d^2(Tx_n,x_n)\\ &\leq \beta_n d^2(x_n,w)+(1-\beta_n)d^2(x_n,w)-\beta_n(1-\beta_n)d^2(Tx_n,x_n)\\ &= d^2(x_n,w)-\beta_n(1-\beta_n)d^2(Tx_n,x_n) \qquad (1)\\ &\leq d^2(x_n,w), \end{aligned}$$

and

$$\begin{aligned} d^2(x_{n+1},w) &= d^2(\alpha_n Sy_n \oplus (1-\alpha_n)x_n,w)\\ &\leq \alpha_n d^2(Sy_n,w)+(1-\alpha_n)d^2(x_n,w)-\alpha_n(1-\alpha_n)d^2(Sy_n,x_n)\\ &\leq \alpha_n d^2(y_n,w)+(1-\alpha_n)d^2(x_n,w)\\ &\quad -\alpha_n(1-\alpha_n)d^2(Sy_n,x_n) \qquad (2)\\ &\leq \alpha_n d^2(x_n,w)+(1-\alpha_n)d^2(x_n,w)-\alpha_n(1-\alpha_n)d^2(Sy_n,x_n)\\ &\leq d^2(x_n,w)-\alpha_n(1-\alpha_n)d^2(Sy_n,x_n) \qquad (3)\\ &\leq d^2(x_n,w). \end{aligned}$$

So $\{d(x_n,w)\}$ is bounded and decreasing sequences. Hence $\lim_{n\to\infty} d(x_n,w)$ exists.

**Lemma 11:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T:E\longrightarrow E$ satisfy condition (C) and $S:E\longrightarrow E$ be a nonspreading mapping such that $F(T)\cap F(S)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined as (A).

If $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, then $\lim_{n\to\infty} d(y_n,x_n)=0$ and $\lim_{n\to\infty} d(y_n,w)$ exists.

**Proof:** Let $\{x_n\}$ be a sequence defined by (A) and $w\in F(T)\cap F(S)$. By Lemma 10, $\lim_{n\to\infty} d(x_n,w)$ exists.

Since $d(y_n,w)\leq d(x_n,w)\leq d(x_1,w)$, so $\{x_n\}$ and $\{y_n\}$ are also bounded. By (3), we have

$$d^2(x_{n+1},w)\leq d^2(x_n,w)-\alpha_n(1-\alpha_n)d^2(Sy_n,x_n).$$

Since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, so there exist $k>0$ and $\exists N\in\mathbb{N}$ such that $\alpha_n(1-\alpha_n)\geq k$ for all $n\geq N$, so

$$\begin{aligned} \limsup_{n\to\infty} kd^2(Sy_n,x_n) &\leq \limsup_{n\to\infty}\alpha_n(1-\alpha_n)d^2(Sy_n,x_n)\\ &\leq \limsup_{n\to\infty}\{d^2(x_n,w)-d^2(x_{n+1},w)\}\\ &= 0. \end{aligned}$$

Hence $0\leq\liminf_{n\to\infty} d^2(Sy_n,x_n)\leq\limsup_{n\to\infty} d^2(Sy_n,x_n)\leq 0.$

Then $\lim_{n\to\infty} d^2(Sy_n,x_n)=0.$

This implies that $\lim_{n\to\infty} d(Sy_n,x_n)=0.$ (4)

Then $\alpha_n[d^2(x_n,w)-d^2(y_n,w)]\leq d^2(x_n,w)-d^2(x_{n+1},w).$

Since $\alpha_n(1-\alpha_n)<\alpha_n$ so $\liminf_{n\to\infty}\alpha_n>0.$

Using the same argument we have $\lim_{n\to\infty}(d^2(x_n,w)-d^2(y_n,w))=0.$

Then

$$\beta_n(1-\beta_n)d^2(Tx_n,x_n)\leq d^2(x_n,w)-d^2(y_n,w).$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0.$

Using the same argument, we have $\lim_{n\to\infty} d^2(Tx_n,x_n)=0.$

This implies that $\lim_{n\to\infty} d(Tx_n,x_n)=0.$ (5)

Hence,

$$\limsup_{n\to\infty} d(y_n,x_n)=\limsup_{n\to\infty}\beta_n d(Tx_n,x_n)\leq\limsup_{n\to\infty} d(Tx_n,x_n)=0.$$

So $\lim_{n\to\infty} d(y_n,x_n)=0.$ Since $\lim_{n\to\infty}(d^2(x_n,w)-d^2(y_n,w))=0,$ and $\lim_{n\to\infty} d(x_n,w)$ exists, we have $\lim_{n\to\infty} d(y_n,w)$ exists.

Now we are ready to prove $\Delta$-convergence theorem for a sequence $\{x_n\}$.

**Theorem 2:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T:E\longrightarrow E$ satisfy condition (C) and $S:E\longrightarrow E$ be a nonspreading mapping such that $F(T)\cap F(S)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined as (A).

If If $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0,$ Then $\Delta-\lim_{n\to\infty} x_n=w\in F(T)\cap F(S).$

**Proof:** Let $\{x_n\}$ be a sequence defined by (A) and $w \in F(T) \cap F(S)$. By Lemma 10, we have $\lim_{n\to\infty} d(x_n, w)$ exists. Then $\{x_n\}$ is bounded. By Lemma 11, we have $\lim_{n\to\infty} d(y_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, w)$ exists. Then $\{y_n\}$ is also bounded. As in the proof of Lemma 11, we get $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and by (4) we have $\lim_{n\to\infty} d(Sy_n, x_n) = 0.$

Since $d(Sy_n, y_n) \le d(Sy_n, x_n) + d(x_n, y_n)$, We have $\lim_{n\to\infty} d(Sy_n, y_n) = 0.$ By Lemma 9, there exist $\bar{x}, \bar{y} \in E$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subset F(T)$ and $\omega_w(\{y_n\}) = \{\bar{y}\} \subset F(S)$. So, $\Delta - \lim_{n\to\infty} x_n = \bar{x}$ and $\Delta - \lim_{n\to\infty} y_n = \bar{y}.$ By Lemma 5, $\bar{x} = \bar{y}$.

**Lemma 12:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \longrightarrow E$ satisfy condition (C) and $S : E \longrightarrow E$ be a nonspreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{z_n\}$ be a sequence defined as (B). Then $\lim_{n\to\infty} d(z_n, v)$ exists for all $v \in F(T) \cap F(S)$.

**Proof:** We can prove this by following the steps of the argument of Lemma 10, simply replacing $\{x_n\}$ with $\{z_n\}$, replacing $w$ with $v$, replacing $T$ with $S$ and replacing $S$ with $T$.

**Lemma 13:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T : E \longrightarrow E$ satisfy condition (C) and $S : E \longrightarrow E$ be a nonspreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{z_n\}$ be a sequence defined as (B).

If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\lim_{n\to\infty} d(y_n, z_n) = 0$ and $\lim_{n\to\infty} d(y_n, v)$ exists.

**Proof:** We can prove this by using Lemma 12 and following the steps of the argument of Lemma 11. Now we are ready to prove $\Delta$-convergence theorem for a sequence $\{z_n\}$.

**Theorem 3:** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$, and let $T : E \to E$ satisfy condition (C) and $S : E \to E$ be a nonspreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{z_n\}$ be a sequence defined as (B).

If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\Delta - \lim_{n\to\infty} z_n = v \in F(T) \cap F(S)$.

**Proof:** We can prove this by using Lemmas 12, 13 and following the steps of the argument of Theorem 2.


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