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The African Journal of Mathematics and Computer Science Research (ISSN 2006-9731) is published bi-monthly (one volume per year) by Academic Journals.

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Ruhr-Universitaet Bochum,  
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<table>
<thead>
<tr>
<th>ARTICLES</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A study of Green’s functions for three-dimensional problem in thermoelastic diffusion media</strong></td>
<td>68</td>
</tr>
<tr>
<td>Rajnesh Kumar and Vijay Chawla</td>
<td></td>
</tr>
<tr>
<td><strong>Multivalent harmonic uniformly convex functions</strong></td>
<td>79</td>
</tr>
<tr>
<td>R. M. EL-Ashwah, M. K. Aouf and F. M. Abdulkarem</td>
<td></td>
</tr>
</tbody>
</table>
Full Length Research Paper

A study of Green’s functions for three-dimensional problem in thermoelastic diffusion media

Rajnesh Kumar\textsuperscript{1*} and Vijay Chawla\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Kurukshetra University, Kurukshetra-136119, Haryana, India.
\textsuperscript{2}Department of Mathematics, Maharaja Agrasen Mahavidyalaya, Jagadhri-135003 Haryana, India.

Received 29 July, 2014; Accepted 9 September, 2014

The purpose of the present paper is to study the three-dimensional general solution and Green’s functions in transversely isotropic thermoelastic diffusion media for static problem. With this objective, two displacement functions are introduced to simplify the basic equation and a general solution is then obtained by using the operator theory. Based on the obtained general solution, the three-dimensional Green’s functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. The components of displacement, stress, temperature distribution and mass concentration are expressed in terms of elementary functions and are convenient to use. When the apex angle $2\alpha$ equals to $\pi$, then we obtain the solution for semi-infinite body with a surface point. From the present investigation, a special case of interest is deduced to depict the effect of diffusion on components of stress and temperature distribution.

Key words: Thermoelastic diffusion media, Green’s function, transversely isotropic.

INTRODUCTION

Fundamental solutions or Green’s functions play an important role in the solution of numerous problems in the mechanics and physics of solids. Green’s functions can be used to construct many analytical solutions of boundary value problems. They are essential in boundary element method as well as the study of cracks, defects and inclusion. They are a basic building block of future works. For example, fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed. Ding et al. (1996) derived the general solutions for coupled equations in piezoelectric media. Dunn and Wienecke (1999) investigated the half space Green’s functions in transversely isotropic piezoelectric solid. Pan and Tanon (2000) studied the Green’s functions for three dimensional problems in anisotropic piezoelectric solids. When thermal effects are considered, Sharma (1958) investigated the fundamental solution in transversely isotropic thermoelastic material in an integral form. Chen et al. (2004) derived the three dimensional general solution in transversely isotropic thermoelastic materials. Hou et al. (2008, 2009) investigated the Green’s function for two and three-dimensional problem for a steady point heat source in the interior of a semi-infinite thermoelastic material. Also, Hou et al. (2011) investigated the two dimensional general solutions and fundamental solutions in orthotropic thermoelastic materials.

Diffusion can be defined as random walk of assembly.

*Corresponding author. E-mail: rajneesh_kumar@rediffmail.com
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of particles from a high concentration region to a low concentration region. An example of diffusion is heat transport or movement transport. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermoelasticity remains a practical process to separate isotopes of noble gases (e.g., xenon) and other light isotopes (e.g., carbon) for research purposes.

Nowaki (1974a, b, c, d) developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief and Saleh (2005) developed the generalized theory of thermoelastic diffusion with one relaxation time which allows finite speeds of propagation of waves. Kumar and Kansal (2008) derived the basic equations for generalized thermoelastic diffusion (G-L model) and discussed the Lamb waves. When diffusion effects are considered, Kumar and Chawla (2011a) derived the Fundamental solution in orthotropic thermoelastic diffusion material. Kumar and Chawla (2011b) discussed the plane wave propagation in the context of anisotropic three-phase-lag and two-phase-lag model of thermoelasticity. Kumar and Chawla (2012) derived the Green’s functions for two-dimensional problem in orthotropic thermoelastic diffusion media. Recently, Kumar and Chawla (2013) discussed the problem of reflection and transmission in thermoelastic media with three-phase-lag model. However, the important Green’s function for three-dimensional problem function in transversely isotropic thermoelastic diffusion material has not been discussed so far.

Keeping in view of these applications, the three dimensional general solution and Green’s function in transversely isotropic thermoelastic diffusion elastic medium for steady state problem was studied. After applying the dimensionless quantities and using the operator theory, the general expression for displacement components, mass concentration and temperature change are derived in terms of four harmonic functions. By virtue of the obtained general solution, the three-dimensional Green’s functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

**Basic equations**

Following Sherief and Saleh (2005) the basic governing equations for homogenous anisotropic generalized thermoelastic diffusion solid in the absence of body forces, heat and mass diffusion sources are:

**(1) Constitutive relations:**

\[ \sigma_{ij} = c_{ijkl}e_{klmn} + a_{ij}T_{ij} + b_{ij}C_{ij} \]  

\[ \alpha_{ij} = c_{ijkl}e_{klmn} + a_{ij}T_{ij} + b_{ij}C_{ij} \]

\[ \rho \dot{C}_{ij} + a_{ij}T_{ij} \dot{r}_{ij} = K_{ij} \dot{T}_{ij} \]

\[ -\alpha_{ij} b_{km} e_{kmij} - \alpha_{ij} b_{C_{ij}} + \alpha_{ij} a_{ij} C_{ij} = -\dot{C}_{ij} \]

Here, \( c_{ijkl} \) and \( \alpha_{ij} \) are elastic parameters; \( a_{ij} \) and \( b_{ij} \) are respectively, the tensor of thermal and diffusion moduli. \( \rho \) is the density and \( C_{ij} \) is the specific heat at constant strain, \( a, b \) are respective coefficients describing the measure of thermoelastic diffusion effects and of diffusion effects, \( T_{ij} \) is the reference temperature assumed to be such that \( T_{ij} \leq T_{0} \).

**Basic equations**

Following Sherief and Saleh (2005) the basic governing equations for homogenous anisotropic generalized thermoelastic diffusion solid in the absence of body forces, heat and mass diffusion sources are:

\[ \rho \dot{C}_{ij} + a_{ij}T_{ij} \dot{r}_{ij} = K_{ij} \dot{T}_{ij} \]

\[ -\alpha_{ij} b_{km} e_{kmij} - \alpha_{ij} b_{C_{ij}} + \alpha_{ij} a_{ij} C_{ij} = -\dot{C}_{ij} \]

Here, \( c_{ijkl} \) are elastic parameters; \( a_{ij} \) and \( b_{ij} \) are respectively, the tensor of thermal and diffusion moduli. \( \rho \) is the density and \( C_{ij} \) is the specific heat at constant strain, \( a, b \) are respective coefficients describing the measure of thermoelastic diffusion effects and of diffusion effects, \( T_{ij} \) is the reference temperature assumed to be such that \( T_{ij} \leq T_{0} \).

In the above equations, the symbol (,) followed by a suffix denotes differentiation with respect to spatial coordinate and a superposed dot (”.”) denotes the derivative with respect to time respectively.

Following Slaughter (2002), applying the transformation, we have:

\[ x' = x \cos \phi + y \sin \phi, \quad y' = -x \sin \phi + y \cos \phi, \quad z' = z, \]

Where \( \phi \) is the angle of rotation in the \( x - z \) plane. In the Equations (1) to (4), the stress-strain-temperature-concentration relation, equations of motion, heat conduction and mass diffusion equation in homogeneous, transversely isotropic thermoelastic diffusion media in cartesian coordinates \((x, y, z)\) can be written as:

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{12} & c_{11} & c_{13} \\
c_{13} & c_{13} & c_{33}
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{xy} \\
e_{xz}
\end{bmatrix}
+ \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\begin{bmatrix}
T_0 \\
T \\
C
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\
\alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\
\alpha_{zx} & \alpha_{zy} & \alpha_{zz}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{12} & c_{11} & c_{13} \\
c_{13} & c_{13} & c_{33}
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{xy} \\
e_{xz}
\end{bmatrix}
+ \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\begin{bmatrix}
T_0 \\
T \\
C
\end{bmatrix}
\]

\[
\begin{bmatrix}
\rho \\
\rho \dot{C}_{xx} \\
\rho \dot{C}_{xy} \\
\rho \dot{C}_{xz} \\
\rho \dot{C}_{yx} \\
\rho \dot{C}_{yy} \\
\rho \dot{C}_{yz} \\
\rho \dot{C}_{zx} \\
\rho \dot{C}_{zy} \\
\rho \dot{C}_{zz}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{12} & c_{11} & c_{13} \\
c_{13} & c_{13} & c_{33}
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{xy} \\
e_{xz}
\end{bmatrix}
+ \begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\begin{bmatrix}
T_0 \\
T \\
C
\end{bmatrix}
\]
\[
c_{i1} \frac{\partial^2 u}{\partial x^2} + c_{i6} \frac{\partial^2 u}{\partial y^2} + c_{i4} \frac{\partial^2 u}{\partial z^2} + (c_{i2} + c_{i6}) \frac{\partial^2 v}{\partial x \partial y} + (c_{i3} + c_{i4}) \frac{\partial^2 w}{\partial x \partial z} - \\
a_1 \frac{\partial T}{\partial x} - b_1 \frac{\partial C}{\partial x} = \frac{\rho \partial^2 u}{\partial t^2}, \tag{7}
\]

\[
(c_{i2} + c_{i6}) \frac{\partial^2 u}{\partial x \partial y} + c_{i6} \frac{\partial^2 v}{\partial x \partial y} + c_{i4} \frac{\partial^2 v}{\partial x \partial z} + (c_{i3} + c_{i4}) \frac{\partial^2 w}{\partial x \partial z} - \\
(a_1 \frac{\partial T}{\partial y} - b_1 \frac{\partial C}{\partial y} = \frac{\rho \partial^2 v}{\partial t^2}, \tag{8}
\]

\[
(c_{i3} + c_{i4}) \frac{\partial^2 u}{\partial x \partial z} + (c_{i3} + c_{i4}) \frac{\partial^2 v}{\partial x \partial z} + c_{i4} \frac{\partial^2 w}{\partial x \partial z} + c_{i4} \frac{\partial^2 w}{\partial y \partial z} - \\
(a_1 \frac{\partial T}{\partial z} - b_1 \frac{\partial C}{\partial z} = \frac{\rho \partial^2 w}{\partial t^2}. \tag{9}
\]

\[
p \rho c_p \frac{\partial T}{\partial t} + a_T \frac{\partial C}{\partial t} + T \left( a_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\partial a_1}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial a_1}{\partial y} \frac{\partial^2 w}{\partial x \partial z} + \right) \\
+ b_1 \left[ a_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\partial a_1}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial a_1}{\partial y} \frac{\partial^2 w}{\partial x \partial z} + \right) \\
+ b_1 \left[ a_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\partial a_1}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial a_1}{\partial y} \frac{\partial^2 w}{\partial x \partial z} + \right) \\
+ \left[ b_1 \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) + \frac{\partial b_1}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial b_1}{\partial y} \frac{\partial^2 w}{\partial x \partial z} + \right) = \frac{\partial C}{\partial t}. \tag{10}
\]

Where

\[a_j = -a_1 \delta_{ij}, \quad b_j = -b_1 \delta_{ij}, \quad K_j = K_1 \delta_{ij}, \quad i \text{ is not summed and}
\]

\[c_{66} = \frac{c_{i1} - c_{i2}}{2}.
\]

**FORMULATION OF THE PROBLEM**

We consider a homogenous transversely isotropic thermoelastic diffusion medium. Let us take Oxyz as the frame of reference in Cartesian coordinates.

For three dimensional problems, we assume the displacement vector, temperature distribution and mass concentration are respectively, of the form:

\[\vec{u} = (u, v, w), \quad T(x, y, z, t), \quad C(x, y, z, t). \tag{12}\]

Moreover, we are discussing steady problem

\[\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial T}{\partial t} = \frac{\partial C}{\partial t} = 0. \tag{13}\]

We define the dimensionless quantities as:

\[a_1 = (x', y' , z', u', v', w', b', r') = \frac{a_1}{v_1} (x, y, z, u, v, w, b, r), \]

\[(T', C') = \frac{1}{c_1} (a_1 T, b_1 C), \]

\[\sigma_{ij}' = \frac{\sigma_{ij}}{a_1 T_0}, \quad H' = \frac{a_1}{c_1} K_3 H. \tag{14}\]

Applying the dimensionless quantities defined by Equation (14) in Equations (7) to (11), after suppressing the primes, we obtain:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u + \left( \delta_1 \frac{\partial^2}{\partial x \partial y} + \delta_1 \frac{\partial^2}{\partial x \partial z} \right) v + \left( \delta_1 \frac{\partial^2}{\partial y \partial z} \right) w - \left( \frac{\partial}{\partial x} \right) T - \left( \frac{\partial}{\partial y} \right) C = 0, \tag{15}\]

\[
\left( \delta_1 \frac{\partial^2}{\partial x \partial y} \right) u + \left( \delta_1 \frac{\partial^2}{\partial x \partial z} + \delta_2 \frac{\partial^2}{\partial y \partial z} \right) v + \left( \delta_1 \frac{\partial^2}{\partial y \partial z} \right) w - \left( \frac{\partial}{\partial x} \right) T - \left( \frac{\partial}{\partial y} \right) C = 0, \tag{16}\]

\[
\left( \delta_1 \frac{\partial^2}{\partial x \partial y} \right) u + \left( \delta_1 \frac{\partial^2}{\partial x \partial z} + \delta_2 \frac{\partial^2}{\partial y \partial z} \right) v + \left( \delta_2 \frac{\partial^2}{\partial y \partial z} \right) w - \left( - \frac{\partial}{\partial x} \right) T - \left( \frac{\partial}{\partial y} \right) C = 0, \tag{17}\]

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T + e_2 \left( \frac{\partial^2}{\partial z^2} \right) T = 0, \tag{18}\]

\[
\left[ b_1 \left( \frac{\partial^2}{\partial x \partial y} \right) u + b_1 \left( \frac{\partial^2}{\partial x \partial z} \right) v + b_1 \left( \frac{\partial^2}{\partial y \partial z} \right) w \right] + \left( \frac{\partial b_1}{\partial x} \right) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] + \left( \frac{\partial b_1}{\partial y} \right) \left( \frac{\partial^2}{\partial z^2} \right) = 0. \tag{19}\]

Where

\[
\left( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \right) = \frac{1}{c_1} (c_{i4} c_{i6} c_{i2} + c_{i6} c_{i1} + c_{i4} c_{i3}), \quad \delta_1 = \frac{a_1}{b_1}, \quad \gamma_1 = \frac{b_1}{b_1}, \quad \gamma_2 = \frac{K_1}{K_1},
\]

\[
\left( \eta_1, \eta_2, \eta_3, \eta_4 \right) = \frac{1}{c_1} (a_1 a_1, a_1 a_2, a_1 a_3, a_i b_1, a_i b_2, a_i b_3, a_i b_4, a_i b_5), \quad \left( \eta_1, \eta_2 \right) = \frac{1}{a_1} (a_1 a_i, a_1 a_i a_i a_i),
\]

\[
\left( \eta_1, \eta_2 \right) = \frac{1}{b_1} (a_i b_1, a_i b_1 b_1).
\]

\[
a_i = (c_{i1} + c_{i2}) a_i + c_{i3} a_i, \quad b_i = (c_{i1} + c_{i2}) a_i + c_{i3} a_i, \quad b_i = 2c_{i1} a_i + c_{i3} a_i, \quad c_{66} = \frac{c_{i1} - c_{i2}}{2}.
\]

**STATIC GENERAL SOLUTIONS**

Two displacements functions \( \Psi \) and \( G \) are introduced as follows:
Using the displacements functions $\Psi$ and $G$ in Equations (15) - (19), we obtain

$$\left[ \delta_{2} \left( \frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} \right) + \delta_{4} \frac{\partial^{2}}{\partial z^{2}} \right] \Psi = 0,$$

(21)

where $D$ is the differential operator matrix given by

$$D =\begin{pmatrix}
\Delta + \delta_{1} \frac{\partial^{2}}{\partial z^{2}} & -\delta_{2} \frac{\partial}{\partial z} & 1 & 1 \\
-\delta_{2} \frac{\partial}{\partial z} & \Delta + \delta_{3} \frac{\partial^{2}}{\partial z^{2}} & -\delta_{4} \frac{\partial}{\partial z} & -\delta_{3} \frac{\partial}{\partial z} \\
-\delta_{1} \frac{\partial^{2}}{\partial z^{2}} + \delta_{2} \frac{\partial^{2}}{\partial z^{2}} + \delta_{3} \frac{\partial^{2}}{\partial z^{2}} & \delta_{2} \frac{\partial}{\partial z} + \delta_{4} \frac{\partial}{\partial z} & \delta_{1} \frac{\partial^{2}}{\partial z^{2}} + \delta_{4} \frac{\partial}{\partial z} & \delta_{1} \frac{\partial^{2}}{\partial z^{2}} + \delta_{4} \frac{\partial}{\partial z} \\
0 & 0 & 0 & \Delta + \delta_{1} \frac{\partial^{2}}{\partial z^{2}}
\end{pmatrix}$$

Equation (22) is a homogeneous set of differential equations in $G,w,T,C$. The general solution by the operator theory is as follows:

$$G = A_{i} F, \quad w = A_{i} F, \quad C = A_{i} F, \quad T = A_{i} F \quad (i=1,2,3,4).$$

(23)

The determinant of the matrix $D$ is given as:

$$|D| = \left( \frac{\partial}{\partial z} \frac{\partial^{6}}{\partial z^{2}} + \bar{b} \Delta \frac{\partial^{4}}{\partial z^{2}} + \bar{c} \Delta' \frac{\partial^{2}}{\partial z^{2}} + \bar{d} \Delta'' \right) \times \left( \Delta + \delta_{i} \frac{\partial^{2}}{\partial z^{2}} \right).$$

(24)

Where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ and $\Delta$ are given in Appendix A. The function $F$ in Equation (23) satisfies the following homogeneous equation:

$$|D|F = 0$$

(25)

It can be seen that if $i=1,2,3$ are taken in Equation (23), three general solutions are obtained in which $T = 0$. These solutions are identical to those without thermal factor and are not discussed here. Therefore if $i = 4$ should be taken in Equation (23), the following solution is obtained:
As known from the generalized Almansi theorem (Ding et al., 1996) the function $F$ can be expressed in terms of four harmonic functions:

1) $F = F_1 + F_2 + F_3 + F_4$ for distinct $s_j (j = 1, 2, 3, 4)$,

2) $F = F_1 + F_2 + F_3 + zF_4$ for $s_1 \neq s_2 \neq s_3 = s_4$,

3) $F = F_1 + F_2 + zF_3 + z^2F_4$ for $s_1 \neq s_2 = s_3 = s_4$,

4) $F = F_1 + zF_2 + z^2F_3 + z^3F_4$ for $s_1 = s_2 = s_3 = s_4$,

where $F_j$ satisfies the following harmonic equation

$$\left( \frac{\partial^2}{\partial x^2} + \sum_{j=1}^{4} \frac{\partial^2}{\partial z_j^2} \right) F_j = 0 \quad (j = 1, 2, 3, 4).$$

(35)

The general solution for the case of distinct roots can be derived as follows:

$$u = \frac{\partial \Psi}{\partial y} - \sum_{j=1}^{4} \frac{\partial^2 F_j}{\partial x \partial z_j}, \quad v = \frac{\partial \Psi}{\partial x} - \sum_{j=1}^{4} \frac{\partial^2 F_j}{\partial y \partial z_j},$$

$$w = \sum_{j=1}^{4} s_j p_{2j} \frac{\partial^5 F_j}{\partial z_j^5}, \quad C = \sum_{j=1}^{4} s_j p_{1j} \frac{\partial^4 F_j}{\partial z_j^4}, \quad T = \sum_{j=1}^{4} s_j p_{3j} \frac{\partial^6 F_j}{\partial z_j^6}.$$  

(36)

Where

$$p_{kj} = \bar{a}_k - \bar{b}_k s^2_j + \bar{c}_k s^4_j \quad (k = 1, 2)$$

$$p_{2j} = -\bar{a}_4 + \bar{b}_4 s^2_j - \bar{c}_4 s^4_j + \bar{d}_4 s^6_j$$

$$p_{4k} = -\bar{d} + \bar{e} s^2_j - \bar{f} s^4_j + \bar{g} s^6_j$$

In the similar way general solution for the other three cases can be derived. Equation (36) can be further simplified by taking

$$p_{kj} \frac{\partial^4 F_j}{\partial z_j^4} = \psi_j, \quad (j = 1, 2, 3, 4)$$

(37)

and writing $\psi_0 = \psi$.

$$u = \frac{\partial \psi}{\partial y} - \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial x}, \quad v = \frac{\partial \psi}{\partial x} - \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial y}, \quad w = \sum_{j=1}^{4} s_j p_{1j} \frac{\partial \psi_j}{\partial z_j}$$

where

$$C = \sum_{j=1}^{4} P_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad T = P_{3k} \frac{\partial^2 \psi_k}{\partial z_k^2},$$

$$\sigma_1 = \sum_{j=1}^{4} (\epsilon s - r_j s^2_j) \Delta \psi_j, \quad \sigma_2 = -2 \epsilon s \Delta F_j \left( \psi_0 + \sum_{j=1}^{4} \psi_j \right),$$

$$\sigma_{zz} = \sum_{j=1}^{4} r_j \Delta \psi_j, \quad \sigma_{zz} = \sum_{j=1}^{4} r_j \frac{\partial \psi_j}{\partial z_j} - i \omega s \frac{\partial \psi_0}{\partial z_0}.$$  

(41)
Boundary Conditions of Cone

We consider a transversely isotropic thermoelastic diffusion cone \( z = \cot \alpha \), where \( 2\alpha \) is the apex angle, whose isotropic plane is perpendicular to \( z \) axis. At the origin of the coordinate system, the apex is to be taken.

At the apex, a concentrated force \( P = p_1 i + p_2 j + p_3 k \), a concentrated moment \( M = M_1 i + M_2 j + M_3 k \) and a point heat source \( H \) are applied, where \( i, j, k \) are three unit vectors of Cartesian coordinates \((x, y, z)\).

In addition, the cone is loaded on the surface with prescribed density of normal heat flux \( \bar{q}_n \) and surface forces \( X = \bar{X}_1 e_r + \bar{X}_2 e_\theta + \bar{X}_3 e_z \), where \( e_r, e_\theta, e_z \) are three unit vectors of cylindrical coordinates \((r, \theta, z)\), which are related to \( i, j, k \) by the following relations:

\[
e_r = i \cos \theta + j \sin \theta, \quad e_\theta = i \sin \theta + j \cos \theta, \quad e_z = k.
\]

The boundary conditions in cylindrical coordinates on the cone \( z/r = \cot \alpha \) are:

\[
\sigma_{rr} \cos \alpha - \sigma_{r\theta} \sin \alpha = \bar{X}_r, \\
\sigma_{r\theta} \cos \alpha - \sigma_{\theta\theta} \sin \alpha = \bar{X}_\theta, \\
\sigma_{r\theta} \cos \alpha - \sigma_{\theta\theta} \sin \alpha = \bar{X}_\theta.
\]

\[
K_1 \frac{\partial T}{\partial r} \cos \alpha - K_3 \frac{\partial C}{\partial z} \sin \alpha = \bar{q}_m, \\
K_1 \frac{\partial T}{\partial r} \cos \alpha - K_3 \frac{\partial C}{\partial z} \sin \alpha = \bar{q}_m.
\]

As shown in Figure 1, when a segment of cone cut off by \( z = b \), its global mechanical concentration and thermal equilibrium equations will be:

\[
P + \int_0^{2\pi} \left[ (\sigma_{rr} + \sigma_{r\theta} + \sigma_{\theta\theta}) \right] r dr d\theta + \int_0^{2\pi} \left[ \bar{X}_1 e_r + \bar{X}_2 e_\theta + \bar{X}_3 e_z \right] d r d\theta \tan \alpha / \cos \alpha = 0,
\]

\[
M + \int_0^{2\pi} \left[ (c \sigma_{rr} e_r + (b \sigma_{r\theta} - \sigma_{\theta\theta}) e_\theta + c \sigma_{\theta\theta} e_\theta) \right] r dr d\theta + \int_0^{2\pi} \left[ \bar{X}_1 e_r + \bar{X}_2 e_\theta + \bar{X}_3 e_z \right] d r d\theta \tan \alpha / \cos \alpha = 0.
\]
\[ u_x = \sum_{j=1}^{4} A_j \frac{r}{R_j^2}, \quad w = \sum_{j=1}^{4} s_j P_j A_j \log R_j', \quad C = \sum_{j=1}^{4} P_j A_j, \] (57)

\[ T = P_m A_1, \quad \sigma_{r\theta} = \frac{\sum_{j=1}^{4} A_j - \sum_{j=1}^{4} w_j A_j}{R_j}, \quad \sigma_{rr} = \sum_{j=1}^{4} \frac{r}{R_j}. \] (58)

\[ \sigma_{00} = 2c_i \sum_{j=1}^{4} A_j - \sum_{j=1}^{4} (s_j' w_j - 2c_i A_j), \quad \sigma_{zr} = \sum_{j=1}^{4} s_j r A_j \frac{r}{R_j/R_j}. \] (59)

For non-torsional axisymmetric problem, the boundary condition in Equation (48) has been satisfied, and Equations (49) to (51) can be deduced from the global mechanical, impermeable and thermal equilibrium condition in Equations (52). The only boundary condition in Equation (47) and the following equations need to be satisfied:

\[ \int_0^{2\pi} \int_0^\infty \sigma_{r\theta} r dr d\theta = 0, \] (60)

\[ K_3 \int_0^{2\pi} \int_0^\infty \frac{\partial T}{\partial z} r dr d\theta = -H, \] (61)

\[ \int_0^{2\pi} \int_0^\infty \frac{\partial C}{\partial z} r dr d\theta = 0. \] (62)

Substituting the values of \( \sigma_{r\theta}, \sigma_{rr}, C \) and \( T \) from Equation (57) in Equations (47) and (60 to 62) yields

\[ \sum_{j=1}^{4} A_j \left( 2c_i s_j w_j + s_j' r_j \frac{1}{W_j} - s_j r_j \frac{1}{W_j V_j} \right) = 0, \] (63)

\[ \sum_{j=1}^{4} \frac{r_j}{H_j} A_j = 0, \] (64)

\[ \sum_{j=1}^{4} \frac{s_j}{H_j \tan \alpha} \left( s_j P_{2j} A_j \right) = 0, \] (65)

\[ \left( \frac{s_j}{H_j \tan \alpha} - 1 \right) s_j P_{2j} A_j = -\frac{H}{2\pi K_3}. \] (66)

Where

\[ H_j = \sqrt{1 + s_j^2 / \tan^2 \alpha}, \quad N_j = H_j + s_j / \tan \alpha \quad (j = 1, 2, 3). \]

The constants \( A_j(j = 1, 2, 3, 4) \) can be determined by solving Equations (63) to (66). When the cone has been
reduced to a semi-infinite body, that is, \( \alpha = \frac{\pi}{2} \) then

\[ H_j = N_j = 1 \quad (j = 1, 2, 3, 4) \]

(67)

Using Equation (49) in Equations (45) to (48) can be simplified as:

\[
\sum_{j=1}^{4} A_j s_j r_j = 0, \tag{68}
\]

\[
\sum_{j=1}^{4} r_j A_j = 0, \tag{69}
\]

\[
\sum_{j=1}^{4} s_j P_{2j} A_j = 0, \tag{70}
\]

\[ A_3 = \frac{H}{2\pi K_3 s_4 P_{24}} \tag{71} \]

We have determined four constants \( A_j (j = 1, 2, 3) \) from three equations including Equations (68) to (71) by the method of Cramer’s rule.

**Special case**

In the absence of diffusion effects, that is, \( b_1 = b_3 = a = b = 0 \). Equations (57) to (59) yields

\[
\begin{align*}
 u_r &= \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{r}{s_j} \sigma_r = 2c_{66} \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{s_j}{s_j} \sigma_z = \frac{3}{3} \frac{A_j}{R_j} \\
\sigma_{zz} &= 2c_{66} \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{r}{s_j} \sigma_{zz} = 2c_{66} \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{s_j}{s_j} \sigma_{zz} = \frac{3}{3} \frac{A_j}{R_j} \\
\sigma_{00} &= 2c_{66} \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{r}{s_j} \sigma_{00} = 2c_{66} \sum_{j=1}^{3} \frac{A_j}{R_j} \frac{s_j}{s_j} \sigma_{00} = \frac{3}{3} \frac{A_j}{R_j}
\end{align*}
\]

(72)

where \( s_1, s_2, s_3, s_4 \) in this case are reduces to \( s_1, s_2, s_3 \)

\[
s_3 = \sqrt{\frac{K_1}{K_3}}
\]

with \( s_1, s_2 \) are two roots (with positive real part) of the equation

\[
\dot{a}s^4 - \dot{b}s^2 + \dot{c} = 0, \tag{73}
\]

and

\[
\dot{a} = \delta_4 \dot{b}, \dot{b} = \delta_2^2 - \dot{d}_2^2, \dot{c} = \delta_3\]

\[
\begin{align*}
\dot{p}_{2j} &= \frac{\dot{p}_{2j}}{p_{1j}}, j = 1, 2, 3, 4 \\
\dot{p}_{4j} &= \dot{a}_k - b_k s_j^2 (k = 1, 2)
\end{align*}
\]

\[
\begin{align*}
\dot{a}_k &= \delta_2 \dot{b}, \dot{b} = \delta_2 \dot{c}, \dot{c} = \delta_3
\end{align*}
\]

Consider the continuity at plane \( z = 0 \) for \( u_z \) and \( \sigma_{zz} \) and substituting the values of \( \sigma_{zz}, \sigma_{zz} \) and \( T \) from Equations (64) with the aid of \( s_3 = \sqrt{K_1/K_3} \) yield the following equations in the absence of diffusion:

\[
\sum_{j=1}^{3} s_j \dot{p}_{2j} A_j = 0. \tag{74}
\]

\[
\sum_{j=1}^{3} s_j A_j = 0. \tag{75}
\]

The constants \( A_j (j = 1, 2) \) are determined by two Equations (74) and (75) using the method of Cramer’s rule.

The above results are similar as obtained by Hou et al. (2005).

**NUMERICAL RESULTS AND DISCUSSION**

Here, the numerical discussions are reported and analysis is conducted for magnesium material. Following Dhaliwal and Singh (2005), the values of physical constants are taken as:

\[
\begin{align*}
c_{11} &= 5.974 \times 10^{10} \text{N.m}^{-2}, c_{12} = 2.624 \times 10^{10} \text{N.m}^{-2}, c_{13} = 2.17 \times 10^{10} \text{N.m}^{-2}, \\
c_{33} &= 6.17 \times 10^{10} \text{N.m}^{-2}, c_{44} = 3.278 \times 10^{10} \text{N.m}^{-2}, T_0 = 298 \times 10^3 \text{K}, \\
a_1 &= 2.68 \times 10^{10} \text{Nm}^{-2} \text{K}^{-1}, a_2 = 2.68 \times 10^{10} \text{Nm}^{-2} \text{K}^{-1}, K_1 = 1.7 \times 10^{12} \text{Wm}^{-1} \text{K}^{-1}, \\
a_2 &= 2.1 \times 10^{12} \text{m}^{-2} \text{K}^{-1}, a_2 = 2.5 \times 10^{12} \text{m}^{-2} \text{K}^{-1}, a_2 = 2.4 \times 10^{12} \text{m}^{-2} \text{K}^{-1}, \\
a_1 &= 9.5 \times 10^8 \text{m}^{-2} \text{s.Kg}^{-1}, a_2 = 9.5 \times 10^8 \text{m}^{-2} \text{s.Kg}^{-1}
\end{align*}
\]

Figures 2 to 5 depict the variations of radial displacement \( u_r \), axial displacement \( u_z \), temperature change \( T \) and
mass concentration $C$ w.r.t. $r$ for thermoelastic diffusion material. The solid and dotted line respectively, corresponds to thermoelastic theory (WTD $z = 5$), (WTD $z = 10$) and centre symbols on these lines, respectively corresponds to thermoelastic theory with mass diffusion (WD $z = 5$), (WD $z = 10$).

Figure 2 shows that the values of $u_r$ in case of WTD slightly decrease for smaller values of $r$ and for higher values of $r$, the values of $u_r$ become dispersionless, although for the case of WD, the values of $u_r$ increase for all values of $r$. It is noticed that the values of $u_r$ in case of WD remain more in comparison with WTD.

Figure 3 depicts that the values of $u_z$ in case of WTD decrease for all values of $r$, whereas for the case of WD,
the values of $u^z$ slightly increase for smaller values of $r$ and finally becomes constant.

It is evident that the values of $u^z$ in case of WD remain more in comparison with WTD. Figure 4 shows that the values of $T$ in case of WTD slightly decreases for all values of $r$. Although for the case of WD, the values of $T$ increase for all values of $r$. It is noticed that the values of $T$ in case of WD remain more in comparison with WTD. Figure 5 depicts that the values of $C$ in case of $z=5$ slightly decrease for all values of $r$, whereas for the case of $z=10$ the values of $C$ increases for all values of $r$. It is evident that the values of $T$ in case of $z=5$ remain more in comparison with $z=10$.

**Conclusion**

The Green's functions for three-dimensional problem in transversely isotropic thermoelastic diffusion medium have been derived for static case. After applying the dimensionless quantities and using the operator theory, we have obtained the general expression for components of displacement, temperature distribution, mass concentration and stress components in Cartesian as well as in cylindrical coordinates. Based on the obtained general solution, the three-dimensional Green's function for a study point heat source on the apex of a transversely isotropic thermoelastic cone in case of steady state problem are derived by four newly introduced harmonic functions. All components of thermoelastic field are expressed in terms of elementary functions and are convenient to use.

From the present investigation, a special case of interest is deduced to depict the effect of diffusion. From numerical results, we conclude that the values of horizontal displacement $u^r$, axial displacement $u^z$ and temperature change $T$ remain more in case of thermoelastic diffusion (WD) in comparison to thermoelastic medium (WTD).

**Conflict of Interest**

The authors have not declared any conflict of interest.

**REFERENCES**


Appendix A

$$\pi = \delta_1(\gamma q_4 - \delta q_4), \; \bar{\pi} = (\delta_1^2 - \delta_2^2)b_{q_4} + \delta_1(q_3 - q_4) + \bar{\delta}_1(\gamma q_4 - \delta q_4) - \delta_2q_4^2
$$

$$c = (\delta_1^2 - \delta_2^2)q_1 + q_1(\gamma - \delta) - \epsilon_1q_2^2 + \delta(q_1 - q_4) + \bar{\delta}_1(q_1 - q_4), \; d = \delta_1(q_1 - q_4)$$

$$\Delta = \frac{\delta_1^2}{\delta_2^2 + \delta_1^2}$$

Appendix B

$$\bar{\alpha}_1 = (q_4^* - q_4)\delta_1, \; \bar{\alpha}_2 = \delta_1(q_3^* - q_3) + \delta_2(q_3 - q_4) + \epsilon_1(\delta_1 q_4^* - q_3) - \gamma_2 q_3^*$$

$$\bar{\alpha}_5 = (\gamma q_4 + \epsilon_2 q_4^* - q_4)\delta_2 + (q_3 - q_3^*) \bar{\delta}_2 - q_3^* \bar{\delta}_2$$

$$\bar{\alpha}_2 = (q_4^* + q_4)\gamma_1 + \epsilon_2(q_3 - q_4) + \delta(q_4^* - q_4), \; \bar{\alpha}_3 = \delta_2(q_3 - q_4)$$

$$\bar{\alpha}_3 = (q_4^* - q_4)\delta_3, \; \bar{\alpha}_4 = (\delta_1^2 - \delta_2^2)q_1 + \delta_1(q_1 - q_4) + \delta_2(q_1 + q_4) - \epsilon_2q_1^*$$

$$\bar{\alpha}_5 = (\delta_1^2 - \delta_2^2)q_5^* + \delta_3(q_5^* + q_5^*) - \delta_2(\epsilon_2 q_5^* + 1) - \delta_1q_5^*$$
Multivalent harmonic uniformly convex functions

R. M. EL-Ashwah¹, M. K. Aouf² and F. M. Abdulkarem¹*

¹Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt.
²Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

Received 10 March, 2014; Accepted 13 October, 2014

In this paper, several properties of the multivalent harmonic uniformly convex classes \( k_u(m, a) \) and \( k_k(m, a) \) were investigated. Coefficient bounds, distortion theorem, extreme points, convolution condition, convex combinations and integral operator for these classes were obtained.

Key words: Harmonic, multivalent functions, convex, convolution.

INTRODUCTION

A continuous complex valued function \( f = u + iv \) which is defined in a simply connected complex domain \( D \) is said to be harmonic in \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain we can write:

\[
f(z) = h(z) + g(z),
\]

where \( h \) and \( g \) are analytic in \( D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( D \) is that \( |h(z)| \leq g(z) | \) in \( D \) (Clunie and Sheil-Small, 1984).

Denote by \( S_H \) the class of functions \( f \) of the form (2) that are harmonic univalent and sense preserving in the unit disc \( U = \{ z ; |z| < 1 \} \) for which \( f(0) = f_1(0) = 0 \). For \( f = h + g \in S_H \), we may express:

\[
f(z) = z + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k, |b_k| < 1.
\]

(2)

where the analytic functions \( h \) and \( g \) are of the form:

\[
h(z) = z + \sum_{k=1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_k| < 1.
\]

(3)

Clunie and Sheil-Small (1984) investigated the class \( S_H \) as well as its geometric subclasses and some coefficient bounds for functions in \( S_H \) were obtained. Since then, various subclasses of \( S_H \) were investigated by several authors (Al-Shaqsi and Darus, 2008; Chandrashekar et al., 2009; Jahangiri, 1999; Murugusundaramoorthy, 2003; Murugusundaramoorthy et al., 2009; Rosy et al., 2001).

Recently, Kanas and Wiśniowska (1999), Kanas and Srivastava (2000) studied the class of \( k \)-uniformly convex analytic functions. For \( m \geq 1 \) and \( 0 \leq \alpha < 1 \), we let
\( H(m) \) denote the class of multivalent harmonic functions
\( f(z) = h(z) + g(z) \), where

\[
h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=m}^{\infty} b_k z^k, \quad |b_m| < 1.
\]

(4)

We consider the class \( K_\nu(m, \alpha) \) of functions of the form (1) where \( h \) and \( g \) are given by Equation (4) satisfying the inequality

\[
\Re \left( \frac{z^2 h'(z) + z h(z) + z^2 g'(z) + z g(z)}{zh'(z) - zg'(z)} \right) \geq \left| \frac{z^2 h'(z) + z h(z) + z^2 g'(z) + z g(z)}{zh'(z) - zg'(z)} - m \right| + m \alpha,
\]

(5)

where \( m \geq 1 \) and \( 0 \leq \alpha < 1 \).

Using the fact that \( \Re(w) > |w - m| + m \alpha \Leftrightarrow \Re[(1+e^{i\varphi})w - me^{i\varphi}] \geq m \alpha \), it follows from the condition (5) that \( f \) is in the class \( K_\nu(m, \alpha) \) if and only if

\[
\Re \left( m + (1+e^{i\varphi}) \left( \frac{z^2 h'(z) + z h(z) + z^2 g'(z) + z g(z)}{zh'(z) - zg'(z)} - m \right) \right) \geq m \alpha,
\]

(6)

where \( m \geq 1 \) and \( 0 \leq \alpha < 1 \).

We note that:

Putting \( m = 1 \), \( K_\nu(1, \alpha) = HCV(1, \alpha) \) (Kim et al., 2002).

Further, let \( \overline{K}_\nu(m, \alpha) \) be the subclass of \( K_\nu(m, \alpha) \) consisting of functions of the form:

\[
f(z) = z^m - \sum_{k=m+1}^{\infty} |a_k| z^k - \sum_{k=m}^{\infty} |b_k| z^k \quad (|b_m| < 1).
\]

(7)

Recent interest in the study of multivalent harmonic functions in the plan prompted the publication of several articles, such as Ahuja and Jahangiri (2001, 2002, 2003), Bshouty et al. (1999), Guney and Ahuja (2006).

In this paper, the coefficient bounds for the classes \( K_\nu(m, \alpha) \) and \( \overline{K}_\nu(m, \alpha) \) as well as distortion theorem, extreme points, convolution, convex combinations and integral operator for functions in the class \( \overline{K}_\nu(m, \alpha) \) were obtained.

COEFFICIENTS BOUNDS AND DISTORTION THEOREM

Unless otherwise mentioned, it was assumed in the course of this study that \( 0 \leq \alpha < 1, m \geq 1 \) and \( z \in U \). We began with a sufficient condition for functions in the classes \( K_\nu(m, \alpha) \) and \( \overline{K}_\nu(m, \alpha) \) and obtained distortion theorem for functions in the class \( \overline{K}_\nu(m, \alpha) \).

Theorem 1

Let \( f = h + g \), where \( h \) and \( g \) are given by Equation (4), and satisfy the condition

\[
\sum_{k=m+1}^{\infty} k [2k - m(1+\alpha)] m(1-\alpha) + 1 - |m(1-\alpha) - 1| \left| a_k \right| \leq \frac{1}{2}.
\]

(8)

Then \( f(z) \in K_\nu(m, \alpha) \).

Proof

Assume that Equation (8) holds. It suffices to prove that

\[
\Re \left( m + (1+e^{i\varphi}) \left( \frac{z^2 h'(z) + z h(z) + z^2 g'(z) + z g(z)}{zh'(z) - zg'(z)} - m \right) \right) = \Re \left( \frac{A(z)}{B(z)} \right) \geq 0.
\]

(9)

Using the fact that \( \Re \left( w \right) \geq 0 \) if and only if \( |1+w| \geq |1-w| \), it suffices to show that

\[
|A(z) + B(z)| - |A(z) - B(z)| \geq 0,
\]

(10)

where

\[
A(z) = [(1-m)(1+e^{i\varphi})+m(1-\alpha)]zh'(z) + (1+e^{i\varphi})z^2 h^*(z)
\]

\[
+ [(1+m)(1+e^{i\varphi})-m(1-\alpha)]zg'(z) + (1+e^{i\varphi})z^2 g^*(z)
\]

and

\[
B(z) = zh'(z) - zg'(z).
\]

Substituting for \( A(z) \) and \( B(z) \) in the left side of Equation (10) we obtain:

\[
|A(z) + B(z)| - |A(z) - B(z)| \leq |m[1+m(1-\alpha)]z^m + \sum_{k=m+1}^{\infty} k [k - m\alpha + (k - m)e^{i\varphi} + 1]a_k z^k|
\]

\[
= |m[1+m(1-\alpha)]z^m + \sum_{k=m+1}^{\infty} k [k - m\alpha + (k - m)e^{i\varphi} + 1]a_k z^k|.
\]

\[
= |m[1+m(1-\alpha)]z^m + \sum_{k=m+1}^{\infty} k [k - m\alpha + (k - m)e^{i\varphi} + 1]a_k z^k|.
\]
holds, then \( f(z) \in K_H(m, \alpha) \).

**Corollary 2**

Let \( f = h + g \), where \( h \) and \( g \) are given by Equation (4). Also, let \( m \geq 1/(1-\alpha) \) and if the condition

\[
\sum_{k=-m}^{\infty} |k[2k-m(1+\alpha)]/m| \leq 1,
\]

holds, then \( f(z) \in K_H(m, \alpha) \).

In the following theorem, it is shown that the condition (12) is also necessary for function \( f = h + g \), where \( f \) is of the form (7).

**Theorem 2**

Let \( f = h + g \) be given by the form (7). Then \( f(z) \in K_H(m, \alpha) \), if and only if the coefficient bound (12) holds.

**Proof**

Since \( K_H(m, \alpha) \subseteq K_H(m, \alpha) \), we only need to prove this part of the theorem. To this end, for functions \( K_H(m, \alpha) \), it was noticed that the necessary and sufficient condition to be in the class \( K_H(m, \alpha) \) is that:

\[
\Re \left\{ \sum_{k=-m}^{\infty} (z^k h(z) + z^{-k} g(z)) \right\} \geq 0.
\]

This is equivalent to

\[
\Re \left\{ \sum_{k=-m}^{\infty} (z^{k+\alpha} + z^{-1}) \sum_{k=-m}^{\infty} k \left| \frac{2k}{m} \right| \left| \frac{k+\alpha}{m} \right| \left| \frac{2k+1}{m} \right| \right\} \geq 0.
\]

This condition must hold for all values of \( z \in U \) and for real \( \alpha \) so that on taking \( z = r < 1 \), the above inequality reduces to:

\[
1 - \sum_{k=-m}^{\infty} \left| \frac{k[2k-m(1+\alpha)]}{m(1-\alpha)} \right| k \left| \frac{k+\alpha}{m} \right| \left| \frac{2k+1}{m} \right| \right\} \geq 0.
\]

This completes the proof of Theorem 2.
Theorem 3

Let the function \( f(z) \) given by Equation (7) be in the class \( K_{\alpha}(m, \alpha) \), then for \( \alpha < r < 1 \)

\[
|f(z)| \leq \left| (1+\|a\|)^{m} + \frac{\|a\|^{m+1}}{2m(1-\alpha)} \sum_{k=1}^{\infty} \left( \frac{1}{k(1-\alpha)} - m \right) \right|^{n+1}, \quad m(1-\alpha) \leq 1
\]

and

\[
|f(z)| \geq \left| (1+\|a\|)^{m} \right|^{n+1}, \quad m(1-\alpha) \geq 1,
\]

where

\[
|p_m| \leq \frac{(1-\alpha)}{(3+\alpha)}.
\]

Proof

If \( m(1-\alpha) \leq 1 \), we have,

\[
|f(z)| \leq (1+\|a\|)^{m} + \sum_{k=1}^{\infty} \left( m_{k} \right) + \|p_{m}\| r^{m+1}
\]

\[
\leq (1+\|a\|)^{m} + \sum_{k=1}^{\infty} \left( m_{k} \right) + \|p_{m}\| r^{m+1},
\]

\[
\leq (1+\|a\|)^{m} + \frac{m(1-\alpha)}{(m+1)(2m+1-\alpha)} \sum_{k=1}^{\infty} \left( \frac{m+1}{m(1-\alpha)} \right) \left( \frac{k}{m(1-\alpha)} \right) \left( |p_{k}| \right) r^{m+1}
\]

\[
\leq (1+\|a\|)^{m} + \frac{m(1-\alpha)}{(m+1)(2m+1-\alpha)} \sum_{k=1}^{\infty} \left( \frac{m+1}{m(1-\alpha)} \right) \left( \frac{k}{m(1-\alpha)} \right) \left( |p_{k}| \right) r^{m+1},
\]

which proves the assertion (16) of Theorem 3. The proof of the assertion (17) is similar, thus, it was omitted.

Remark 1

Putting \( m=1 \) in Theorem 3, we improve the result obtained by Kim et al. (2002) by adding the condition

\[
|p_{m}| \leq \frac{(1-\alpha)}{(3+\alpha)}.
\]

The following covering result follows the left hand inequality Theorem 3.

Corollary 3

Let the function \( f(z) \) given by (7) be in the class \( K_{\alpha}(m, \alpha) \), then for \( |z|=r<1 \), we have:

\[
|f(z)| \leq \left( \frac{2m(1-\alpha)}{(m+1)(2m+1-\alpha)} \right) \left( |p_{m}| \right) , \quad m(1-\alpha) \leq 1
\]

\[
|f(z)| \geq \left( \frac{2m(1-\alpha)}{(m+1)(2m+1-\alpha)} \right) \left( |p_{m}| \right) , \quad m(1-\alpha) \geq 1
\]

where

\[
|p_{m}| \leq \frac{2+k(3-\alpha)}{2m(3-2m-\alpha)},
\]

or

\[
|p_{m}| \leq \frac{1+m[3m(1-\alpha)-\alpha]}{2m(3-2m-\alpha)(2m+1)}.
\]

EXTREME POINTS

Here, the extreme points of the closed convex hull of the class \( K_{\alpha}(m, \alpha) \) denoted by \( clcK_{\alpha}(m, \alpha) \) was determined.

Theorem 4

Let \( f(z) \) be given by (7), then \( f(z) \in clcK_{\alpha}(m, \alpha) \) if and only if

\[
f(z) = \sum_{k=1}^{\infty} \left( x_{k} h_{k}(z) + y_{k} g_{k}(z) \right),
\]

where

\[
h_{k}(z) = z^{k},
\]

\[
g_{k}(z) = \left[ \frac{m(1-\alpha)}{k} \right] z^{k},
\]

or

\[
g_{k}(z) = \left[ \frac{m(1-\alpha)}{k} \right] z^{k}.
\]

And

\[
\sum_{k=1}^{\infty} \left( x_{k} + y_{k} \right) = 1, \quad x_{k} \geq 0 \quad and \quad y_{k} \geq 0.
\]

In particular, the extreme points of the class \( K_{\alpha}(m, \alpha) \) are \( \{h_{k}\}_{k \geq m+1} \) and \( \{g_{k}\}_{k \geq m} \), respectively.
Proof

For a function \( f(z) \) of the form (18), we have:

\[
f(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k! [2k - m(1 - \alpha)]} + \sum_{k=0}^{\infty} b_k \frac{z^k}{k! [2k + m(1 + \alpha)]}
\]

\[
= z^n - \sum_{k=0}^{\infty} k \frac{m^{-1} z^{-1}}{k! [2k - m(1 - \alpha)]} x^k + \sum_{k=0}^{\infty} k \frac{m^{-1} z^{-1}}{k! [2k + m(1 + \alpha)]} y^k,
\]

but,

\[
\sum_{k=0}^{\infty} k \frac{m^{-1} z^{-1}}{k! [2k - m(1 - \alpha)]} x^k + \sum_{k=0}^{\infty} k \frac{m^{-1} z^{-1}}{k! [2k + m(1 + \alpha)]} y^k = 1 - x_k \leq 1,
\]

and \( f(z) \in clco \overline{K}_H(m, \alpha). \)

Conversely, assume that \( f(z) \in clco \overline{K}_H(m, \alpha). \) Then

\[
a_k = m^{-1} \left( \frac{1 - \alpha}{k! [2k - m(1 + \alpha)]} \right),
\]

and

\[
b_k = m^{-1} \left( \frac{1 - \alpha}{k! [2k + m(1 + \alpha)]} \right)
\]

set

\[
x_k = k \frac{[2k - m(1 + \alpha)]}{m^2(1 - \alpha)} \left| \frac{a_k}{1} \right|,
\]

and

\[
y_k = k \frac{[2k + m(1 + \alpha)]}{m^2(1 - \alpha)} \left| \frac{b_k}{1} \right|
\]

Then by using Equation (12), we have \( 0 \leq x_k \leq 1(k = m + 1, m + 2, \ldots) \) \( 0 \leq y_k \leq 1(k = m, m + 1, \ldots). \)

\[
x_m = 1 - \sum_{k=m+1}^{\infty} x_k = \sum_{k=m}^{\infty} y_k,
\]

is defined and the equation:

\[
f(z) = \sum_{k=0}^{\infty} (x_k b_{k+1} + y_k g_k)\]

is obtained. This completes the proof of Theorem 4.

CONVOLUTION AND CONVEX COMBINATION

In this section, the convolution properties and convex combination were determined.

Let the functions \( f_j(z) \) be defined by:

\[
f_j(z) = z^n - \sum_{k=1}^{\infty} |a_{k,j}| |z|^{-1} - \sum_{k=1}^{\infty} |b_{k,j}| |z|^{-1} (j = 1, 2),
\]

be in the class \( \overline{K}_H(m, \alpha) \), we denote by \( f_1 * f_2(z) \) the convolution or (Hadamard Product) of the function \( f_1(z) \) and \( f_2(z) \), that is,

\[
(f_1 * f_2)(z) = z^n - \sum_{k=1}^{\infty} |a_{k,1}||a_{k,1}| |z|^{-1} - \sum_{k=1}^{\infty} |b_{k,1}||b_{k,1}| |z|^{-1}.
\]

while the integral convolution is defined by

\[
(f_1 \ast f_2)(z) = z^n - \sum_{k=1}^{\infty} m \frac{|a_{k,1}|}{k} |z|^{-1} - \sum_{k=1}^{\infty} m \frac{|b_{k,1}|}{k} |z|^{-1}.
\]

We first show that the class \( \overline{K}_H(m, \alpha) \) is closed under convolution.

Theorem 5

For \( 0 \leq \delta \leq \alpha < 1 \), let the functions \( f_j(z) \in \overline{K}_H(m, \alpha) \) and \( f_j(z) \in \overline{K}_H(m, \delta) \).

Then

\[
(f_1 * f_2)(z) \in \overline{K}_H(m, \alpha) \subset \overline{K}_H(m, \delta),
\]

\[
(f_1 \ast f_2)(z) \in \overline{K}_H(m, \alpha) \subset \overline{K}_H(m, \delta).
\]

Proof

Let \( f_j(z)(j = 1, 2) \) given by Equation (19), where \( f_j(z) \) is in the class \( \overline{K}_H(m, \alpha) \) and \( f_j(z) \) be in the class \( \overline{K}_H(m, \delta) \). It therefore shows that the coefficients of \( f_1 * f_2(z) \) satisfy the required condition given in Equation (12). For \( f_j(z) \in \overline{K}_H(m, \delta) \), we note that \( |a_{k,1}| < 1 \) and \( |b_{k,1}| < 1 \).

Now for the convolution functions \( f_1 * f_2(z) \), we obtain

\[
\sum_{k=1}^{\infty} m \frac{|a_{k,1}|}{k} |z|^{-1} - \sum_{k=1}^{\infty} m \frac{|b_{k,1}|}{k} |z|^{-1} \leq \sum_{k=1}^{\infty} m \frac{|a_{k,1}|}{k} |z|^{-1} - \sum_{k=1}^{\infty} m \frac{|b_{k,1}|}{k} |z|^{-1} \leq 1.
\]
since \( 0 \leq \delta \leq \alpha < 1 \) and \( f_i(z) \in \overline{K}_n(m, \alpha) \).

Thus \( (f_1, f_2)(z) \in \overline{K}_n(m, \alpha) \subset \overline{K}_n(m, \delta) \). The proof of the assertion (23) is similar, thus, it was omitted. This completes the proof of Theorem 5.

Next we show that \( \overline{K}_n(m, \alpha) \) is closed under convex combinations of its members.

\[ \begin{align*}
\text{Theorem 6} \\
\text{The class } \overline{K}_n(m, \alpha) \text{ is closed under convex combination.}
\end{align*} \]

**Proof**

For \( i = 1, 2, \ldots \), let \( f_i(z) \in \overline{K}_n(m, \alpha) \), where

\[ f_i(z) = z^m - \sum_{k=m}^{\infty} \left[ \sum_{l=0}^{\infty} |a_{i, l}| z^l \right] \sum_{l=1}^{\infty} \left[ \sum_{j=1}^{\infty} |b_{i, j}| z^j \right] (z \in U; i = 1,2,\ldots), \tag{24} \]

then from (12), for \( \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i(z) \) may be written as:

\[ \sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{k=m}^{\infty} \left[ \sum_{l=0}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i, l}| \right) \right] z^l - \sum_{k=m}^{\infty} \left[ \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{i, j}| \right) \right] z^j. \tag{25} \]

Then by using Equation (12), we have

\[ \begin{align*}
\sum_{k=m}^{\infty} k [2k - m(1 + \alpha)] & \left( \sum_{i=1}^{\infty} t_i |a_{i, l}| \right) \sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} |b_{i, j}| \right] \\
\leq & \sum_{k=m}^{\infty} k [2k - m(1 + \alpha)] |a_{1, l}| \sum_{i=1}^{\infty} \left[ \sum_{j=1}^{\infty} |b_{i, j}| \right] \leq 1.
\end{align*} \]

This completes the proof of Theorem 6.

\[ \text{Integral operator} \]

Here, a closure property of the class \( \overline{K}_n(m, \alpha) \) was examined under the generalized Bernardi-Libera-Livingston integral operator (Saitoh et al., 1992), \( L_{c, m}(f(z)) \) which is defined by:

\[ L_{c, m}(f(z)) = \left( \frac{c + m}{z^c} \right) \int_0^z f \left( \frac{t}{c} \right) \frac{dt}{t^c}, \quad c > -m. \tag{26} \]

**Theorem 7**

Let \( \overline{K}_n(m, \alpha) \).

Then

\[ L_{c, m}(f(z)) \in \overline{K}_n(m, \alpha). \]

**Proof**

From Equation (26), it follows that

\[ L_{c, m}(f(z)) = \left( \frac{c + m}{z^c} \right) \int_0^z f \left( \frac{t}{c} \right) \frac{dt}{t^c} \]

\[ = \left( \frac{c + m}{z^c} \right) \int_0^z \left( \int_0^m \sum_{k=m+1}^{\infty} a_{k, t^k} dt \right) \frac{dt}{t^c} \]

\[ = z^m - \sum_{k=m+1}^{\infty} A_k z^k - \sum_{k=m}^{\infty} B_k z^k, \]

where

\[ A_k = \left( \frac{c + m}{c + k} \right) a_k, \quad B_k = \left( \frac{c + m}{c + k} \right) b_k. \]

Therefore,

\[ \sum_{k=m+1}^{\infty} \left| k [2k - m(1 + \alpha)] \left( \sum_{i=1}^{\infty} t_i |a_{i, l}| \right) \right| \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} |b_{i, j}| \right) \]

\[ \leq \sum_{k=m+1}^{\infty} \left| k [2k - m(1 + \alpha)] \right| \left| a_{1, l} \right| \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} |b_{i, j}| \right) \leq 1. \]

Since \( f(z) \in \overline{K}_n(m, \alpha) \), by using Corollary 1, then \( L_{c, m}(f(z)) \in \overline{K}_n(m, \alpha) \).

This completes the proof of Theorem 7.

**Remark 2**

Putting \( m = 1 \) in the above results, the corresponding results by Kim et al. (2002), with \( k = 1 \) is obtained.

**Conflict of Interest**

The authors have not declared any conflict of interest.

**REFERENCES**


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