ABOUT AJMCSR

The African Journal of Mathematics and Computer Science Research (ISSN 2006-9731) is published bi-monthly (one volume per year) by Academic Journals.

The African Journal of Mathematics and Computer Science Research (AJMCSR) (ISSN:2006-9731) is an open access journal that publishes high-quality solicited and unsolicited articles, in all areas of the subject such as Mathematical model of ecological disturbances, properties of upper fuzzy order, Monte Carlo forecast of production using nonlinear econometric models, Mathematical model of homogenous tumour growth, Asymptotic behavior of solutions of nonlinear delay differential equations with impulse etc. All articles published in AJMCSR are peer-reviewed.

Contact Us

Editorial Office: ajmcsr@academicjournals.org
Help Desk: helpdesk@academicjournals.org
Website: http://www.academicjournals.org/journal/AJMCSR
Submit manuscript online http://ms.academicjournals.me/
Editors

Prof. Mohamed Ali Toumi
Département de Mahtématiques
Faculté des Sciences de Bizerte
7021, Zarzouna, Bizerte
Tunisia.

Associate Professor Kai-Long Hsiao,
Department of Digital Entertainment,
and Game Design,
Taiwan Shoufu University,
Taiwan,
R. O. C.

Dr. Marek Galewski
Faculty of Mathematics and Computer Science,
Lodz University
Poland.

Prof. Xianyi Li
College of Mathematics and Computational Science
Shenzhen University
Shenzhen City
Guangdong Province
P. R. China.
## Editorial Board

**Dr. Rauf, Kamilu**  
Department of mathematics,  
University of Ilorin,  
Ilorin, Nigeria.

**Dr. Adewara, Adedayo Amos**  
Department of Statistics,  
University of Ilorin.  
Ilorin, Kwara State, Nigeria.

**Dr. Johnson Oladele Fatokun,**  
Department of Mathematical Sciences  
Nasarawa State University, Keffi.  
P. M. B. 1022, Keffi, Nigeria.

**Dr. János Toth**  
Department of Mathematical Analysis,  
Budapest University of Technology and Economics.

**Professor Aparajita Ojha,**  
Computer Science and Engineering,  
PDPM Indian Institute of Information Technology,  
Design and Manufacturing, IT Building,  
JEC Campus, Ranjhi, Jabalpur 482 011 (India).

**Dr. Elsayed Elrifai,**  
Mathematics Department,  
Faculty of Science,  
Mansoura University, Mansoura, 35516, Egypt.

**Prof. Reuben O. Ayeni,**  
Department of Mathematics,  
Ladoke Akintola University, Ogbomosho, Nigeria.

**Dr. B. O. Osu,**  
Department of Mathematics,  
Abia State University,  
P. M. B. 2000, Uturu, Nigeria.

**Dr. Nwabueze Joy Chioma,**  
Abia State University,  
Department of Statistics,  
Uturu, Abia State, Nigeria.

**Dr. Marchin Papzhytski,**  
Systems Research Institute,  
Polish Academy of Science,  
ul. Newelska 6 0-60-66-121-66,  
03-815 Warszawa, POLAND.

**Amjad D. Al-Nasser,**  
Department of Statistics,  
Faculty of Science, Yarmouk University,  
21163 Irbid, Jordan.

**Prof. Mohammed A. Qazi,**  
Department of Mathematics,  
Tuskegee University, Tuskegee,  
Alabama 36088, USA.

**Professor Gui Lu Long**  
Dept. of Physics,  
Tsinghua University, Beijing 100084,  
P. R. China.

**Prof. A. A. Mamun, Ph. D.**  
Ruhr-Universitaet Bochum,  
Institut fuer Theoretische Physik IV,  
Fakultaet fuer Physik und Astronomie,  
Bochum-44780, Germany.

**Prof. A. A. Mamun, Ph. D.**  
Ruhr-Universitaet Bochum,  
Institut fuer Theoretische Physik IV,  
Fakultaet fuer Physik und Astronomie,  
Bochum-44780, Germany.
ARTICLES

Solvability of nonlinear Klein-Gordon equation by Laplace Decomposition Method
Mohammed E. A. Rabie
Full Length Research Paper

Solvability of nonlinear Klein-Gordon equation by Laplace Decomposition Method

Mohammed E. A. Rabie

Mathematics Department, Faculty of Education-Arif, Shaqra University, Saudi Arabia
Mathematics Department, Faculty of Science, Sudan University of Science and Technology, Sudan.

Received 9 September, 2014; Accepted 16 June 2015

In this study, Adomian Decomposition Method (ADM), Modified Decomposition Method (MD) and Laplace Decomposition Method (LDM) were used in solving nonlinear Klein-Gordon equation. It can be easily concluded that these three methods yielded exactly the same results.

Key words: Adomian Decomposition Method, Modified Decomposition Method, Laplace Decomposition Method, Klein-Gordon equation and Noise terms phenomena.

INTRODUCTION

A wide variety of physically significant problems such as nonlinear Klein-Gordon equation, modeled by linear and nonlinear partial differential equations has been the focus of extensive studies for the last decades. A huge number of research and investigations have been invested in these scientific applications.

Several approaches such as the characteristics method, spectral methods and perturbation techniques have been extensively used to examine these problems (Wazwaz, 2009). Solving of nonlinear equations using Adomian decomposition method (ADM) has been done in Wazwaz 2009; 2006; El-Wakil et al. 2006; Adomian 1994;1984;1986; Abassy et al. 2007; 2004; Cherrualt 1990; Lesnic 2006; 2007; Wazwaz 2001; Mohammed and Tarig 2013; 2014 and modified decomposition method (MD) in Mohammed and Tarig 2013; 2014.

The aim of this paper is in two folds: firstly, to solve the nonlinear Klein-Gordon equation via LDM, ADM and MD. Secondly, to show these three methods yielded exactly the same result.

As we know the nonlinear Klein-Gordon, equation comes from quantum field theory and describes nonlinear wave interaction. The nonlinear Klein-Gordon equation in its standard form is

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = h(x,t)$$

Subject to the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

Where $a$ is a constant, $h(x,t)$ is a source term and
\[ F(u(x,t)) \] is a nonlinear function of \( u(x,t) \).

In this work, the noise terms phenomenon was used (Wazwaz, 2009), which provides a major advantage in that it demonstrates a fast convergence of the solution. It is important to note that the noise terms phenomenon may appear only for inhomogeneous partial differential equations; in addition, this phenomenon is applicable to all inhomogeneous PDEs of any order. The noise terms, if existed in the components \( u_0 \) and \( u_1 \), will provide, in general, the solution in a closed form with only two successive iterations.

**Solution of Nonlinear Klein–Gordon Equation by ADM**

The decomposition method will be employed. The nonlinear term \( F(u(x,t)) \) will be equated to the infinite series of Adomian polynomials (Adomian, 1994).

\[
L_t(x,t) = L_x(x,t) - au(x,t) - F(u(x,t)) + h(x,t)
\]

Where
\[
L_t = \frac{\partial^2}{\partial t^2}, \quad L_x = \frac{\partial^2}{\partial x^2} \quad \text{and} \quad L_t^{-1} = \int_0^t \int_0^t \cdots dt \]  

Applying \( L_t^{-1} \) to both sides of (3) and using the initial conditions to obtain,

\[
u(x,t) = f(x) + tg(x) + L_t^{-1}(h(x,t))
\]

\[
+ L_t^{-1}(u_{ss}(x,t) - au(x,t)) - L_t^{-1}(F(u(x,t)))
\]

Using the decomposition series for the linear term by

\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).
\]

and the infinite series of Adomian polynomials for the nonlinear term \( F(u(x,t)) \) by

\[
F(u(x,t)) = \sum_{n=0}^{\infty} A_n(x,t)
\]

Where \( A_n \) are Adomian polynomials which calculated by

\[
A_n = \frac{1}{n!} \frac{d^n}{da^n} \left[ F \left( \sum_{i=0}^{\infty} a^i u_i \right) \right]_{a=0}, \quad n = 0, 1, 2, ...
\]

We obtain the recursive relation:

\[
u_o(x,t) = f(x) + tg(x) + L_t^{-1}(h(x,t))
\]

\[
u_{k+1}(x,t) = L_t^{-1}(u_{ss}(x,t) - u_k(x,t)) - L_t^{-1}(A_k), \quad k \geq 0
\]

that leads to:

\[
u_0(x,t) = f(x) + tg(x) + L_t^{-1}(h(x,t))
\]

\[
u_1(x,t) = L_t^{-1}(u_0(x,t) - u_0(x,t)) - L_t^{-1}(A_0)
\]

\[
u_2(x,t) = L_t^{-1}(u_{ss}(x,t) - u_1(x,t)) - L_t^{-1}(A_1)
\]

This completes the determination of the first few components of the solution. Based on this determination, the solution in a series form is readily obtained. In many cases, a closed form solution obtained conductively.

**Example**

Given the following nonlinear Klein-Gordon equation:

\[
u_{ss} - u + u^2 = xt + \frac{x^3}{3!} + \frac{x^4}{12}
\]

following the discussion presented above, we find:

\[
u_0 = 1 + xt + \frac{x^3}{3!} + \frac{x^4}{12}
\]

\[
u_1 = \cdots - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{12} + \cdots
\]

Canceling the noise terms \( \frac{xt^3}{3!} \) and \( \frac{x^7}{12} \) from the component \( u_0 \), and verifying that the remaining non-canceled terms (from the component \( u_0 \)) satisfies the Equation (7), then the exact solution is

\[
u(x,t) = 1 + xt
\]

**Solution of Nonlinear Klein–Gordon Equation by MD**

In an operator form Equation (1) becomes

\[
L_t(x,t) = L_x(x,t) - au(x,t) - F(u(x,t)) + h(x,t)
\]
Where \( L_t = \frac{\partial^2}{\partial x^2}, \quad L_x = \frac{\partial^2}{\partial t^2} \) and \( L_t^{-1} = \int_0^t \int_0^r (\cdot) dt dr \\

The modified decomposition method suggests that

\[
 u(x,t) = \sum_{m=0}^{\infty} a_m(x) t^m
\]

\[
 h(x,t) = \sum_{r=0}^{\infty} r_m(x) t^m
\]

\[
 F[u(x,t)] = \sum_{m=0}^{\infty} A_m(x) t^m
\]

Operating with \( L_t^{-1} \) on both sides of Equation (9) subject to Equation (2) and using the above assumption, we obtain:

\[
 \sum_{m=0}^{\infty} a_m(x) t^m = f(x) + t g(x) + \sum_{m=0}^{\infty} t r_m(x) \frac{t^{m+2}}{(m+1)(m+2)} + \frac{\partial^2}{\partial x^2} \sum_{m=0}^{\infty} a_m(x) \frac{t^m}{(m+1)(m+2)}
 - a_0 \sum_{m=0}^{\infty} a_m(x) - \sum_{m=0}^{\infty} A_m(x) \frac{t^{m+2}}{(m+1)(m+2)}
\]

Let \( m = m - 2 \) be on the right side of Equation (10)

and equate the coefficients of like power of \( t \) on both sides, we get:

\[
 \sum_{m=0}^{\infty} a_m(x) t^m = f(x) + t g(x) + \sum_{m=0}^{\infty} r_m(x) \frac{t^{m+2}}{(m+1)} + \frac{\partial^2}{\partial x^2} \sum_{m=0}^{\infty} a_m(x) \frac{t^m}{(m+1)}
 - a_0 \sum_{m=0}^{\infty} a_m(x) - \sum_{m=0}^{\infty} A_m(x) \frac{t^m}{(m+1)}
\]

And then, the recurrence relation given by:

\[
 a_0(x) = f(x) , \quad a_1(x) = g(x)
 a_m(x) = \frac{1}{m(m-1)} [ r_m(x) + (a_{m-2}(x))_x - a(a_{m-2}(x)) - A_{m-2}(x) ] \quad m \geq 2
\]

Having determined the coefficients, \( a_m(x) \), the solution \( u(x,t) \) in a series form, follow immediately.

**Example:**

Consider the initial nonlinear Klein-Gordon problem (7)
So,
\[
L[u(x, t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} L[u_{xx}(x, t)] - \frac{a}{s^2} L[u(x, t)] + \frac{1}{s^2} L[h(x, t)] - \frac{1}{s^2} L[F(u(x, t))]
\]

Secondly using the decomposition series for the linear term \( u(x, t) \) and the infinite series of Adomian polynomials for the nonlinear term \( F(u(x, t)) \) which gives,
\[
L\left[ \sum_{n=0}^{\infty} u_n(x, t) \right] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} L[h(x, t)] + \frac{1}{s^2} L\left[ \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right] - \frac{a}{s^2} L\left[ \sum_{n=0}^{\infty} u_n(x, t) \right] - \frac{1}{s^2} L\left[ \sum_{n=0}^{\infty} A_n \right]
\]

Where \( A_n \) is the Adomian Polynomials given by Equation (5). Then Equation (14) becomes
\[
\sum_{n=0}^{\infty} L[u_n(x, t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} L[h(x, t)] + \frac{1}{s^2} \sum_{n=0}^{\infty} L[u_{xx}(x, t)] - \frac{a}{s^2} \sum_{n=0}^{\infty} L[u_n(x, t)] - \frac{1}{s^2} \sum_{n=0}^{\infty} L[A_n]
\]

This gives the recurrence relation,
\[
L[u_0(x, t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} L[h(x, t)]
\]
\[
L[u_{n+1}(x, t)] = -\frac{1}{s^2} \sum_{n=0}^{\infty} L[u_{xx}(x, t)] - \frac{a}{s^2} \sum_{n=0}^{\infty} L[u_n(x, t)] - \frac{1}{s^2} \sum_{n=0}^{\infty} L[A_n], n \geq 0
\]

Applying the inverse Laplace transform to the Equation (16), then the required recurrence relation is immediately obtained which complete the solution
\[
u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).
\]

**Application 1:**

Consider the nonlinear Klein-Gordon Equation (7), using Equation (16) subject to the initial condition, we get

\[
L[u(x, t)] = \frac{1}{s^2} \left( s + x + \frac{x^3}{3!} + \frac{2x^2}{s^3} + \frac{1}{s^2} (L[u_{xx}] + L[u] - L[u^2]) \right)
\]

Secondly, using the decomposition series for the linear term \( u(x, t) \) and the infinite series of Adomian polynomials for the nonlinear term \( u^2 \) which gives,
\[
L \left[ \sum_{n=0}^{\infty} u_n(x, t) \right] = \frac{1}{s^2} \left( s + x + \frac{x^3}{3!} + \frac{2x^2}{s^3} \right) + \frac{1}{s^2} \left( L \left[ \sum_{n=0}^{\infty} A_n \right] + L \left[ \left( \sum_{n=0}^{\infty} u_n \right) \right] - L \left[ \sum_{n=0}^{\infty} A_n \right] \right)
\]

That leads to the recurrence relation below
\[
L[u_0] = \frac{1}{s} + \frac{x}{s^2} + \frac{x^2}{s^3} + \frac{2x^2}{s^4}
\]
\[
L[u_{n+1}] = \frac{1}{s^2} (L[u_{xx}] + L[u] - L[A_n]), n \geq 0
\]

Calculate: \( u_0 \)

Applying inverse Laplace transform on the first equation of Equation (20) which gives
\[
u_0 = 1 + xt + \frac{x^3 t^3}{3!} + \frac{2x^2 t^4}{4!}
\]

Calculate: \( u_1 \)

From Equation (18):
\[
u_0 = u_0 = 1 + xt + \frac{x^2 t^2}{3!} + \frac{2x t^4}{4!}
\]
\[
u_1 = u_1 = \frac{x^3}{3!} + \frac{4x^2}{12} + \frac{2x t^4}{3!} + \frac{x^2 t^5}{12} + \frac{x t^7}{3!} + \frac{x^2 t^6}{36}
\]

From Equation (20):
\[
L[u_1] = \frac{1}{s^2} L[u_{xx} + u_0 - A_0]
\]
\[
= \frac{1}{s^2} \left[ \frac{x^3}{3} - \frac{x^2 t^6}{36} - \frac{x^2 t^7}{36} - \frac{x^3}{3} - \frac{x^2 t^6}{36} + \frac{x^2 t^7}{36} \right]
\]
\[
\left. \frac{1}{s^2} \left[ 4! x^4 - 4! x^2 x + 2! x^3 x^2 - 5! x^3 \frac{1}{s^2} + 7! x^3 \frac{1}{s^3} - 6! x^2 \frac{1}{s^4} - 8! x^4 \right] \right|
\]

Applying inverse Laplace to obtain
\[
u_1 = \frac{4! x^4}{6!} - \frac{8! x^2 x}{6!} - \frac{2! x^3 x^2}{4!} - \frac{20! x^3 x^2}{7!} - \frac{5! x^3}{9!} + \frac{144! x^3}{8!} - \frac{20! x^2}{8!}
\]

Canceling the noise terms \(\frac{x^3}{3!}\) and \(\frac{2x^4}{4!}\) from the component \(u_0\) and verifying that the remaining non-canceled terms satisfies the equation, the exact solution
\[u(x,t) = 1 + xt .\]

**Application 2:**
Consider the nonlinear Equation
\[u_{tt} - u_{xx} + u^2 = 6x^3 t - 6xt^3 + x^6 t^6 \quad (21)\]
Subject to the initial conditions
\[u(x,0) = u_t(x,0) = 0\]
Following the analysis presented above and using the given initial conditions, we obtain the recursive relation in the form
\[L[u_0] = \frac{6x^3}{s^4} - \frac{36x}{s^6} + \frac{6! x^6}{s^9} \]
\[L[u_{k+1}] = \frac{1}{s^2} (L[u_{k,xx}] - L[A_k]), \quad k \geq 0 \quad (22)\]
Calculate: \(u_0\)
Applying inverse Laplace transform on the first equation of (22) that leads to
\[u_0 = x^3 t^3 - \frac{3x t^5}{10} + \frac{x^6 t^8}{56}\]
Calculate: \(u_1\)
\[A_0 = u_0^2 = \left( x^3 t^3 - \frac{3x t^5}{10} + \frac{x^6 t^8}{56} \right)^2 \]

From the second Equation (22)
\[L[u_1] = \frac{1}{s^2} (L[u_{0,xx}] - L[A_0])\]
Applying inverse Laplace transform on the last equation leads to
\[u_1 = \frac{3x t^5}{10} + \frac{x^4 t^{10}}{168} - \frac{x^6 t^8}{56} + \frac{x^4 t^{10}}{300} - \frac{11! x^6 t^{13}}{(56) 13!} + \ldots \]
By canceling the noise terms \(-\frac{3x t^5}{10}\) and \(\frac{x^6 t^8}{56}\) from the component \(u_0\) and verifying that the remaining non-canceled terms of \(u_0\) satisfies Equation (21), we find that the exact solution is given by
\[u(x,t) = x^3 t^3 \]

**Conclusion**
In this paper, we introduced Klein-Gordon equation, and solved it by using ADM, and MD then applied LDM. Clearly, these three methods are very effective, it accelerates the solutions. If we compare it with the other methods, it will be the best. In addition, the LDM may give the exact solutions for nonlinear PDEs. Moreover, the noise terms may appear if the exact solution is a part of the 'theroth component \(\overline{\mathcal{L}}O\).'

**Conflict of Interest**
The author has not declared any conflict of interest.

**REFERENCES**


Related Journals Published by Academic Journals

- International Journal of Physical Sciences
- Journal of Geology and Mining Research
- Journal of Environmental Chemistry and Ecotoxicology
- Journal of Internet and Information Systems
- Journal of Oceanography and Marine Science
- Journal of Petroleum Technology and Alternative Fuels