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Bochum-44780,  
Germany.

Prof. A. A. Mamun, Ph. D.  
Ruhr-Universitaet Bochum,  
Institut fuer Theoretische Physik IV,  
Fakultaet fuer Physik und Astronomie,  
Bochum-44780,  
Germany.
Results on generalized fuzzy soft topological spaces
F. H. Khedr, S. A. Abd El-Baki and M. S. Malfi
Full Length Research Paper

Results on generalized fuzzy soft topological spaces

F. H. Khedr*, S. A. Abd El-Baki and M. S. Malfi

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.

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In this manuscript, the concept of a generalized fuzzy soft point is introduced and some of its basic properties were studied. Also, the concepts of a generalized fuzzy soft base (subbase) and a generalized fuzzy soft subspace were introduced and some important theorems were established. Finally, the relationship between fuzzy soft set, intuitionistic fuzzy soft set, generalized fuzzy soft set and generalized intuitionistic fuzzy soft set were investigated.

Key words: Fuzzy soft set, generalized fuzzy soft set, generalized fuzzy soft topology, generalized fuzzy soft base (subbase), generalized fuzzy soft subspace, intuitionistic fuzzy soft set.

INTRODUCTION

Most of our real life problems in engineering, social and medical science, economics, environment, etc., involve imprecise data and their solutions involve the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties, Zadeh (1965) introduced the concept of fuzzy set (FS) and fuzzy set operations. The analytical part of fuzzy set theory was practically started with the paper of Chang (1968) who introduced the concept of fuzzy topological spaces. However, this theory is associated with an inherent limitation, which is the inadequacy of the parametrization tool associated with this theory as it was mentioned by Molodtsov (1999). Molodtsov (1999) introduced the concept of the soft set (SS) theory which is free from the aforementioned problems and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. Shabir and Naz (2011) studied the topological structures of soft sets. Intuitionistic fuzzy set theory was introduced by Atanassov (1986). In recent times, the process of fuzzification of soft set theory is rapidly progressed. Maji et al. (2001a, b) combined the theory of SS with the fuzzy and intuitionistic fuzzy set theory and called as fuzzy soft set (FSS) and intuitionistic fuzzy soft set (IFSS). Topological structure of fuzzy soft sets was started by Tanay and Burc Kandemir (2011). The study was pursued by some others researchers (Chakraborty et al., 2014; Gain et al., 2013; Mukherjee et al., 2015). Majumdar and Samanta (2010) introduced generalized fuzzy soft set (GFSS) and successfully applied their notion in a decision making problem. Yang (2011) pointed out that some results of Majumdar and Samanta (2010) are not valid in general. Chakraborty and Mukherjee (2015) introduced generalized fuzzy soft union, generalized fuzzy soft intersection and several...

*Corresponding author. E-mail: khedrfathi@gmail.com.

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other properties of generalized fuzzy soft sets. Also, they introduced generalized fuzzy soft topological spaces, generalized fuzzy soft closure, generalized fuzzy soft interior and studied some of their properties. Arora and Garg (2017a, b) solved the MCDM problem and established the IFSWA operator and the IFSWG operator under the IFSS environment. Garg (2017) introduced some series of averaging aggregation operators have been presented under the intuitionistic fuzzy environment. Garg and Arora (2017a, b) introduced distance and similarity measures for dual hesitant fuzzy environment. Garg and Arora (2017b) have been presented under the intuitionistic fuzzy environment. Garg and Arora (2017a, b) introduced distance and similarity measures for dual hesitant fuzzy environment. Garg and Arora (2017b) have been presented under the intuitionistic fuzzy environment. Garg and Arora (2017a, b) introduced distance and similarity measures for dual hesitant fuzzy environment. Garg and Arora (2017b) have been presented under the intuitionistic fuzzy environment. Garg and Arora (2017a, b) introduced distance and similarity measures for dual hesitant fuzzy environment. Garg and Arora (2017b) have been presented under the intuitionistic fuzzy environment.

**Preliminaries**

Here, the basic definitions and results which will be needed in the sequel were presented.

**Definition 1**

Let $X$ be a non-empty set. A fuzzy set $A$ in $X$ is defined by a membership function $\mu_A : X \rightarrow [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of $x$ in $A$ for $x \in X$. The set of all fuzzy sets in a set $X$ is denoted by $I^X$, where $I$ is the closed unit interval $[0,1]$.

**Theorem**

If $A, B \in I^X$, then, we have:

(i) $A \preceq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, $\forall x \in X$;

(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, $\forall x \in X$;

(iii) $C = A \lor B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x))$, $\forall x \in X$;

(iv) $D = A \land B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x))$, $\forall x \in X$;

(v) $E = A^C \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, $\forall x \in X$.

**Definition 2**

Let $X$ be an initial universe and $E$ be a set of parameters (Molodtsov, 1999). Let $P(X)$ denotes the power set of $X$ and $A \subseteq E$. A pair $(f, A)$ is called a soft set over $X$ if $f$ is a mapping from $A$ into $P(X)$, that is, $f : A \rightarrow P(X)$.

In other words, a soft set is a parameterized family of subsets of the set $X$. For $e \in A$, $f(e)$ may be considered as the set of $e$—approximate elements of the soft set $(f, A)$.

**Definition 3**

Let $X$ be an initial universe and $E$ be a set of parameters. Let $I^X$ be the set of all fuzzy sets in $X$ and $A \subseteq E$. A pair $(F, A)$ is called a fuzzy soft set over $X$, where $F : A \rightarrow I^X$ is a function, that is, for each $e \in A$, $F(e) = F_e : X \rightarrow I$ is a fuzzy set in $X$.

**Definition 4**

Let $X$ be a universal set of elements and $E$ be a universal set of parameters for $X$ (Majumdar and Samanta, 2010). Let $F : E \rightarrow I^X$ and $\mu$ be a fuzzy subset of $E$, that is, $\mu : E \rightarrow I$. Let $F^\mu : E \rightarrow I^X \times I$ defined as follows:

$F^\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then $F^\mu$ is called a generalised fuzzy soft set (GFSS in short) over $(X, E)$.

**Definition 5**

Let $F^\mu$ and $G^\delta$ be two GFSSs over $(X, E)$ (Majumdar and Samanta, 2010). Now $F^\mu$ is said to be a GFS subset of $G^\delta$ or $G^\delta$ is said to be a GFS super set of $F^\mu$ if:

(i) $\mu$ is a fuzzy subset of $\delta$;

(ii) $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.

In this case, we write $F^\mu \subseteq G^\delta$.

**Definition 6**

Let $F^\mu$ be a GFSS over $(X, E)$. The complement of $F^\mu$, denoted by $F^c_\mu$, is defined by $F^c_\mu = G^\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e)$, $\forall e \in E$ (Majumdar and Samanta, 2010). Obviously

$(F^c_\mu)^c = F_\mu$.

**Definition 7**

Let $F^\mu$ and $G^\delta$ be two GFSSs over $(X, E)$ (Chakraborty and Mukherjee, 2015). The union of $F^\mu$ and $G^\delta$, denoted by $F^\mu \cup G^\delta$, is the GFSS $H_{uv}$, defined as
\[ H_\psi : E \rightarrow I^X \times I \text{ such that } H_\psi(e) = (H(e), \nu(e)), \]
where
\[ H(e) = F(e) \vee G(e), \] and
\[ \nu(e) = \mu(e) \vee \delta(e), \forall e \in E. \]

Let \( \{ (F_\mu)_{\lambda}, \lambda \in \Lambda \} \), where \( \Lambda \) is an index set, be a family of GFSSs. The union of these family, denoted by \( \cup_{\lambda \in \Lambda} (F_\mu)_{\lambda} \), is the GFSS \( H_\psi \), defined as
\[ H_\psi : E \rightarrow I^X \times I \text{ such that } H_\psi(e) = (H(e), \nu(e)), \]
where
\[ H(e) = \bigvee_{\lambda \in \Lambda} (F(e))_{\lambda}, \]
and \( \nu(e) = \bigvee_{\lambda \in \Lambda} (\mu(e))_{\lambda}, \forall e \in E. \)

**Definition 8**

Let \( F_\mu \) and \( G_\delta \) be two GFSSs over \((X, E)\) (Chakraborty and Mukherjee, 2015). The intersection of \( F_\mu \) and \( G_\delta \), denoted by \( F_\mu \cap G_\delta \), is the GFSS \( M_\sigma \), defined as
\[ M_\sigma : E \rightarrow I^X \times I \text{ such that } M_\sigma(e) = (M(e), \sigma(e)), \]
where
\[ M(e) = F(e) \wedge G(e) \] and \( \sigma(e) = \mu(e) \wedge \delta(e), \forall e \in E. \)

Let \( \{ (F_\mu)_{\lambda}, \lambda \in \Lambda \} \), where \( \Lambda \) is an index set, be a family of GFSSs. The intersection of these family, denoted by \( \cap_{\lambda \in \Lambda} (F_\mu)_{\lambda} \), is the GFSS \( M_\sigma \), defined as
\[ M_\sigma : E \rightarrow I^X \times I \text{ such that } M_\sigma(e) = (M(e), \sigma(e)), \]
where
\[ M(e) = \bigwedge_{\lambda \in \Lambda} (F(e))_{\lambda}, \] and \( \sigma(e) = \bigwedge_{\lambda \in \Lambda} (\mu(e))_{\lambda}, \forall e \in E. \)

**Definition 9**

A GFSS is said to be a generalized null fuzzy soft set, denoted by \( \emptyset_\theta \), if \( \emptyset_\theta : E \rightarrow I^X \times I \) such that
\[ \emptyset_\theta(e) = (\emptyset(e), \theta(e)) \] where \( \emptyset(e) = \emptyset \forall e \in E \) and \( \theta(e) = 0 \forall e \in E \) (Where \( \emptyset(x) = 0, \forall x \in X \)) (Majumdar and Samanta, 2010).

**Definition 10**

A GFSS is said to be a generalized absolute fuzzy soft set, denoted by \( \mathbf{1}_\Delta \), if \( \mathbf{1}_\Delta : E \rightarrow I^X \times I \), where
\[ \mathbf{1}_\Delta(e) = (\mathbf{1}(e), \Delta(e)) \] is defined by \( \mathbf{1}(e) = \mathbf{1}, \forall e \in E \) and \( \Delta(e) = 1, \forall e \in E \) (Where \( \mathbf{1}(x) = 1, \forall x \in X \)) (Majumdar and Samanta, 2010).

**Definition 11**

Let \( T \) be a collection of generalized fuzzy soft sets over \((X, E)\). Then \( T \) is said to be a generalized fuzzy soft topology (GFST, in short) over \((X, E)\) if the following conditions are satisfied:

(i) \( \emptyset_\theta \) and \( \mathbf{1}_\Delta \) are in \( T \);
(ii) Arbitrary unions of members of \( T \) belong to \( T \);
(iii) Finite intersections of members of \( T \) belong to \( T \).

The triplet \((X, T, E)\) is called a generalized fuzzy soft topological space (GFST- space, in short) over \((X, E)\).

The members of \( T \) are called a GFST open sets in \((X, T, E)\). The complement of a GFST open set is called GFST closed.

**Definition 12**

Let \((X, T, E)\) be a GFST-space and \( F_\mu \) be a GFSS over \((X, E)\). Then the generalized fuzzy soft closure of \( F_\mu \), denoted by \( \overline{F_\mu} \), is the intersection of all GFST closed supper sets of \( F_\mu \).

Clearly, \( \overline{F_\mu} \) is the smallest GFST closed set over \((X, E)\) which contains \( F_\mu \) (Chakraborty and Mukherjee, 2015).

**Definition 13**

Let \( F_\mu \) be a GFSS over \((X, E)\) (Chakraborty and Mukherjee, 2015). We say that \((x_\alpha, e_\lambda) \in F_\mu \) read as \((x_\alpha, e_\lambda) \) belongs to the GFSS \( F_\mu \) if
\[ F(e)(x) = \alpha (0 < \alpha \leq 1) \]
and
\[ F(e)(y) = 0, \forall y \in X \setminus \{x\}, \mu(e) > \lambda. \]

**Definition 14**

A GFSS \( F_\mu \) in a GFST-space \((X, T, E)\) is called a generalized fuzzy soft neighborhood [GFS-nbd, in short] of the GFSS \( G_\delta \) if there exists a GFST open set \( H_\psi \) such that \( G_\delta \subseteq H_\psi \subseteq F_\mu \) (Chakraborty and Mukherjee, 2015).

**Definition 15**

A GFSS \( F_\mu \) in a GFST-space \((X, T, E)\) is called a generalized fuzzy soft neighborhood of the generalized fuzzy soft point \((x_\alpha, e_\lambda) \subseteq H_\psi \) if there exists a GFST open set \( G_\delta \) such that \((x_\alpha, e_\lambda) \subseteq G_\delta \subseteq F_\mu \)
Difference of two GFSS $F_\mu$ and $G_\delta$, denoted by $F_\mu \setminus G_\delta$, is a GFSS $H(e) = F(e) \wedge G_\delta(e)$, defined as $H(e) = F(e) \wedge G_\delta(e)$ and $\nu(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$ (Mukherjee, 2015).

Definition 16

Let a set $A$ be fixed. An intuitionistic fuzzy set or IFS ‘$A$’ in $E$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x) : x \in E)\}$ where the functions $\mu_A : E \rightarrow [0,1] \& \nu_A : E \rightarrow [0,1]$ define the degree of membership and non-membership, respectively, of the element $x \in E$ to the set $A$ and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ (Atanassov, 1986).

Definition 17

For any $e \in A$, $F(e)$ is an intuitionistic fuzzy subset of $X$. Let us denote $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ by the membership degree and non-membership degree, respectively, that object $x$ holds parameter $e$, where $x \in X$ and $e \in A$. Then, $F(e)$ can be written as an intuitionistic fuzzy set such that $F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x)) : x \in X\}$.

Definition 18

Let $X$ be an initial universe and $E$ be a set of parameters (Maji et al., 2001a,b). Let $F^X$ be the set of all intuitionistic fuzzy subsets of $X$ and $A \subseteq E$. Then, the pair $(F, A)$ is called an intuitionistic fuzzy soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow F^X$.

The complement of a generalized fuzzy soft point $(x_\alpha, e_\lambda)$, denoted by $(x_\alpha, e_\lambda)^c$, is defined as follows $(x_\alpha, e_\lambda)^c = (x_{1-\alpha}, e_{1-\lambda})$.

Example 1

Let $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$ the set of parameters. Then $(x_3(0.1), e_2(0.6)) = ((x_3, x_2, x_4, x_4, 0.6))$ is a generalized fuzzy soft point whose complement is $(x_3(0.1), e_2(0.6))^c = ((x_3, x_2, x_3, x_4, 0.4))$. The belongingness of a generalized fuzzy soft point to a generalized fuzzy soft set in Definition 13 as follows was redefined:

Definition 3

Let $F_\mu$ be a GFSS over $(X, E)$. We say that $(x_\alpha, e_\lambda) \in F_\mu$ read as $(x_\alpha, e_\lambda)$ belongs to the GFSS $F_\mu$ if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

Definition 4

Let $(x_\alpha, e_\lambda)$ and $(y_\beta, e_\delta)$ be two generalized fuzzy soft points, we say that $(x_\alpha, e_\lambda) \in (y_\beta, e_\delta) \leftrightarrow (x, e) = (y, e')$ and $\alpha < \beta$. 

GENERALIZED FUZZY SOFT POINTS AND NEIGHBORHOOD SYSTEMS

Here, a generalized fuzzy soft point was introduced and some of its basic properties were studied. Also, we discuss the concept of a neighborhood of a generalized fuzzy soft point in a generalized fuzzy soft topological space.
\( \lambda < \delta. \)

**Theorem 1**

Let \( F_\mu \) be a GFSS over \((X, E)\), then:

1. \( F_\mu \cup F^c_\mu \neq \mathbb{1}_\Delta; \)
2. \( \mathbb{1}_\sigma \nsubseteq F_\mu \cap F^c_\mu \)

\[
\begin{align*}
F_\mu &= \{ F_\mu(e_1) = \{ (x_1, x_2) : (0.2, 0.7), (0.3) \}, F_\mu(e_2) = \{ (x_1, x_2) : (0.5', 0.4), 0.9 \} \}, \\
F^c_\mu &= \{ F_\mu(e_1) = \{ (x_1, x_2) : (0.8', 0.3), 0.7 \}, F_\mu(e_2) = \{ (x_1, x_2) : (0.5, 0.6'), 0.1 \} \}.
\end{align*}
\]

Then

\[
\begin{align*}
F_\mu \cup F^c_\mu &= \{ F_\mu(e_1) = \{ (x_1, x_2) : (0.8', 0.3), 0.7 \}, F_\mu(e_2) = \{ (x_1, x_2) : (0.5, 0.6'), 0.1 \} \} \neq \mathbb{1}_\Delta, \\
F_\mu \cap F^c_\mu &= \{ F_\mu(e_1) = \{ (x_1, x_2) : (0.2', 0.3), 0.3 \}, F_\mu(e_2) = \{ (x_1, x_2) : (0.5', 0.4), 0.9 \} \} \neq \mathbb{1}_\sigma.
\end{align*}
\]

Consider the GFS point

\[
(x_{1(0.1)}, e_{1(0.2)}) = \{ (x_1, 0.1), (0.2) \}
\]

then

\[
(x_{1(0.1)}, e_{1(0.2)}) \in F_\mu \quad \text{and} \quad (x_{1(0.1)}, e_{1(0.2)}) \in F^c_\mu.
\]

For Theorem 1, we have:

\[
\begin{align*}
(x_{1(0.1)}, e_{1(0.2)}) &= \{ (x_1, 0.1), (0.2) \} \in F_\mu \quad \text{but} \quad (x_{1(0.1)}, e_{1(0.2)}) \notin F^c_\mu.
\end{align*}
\]

**Definition 5**

Let \((X, T, E)\) be a GFST-space. The set of all GFS neighborhoods of a generalized fuzzy soft point \((x_\alpha, e_\lambda)\) is called the GFS neighborhood system of \((x_\alpha, e_\lambda)\) and is denoted by \(N_T(x_\alpha, e_\lambda)\).

**Theorem 2**

Let GFSS \((X, E)\) be a family of all generalized fuzzy soft sets over soft universe \((X, E)\) and \((X, T, E)\) be GFST-space. Then the following properties are satisfied:

1. \( F_\mu = \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda); \)
2. \( (x_\alpha, e_\lambda) \in \bigcup \{(F_\mu)_i : i \in I\} \Rightarrow \exists i_0 \in I \text{ such that } (x_\alpha, e_\lambda) \in (F_\mu)_{i_0}; \)
3. \( (x_\alpha, e_\lambda) \in \bigcap \{(F_\mu)_i : i \in J\} \Rightarrow \forall i \in J, (x_\alpha, e_\lambda) \in (F_\mu)_i; \)
4. \( (x_\alpha, e_\lambda) \in (x_\beta, e_\delta) \quad \text{and} \quad (x_\beta, e_\delta) \in \bigcap \{(F_\mu)_i : i \in J\}. \)

Then \((x_\alpha, e_\lambda) \in \bigcap \{(F_\mu)_i : i \in J\};

(3) if \((x_\alpha, e_\lambda) \notin F_\mu \quad \text{then} \quad (x_\alpha, e_\lambda) \notin F^c_\mu \text{ is not hold};
(4) \((x_\alpha, e_\lambda) \in F_\mu \Rightarrow (x_\alpha, e_\lambda) \notin F^c_\mu \).

**Example 2**

Let \( X = \{ x_1, x_2 \} \) and \( E = \{ e_1, e_2 \}, \) consider the GFSS \( F_\mu \) over \((X, E), \) as:

\[
F_\mu \subseteq \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda) \quad \text{straightforward. Let}
\]

\[
(x_\beta, e_\delta) \subseteq \bigcup_{(x_\alpha, e_\lambda) \in F_\mu} (x_\alpha, e_\lambda), \quad \text{then there exists } (x_\alpha, e_\mu) \in F_\mu \quad \text{such that } (x_\beta, e_\delta) \subseteq (x_\alpha, e_\mu).
\]

Therefore, \( \beta \leq \alpha', \delta \leq \lambda' \) put \( \alpha' \leq F(e)(x) \) and \( \lambda' \leq \mu(e), \) then \( \beta \leq F(e)(x) \) and \( \delta \leq \mu(e). \) This shows that \( (x_\beta, e_\delta) \subseteq F_\mu. \)

(2) Let \( (x_\alpha, e_\lambda) \in \bigcup_{i \in I} (F_\mu)_i = H_v. \) Then \( \alpha \leq H(e)(x) = \bigvee_{i \in I} \{ F(e)(x) \}_i \) and \( \lambda \leq v(e) = \bigvee_{i \in J} \{ \mu(e) \}_i \forall e \in E, x \in X. \) Then there exists \( i_0 \in J \) such that \( \alpha \leq (F(e)(x))_{i_0} \) and \( \lambda \leq (\mu(e))_{i_0}, \) which shows that \( (x_\alpha, e_\lambda) \in (F_\mu)_{i_0}. \)

Conversely, let \( (x_\alpha, e_\lambda) \in (F_\mu)_{i_0} \) such that \( (F_\mu)_{i_0} (e) = \{ (F(e))_{i_0}, (\mu(e))_{i_0} \} \) for some \( i_0 \in J, \) thus \( \alpha \leq (F(e)(x))_{i_0} \) and \( \lambda \leq (\mu(e))_{i_0} \) which implies that: \( \alpha \leq \bigvee_{i \in J} \{ F(e)(x) \}_i \) and \( \lambda \leq \bigvee_{i \in J} \{ \mu(e) \}_i. \) Therefore \( \alpha \leq H(e)(x) \) and \( \lambda \leq v(e). \) So \( (x_\alpha, e_\lambda) \in H_v = \bigcup_{i \in I} (F_\mu)_i. \)

(3) Let \( (x_\alpha, e_\lambda) \notin \bigcap_{i \in J} (F_\mu)_i = M_\sigma. \) Therefore:

\[
(x_\alpha, e_\lambda) \notin (F_\mu)_i \text{ for } \forall i \in J.
\]
\[ \alpha \leq M(e)(x) = \bigwedge_{i \in I}(F(e)(x))_i \quad \text{and} \quad \lambda \leq \sigma(e) = \bigwedge_{i \in J}(\mu(e))_i \].

Thus \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \) for all \( i \in I \). Therefore 
\( (x_{\alpha}, e_\lambda) \in (F_i)_i \quad \text{for all} \quad i \in I \). Conversely, let
\[ (x_{\alpha}, e_\lambda) \in (F_i)_i \quad \text{for all} \quad i \in I \].
then \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \) for all \( i \in I \). Therefore \( \alpha \leq \bigwedge_{i \in I}(F(e)(x))_i \) and \( \lambda \leq \bigwedge_{i \in J}(\mu(e))_i \). This implies that \( \alpha \leq M(e)(x) \) and \( \lambda \leq \sigma(e) \). Hence
\( (x_{\alpha}, e_\lambda) \in M \). 

(4) Let \( (x_{\alpha}, e_\lambda) \in (F_i)_i \). This implies that \( \alpha \leq (F(e)(x))_i \) and \( \lambda \leq (\mu(e))_i \) for all \( i \in J \). Then
\( \alpha \leq \min\{ (F(e)(x))_i : i \in J \} \) and \( \lambda \leq \min\{ (\mu(e))_i : i \in J \} \). Therefore

\[ F_\mu = \left\{ F_\mu(e) = \left( \left\{ \frac{x}{0.6} : \frac{y}{0.5} : \frac{z}{0.4} \right\}, 0.2 \right), F_\mu(d) = \left( \left\{ \frac{x}{0.2} : \frac{y}{0.3} : \frac{z}{0.8} \right\}, 0.5 \right), F_\mu(h) = \left( \left\{ \frac{x}{0.7} : \frac{y}{0.4} : \frac{z}{0.3} \right\}, 0.6 \right) \right\}, \]

\[ G_\delta = \left\{ G_\delta(e) = \left( \left\{ \frac{x}{0.3} : \frac{y}{0.2} : \frac{z}{0.2} \right\}, 0.1 \right), G_\delta(d) = \left( \left\{ \frac{x}{0.1} : \frac{y}{0.5} : \frac{z}{0.5} \right\}, 0.3 \right), G_\delta(h) = \left( \left\{ \frac{x}{0.5} : \frac{y}{0.3} : \frac{z}{0.3} \right\}, 0.5 \right) \right\}. \]

Consider \( T = \left\{ \emptyset, \mathbb{I}_\Delta, F_\mu, G_\delta \right\} \). Then \( T \) forms a GFS topology over \( (X, E) \). Consider the following GFS over \( (X, E) \),

\[ H_v = \left\{ H_v(e) = \left( \left\{ \frac{x}{0.7} : \frac{y}{0.6} : \frac{z}{0.5} \right\}, 0.3 \right), H_v(d) = \left( \left\{ \frac{x}{0.3} : \frac{y}{0.4} : \frac{z}{0.8} \right\}, 0.5 \right), H_v(h) = \left( \left\{ \frac{x}{0.9} : \frac{y}{0.4} : \frac{z}{0.7} \right\}, 0.8 \right) \right\}. \]

If \( (x_{(0.4)}, e_{(0.1)}) \in \mathbb{I}_\Delta \), then there exists a GFS open set \( e_\mu \) such that \( (x_{(0.4)}, e_{(0.1)}) \in F \subseteq H_v \), that is, \( H_v \) is a GFS neighborhood of \( (x_{(0.4)}, e_{(0.1)}) \). Also, if \( M \subseteq F \subseteq H_v \), then \( H_v \) is a GFS neighborhood of \( M \subseteq GFS(X, E) \), where GFS \( (X, E) \) the family of all generalized fuzzy soft sets over \( (X, E) \).

### GENERALIZED FUZZY SOFT BASE AND GENERALIZED FUZZY SOFT SUBBASE

**Definition 1**

Let \( (X, T, E) \) be GFS topological space. A collection \( \mathcal{R} \) of generalized fuzzy soft sets over \( (X, E) \) is called a generalized fuzzy soft open base or simply a base for generalized fuzzy soft topology on \( (X, E) \), if the following conditions hold:

1. \( \emptyset \in \mathcal{R} \)

2. \( \bigcup \mathcal{R} = \mathbb{I}_\Delta \) i.e. for each \( e \in E \) and \( x \in X \), there exists \( R_\mu \in \mathcal{R} \) such that \( R_\mu(x) = 1 \) and \( \mu(e) = 1 \).

3. If \( F_\mu \), \( G_\delta \in \mathcal{R} \) then for each \( e \in E \) and \( x \in X \), there exists \( H_v \in \mathcal{R} \) such that \( H_v \subseteq F_\mu \cap \bigcup G_\delta \) and \( H_v(x) = \min\{ F_\mu(x), G_\delta(x) \} \).

**Example 1**

Let \( X = \{ x_1, x_2, x_3, x_4 \} \), \( E = \{ e_1, e_2, e_3, e_4 \} \). Let us consider the collection

\[ \mathcal{R} = \left\{ \emptyset, \mathbb{I}_\Delta, G_\delta, H_v, K_\gamma, J_\rho, M_\eta, N_\psi \right\}. \]

Where

\[ G_\delta = \left\{ G_\delta(e_2) = \left( \left\{ \frac{x}{0.2} \right\}, 0.2 \right), \right\}, \]

\[ H_v = \left( \left\{ H_v(e_3) = \left( \left\{ \frac{x}{0.4} \right\}, 0.4 \right), \right\}. \]

\[ v(e) = \min\{ \mu(e), \delta(e) \}. \]
We can see that \( e \) satisfies the conditions 1 to 3 of Definition (4-1). Therefore, \( e \) forms a GFS base for a topology on \((X, E)\).

**Definition 2**

\[
T_{\mathcal{R}} = \left\{ H_{\emptyset}, I_{\Delta}, F_\mu, G_\delta, H_\varphi \right\}
\]

\[
\{ F_\mu \cup G_\delta (e_1) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}, F_\mu \cap G_\delta (e_2) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}
\]

\[
\{ F_\mu \cap H_\varphi (e_1) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}, F_\mu \cap H_\varphi (e_2) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}
\]

Then obviously, \( \mathcal{R} \) is a GFS base for a GFS topology on \((X, E)\). The GFS topology generated by \( \mathcal{R} \) is \( \mathcal{T}_{\mathcal{R}} \), where

**Theorem 1**

Let \((X, T, E)\) be a GFST-space and \( \mathcal{R} \) be a subcollection of \( T \) such that every member of \( T \) is union of some members of \( \mathcal{R} \). Then \( \mathcal{R} \) is GFS base for the GFS topology \( T \) on \((X, E)\).

**Proof 1**

Since \( \emptyset \in T \), \( \emptyset \in \mathcal{R} \). Again since \( \bigcup \mathcal{R} \in T \), \( \bigcup \mathcal{R} \in \mathcal{R} \). Let \( (R_\mu)_1, (R_\mu)_2 \in \mathcal{R} \). Then \( (R_\mu)_1, (R_\mu)_2 \in T \) and so \( (R_\mu)_1 \cap (R_\mu)_2 \in T \). Then there exists \( (R_\mu)_\alpha \in \mathcal{R}, \alpha \in \mathcal{V} \) such that \( (R_\mu)_1 \cap (R_\mu)_2 = \bigcup \{ (R_\mu)_\alpha : \alpha \in \mathcal{V} \} \). Therefore, \( (R_\mu)_1 (e) \cap (R_\mu)_2 (e) = \bigcup \{ (R_\mu)_\alpha (e) : \alpha \in \mathcal{V} \} \), for \( e \in E \). That is, for each \( e \in E \) and \( x \in X \), min \{ \( R_\mu (e) (x) \) \}, \( (R_\mu (e) (x)) \) = max\( (R_\mu (e) (x)) \); \( \alpha \in \mathcal{V} \) and min \{ \( \mu (e) \) \}, \( \mu (e) \) = max\( \mu (e) \); \( \alpha \in \mathcal{V} \).

Therefore there exists \( \alpha \in \mathcal{V} \) such that: min \{ \( (R_\mu (e) (x)) \), \( (R_\mu (e) (x)) \) \} = max\( (R_\mu (e) (x)) \); \( \alpha \in \mathcal{V} \).

Let \( \mathcal{R} \) be a GFS base for a GFS topology on \((X, E)\). Then, the GFS topology generated by GFS base \( \mathcal{R} \), is denoted by \( T_{\mathcal{R}} \) and is defined as follows:

\[
T_{\mathcal{R}} = \{ H_\varphi : H_\varphi = \bigcup (R_\mu)_\alpha, (R_\mu)_\alpha \in \mathcal{R}, \alpha \in \mathcal{V}, \forall \alpha \in \mathcal{V} \text{ an index set} \}
\]

**Example 2**

Let \( E = \{ e_1, e_2, e_3 \} \), \( X = \{ x_1, x_2, x_3, x_4 \} \) and \( \mathcal{R} = \{ H_\emptyset, F_\mu, G_\delta, H_\varphi \} \), where

\[
F_\mu = \{ f_\mu (e_1) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 0.1 \right\}, F_\mu (e_2) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 0.1 \right\},
\]

\[
G_\delta (e_1) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}, G_\delta (e_2) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\},
\]

\[
H_\varphi (e_1) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}, H_\varphi (e_2) = \left( \left\{ \frac{x_1}{x_2}, \frac{x_3}{x_4}, \frac{x_5}{x_6}, \frac{x_7}{x_8}, \frac{x_9}{x_{10}} \right\}, 1 \right\}.
\]

Then obviously, \( \mathcal{R} \) is a GFS base for a GFS topology on \((X, E)\). The GFS topology generated by \( \mathcal{R} \) is \( T_{\mathcal{R}} \), where

**Definition 3**

A collection \( \Omega \) of some members of GFST-space \((X, T, E)\) is said to be a subbase of \( T \) if and only if the collection of all finite intersection of members of \( \Omega \) is a base for \( T \).

**Example 3**

Let \( X = \{ x_1, x_2, x_3, x_4 \}, E = \{ e_1, e_2, e_3 \} \) and \( \mathcal{R} = \{ f_\mu, G_\delta, N_\varphi \} \), where
\[ I_\sigma = \begin{cases} & \{ (x_1, x_2, x_3) \}, \ 0.2 \}, \\
I_\sigma(e_2) = \left( \left\{ \frac{x_1}{1}, \frac{x_3}{1}, \frac{x_4}{0.4} \right\} \right), \\
M_\eta = \left( \left\{ \frac{x_1}{1}, \frac{x_3}{1}, \frac{x_4}{0.4} \right\} \right), \\
M_\eta(e_2) = \left( \left\{ \frac{x_1}{1}, \frac{x_3}{1}, \frac{x_4}{1} \right\} \right), \\
M_\eta(e_3) = \left( \left\{ \frac{x_3}{0.3}, \frac{x_4}{1} \right\} \right), \\
N_\psi = \left( \left\{ \frac{x_3}{0.1}, \frac{x_4}{1} \right\} \right), \\
N_\psi(e_1) = \left( \left\{ \frac{x_3}{0.1}, \frac{x_4}{1} \right\} \right), \\
N_\psi(e_3) = \left( \left\{ \frac{x_3}{1}, \frac{x_4}{1} \right\} \right), \\
N_\psi(e_4) = \left( \left\{ \frac{x_3}{1}, \frac{x_4}{1}, \frac{x_5}{0.4} \right\} \right). \]

The collection of all finite intersection of members of \( \Omega \) is the base \( \mathcal{R} \) in Example 1. So \( \Omega \) is a subbase for a GFS topology on \((X, E)\).

**Theorem 2**

A collection \( \Omega \) of GFSSs over \((X, E)\) is a subbase for a suitable GFS topology \( T \) if and only if:

1. \( \emptyset \in \Omega \) or \( \emptyset \in \Omega \) is the intersection of finite number of members of \( \Omega \).
2. \( \mathcal{I}_\Delta = \bigcup \Omega \).

**Proof 2**

First let \( \Omega \) be a subbase for \( T \) and \( \mathcal{R} \) be a base generated by \( \Omega \). Since \( \emptyset \in \mathcal{R} \), either \( \emptyset \in \Omega \) or \( \emptyset \in \Omega \) is expressible as an intersection of many finite members of \( \Omega \). Now let \( x \in X \) and \( e \in E \). Since \( \bigcup \mathcal{R} = \mathcal{I}_\Delta \), there exists \( R_\mu \in \mathcal{R} \) such that \( R(e)(x) = 1 \) and \( \mu(e) = 1 \).

Since \( R_\mu \in \mathcal{R} \), there exists \( (K_\gamma)_i \in \Omega, i = 1,2, ..., n \) such that \( R_\mu = \bigcap_{i=1}^n (K_\gamma)_i \).

Therefore \( R(e)(x) = \min_{i=1}^n (K_\gamma(x))_i \), \( \mu(e) = \min_{i=1}^n (\gamma(e))_i \), and so \( R(e)(x) = (K_\gamma(x))_i \) for some \( i \in \{1,2, ..., n\} \), \( \mu(e) = (\gamma(e))_i \), for some \( i \in \{1,2, ..., n\} \).

Thus \( (K_\gamma(x))_i = 1 \) and \( (\gamma(e))_i = 1 \). Hence \( \mathcal{I}_\Delta = \bigcup \Omega \).

Conversely, let \( \Omega \) be collection of GFSSs over \((X, E)\) satisfying the conditions 1 and 2. Let \( \mathcal{R} \) be the collection of all finite intersection of members of \( \Omega \). Now it enough to show that \( \mathcal{R} \) forms base for suitable GFS topology.

Since \( \mathcal{R} \) is the collection of all finite intersection of members of \( \Omega \), by assumption 1 we get \( \emptyset \in \mathcal{R} \) and by 2 we get \( \bigcup \mathcal{R} = \mathcal{I}_\Delta \). Again let \( F_\mu, G_\delta \in \mathcal{R} \) and \( x \in X, e \in E \). Since \( F_\mu \in \mathcal{R} \), there exists \( (F_\mu)_i \in \Omega \) for \( i = 1,2, ..., n \) such that \( F_\mu = \bigcap_{i=1}^n (F_\mu)_i \). Again since \( G_\delta \in \mathcal{R} \), there exists \( (G_\delta)_j \in \Omega \) for \( j = 1,2, ..., m \) such that \( G_\delta = \bigcap_{j=1}^m (G_\delta)_j \).

Therefore, \( F_\mu \cap G_\delta = \bigcap_{i=1}^n (F_\mu)_i \cap \bigcap_{j=1}^m (G_\delta)_j \in \mathcal{R} \).

That is, \( F_\mu \cap G_\delta \in \mathcal{R} \). This completes the proof.

**GENERALIZED FUZZY SOFT TOPOLOGICAL SUBSPACES**

**Definition 1**

Let \((X, T, E)\) be a GFS topological space. Let \( Y \) be an ordinary subset of \( X \) and \( Y_\nu \) be GFSS over \((Y, E)\) such that:

\[ \forall e \in E, Y(e)(x) = \begin{cases} 1 & \text{if } x \in Y, \\
0 & \text{if } x \notin Y \end{cases} \]

That is \( Y(e) = Y, \forall e \in E \) and \( \nu(e) = 1 \).

Let \( T_Y = \{ Y_\nu \cap G_\delta : G_\delta \in T \} \). We can show that \( T_Y \) is a GFS topology on \((Y, E)\) as follows:

(i) Since \( \emptyset_\nu \in T \), \( \mathcal{I}_\Delta \in T \), \( (\mathcal{I}_\Delta)_Y = Y_\nu \cap \mathcal{I}_\Delta \) and \( (\emptyset_\nu)_Y = Y_\nu \cap \emptyset_\nu \), then \( (\emptyset_\nu)_Y, (\mathcal{I}_\Delta)_Y \in T_Y \).

(ii) Suppose that \( (F_\mu)_1, (F_\mu)_2 \in T_Y \). Then for each \( i = 1,2, \) there exist \( (G_\delta)_i \in T \) such that \( (F_\mu)_i = Y_\nu \cap (G_\delta)_i \). We have \( (F_\mu)_1 \cap (F_\mu)_2 = (Y_\nu \cap (G_\delta)_1) \cap (Y_\nu \cap (G_\delta)_2) = Y_\nu \cap [(G_\delta)_1 \cap (G_\delta)_2] \).

(iii) Let \( \{ (G_\delta)_i : i \in J \} \) be a subfamily of \( T_Y \). Then for each \( i \in J \), there is a GFSS \( (M_\sigma)_i \) of \( T \) such that \( (G_\delta)_i = Y_\nu \cap (M_\sigma)_i \). We have \( \bigcup_{i \in J} (G_\delta)_i \in T \) and \( \bigcup_{i \in J} (M_\sigma)_i \in T \).

Since \( \bigcup_{i \in J} (M_\sigma)_i \in T \), we have \( \bigcup_{i \in J} (G_\delta)_i \in T \).
$T_Y$ is called the GFS subspace topology on $(Y, E)$ and $(Y, T_Y, E)$ is called a GFS subspace of $(X, T, E)$.

$F_\mu = \left\{ F_\mu(e_1) = \left( \left( \frac{x_1}{0.04}, \frac{x_2}{0.3} \right), 0.1 \right), F_\mu(e_2) = \left( \left( \frac{x_1}{0.35}, \frac{x_2}{0.5} \right), 0.2 \right), F_\mu(e_3) = \left( \left( \frac{x_1}{0.5}, \frac{x_2}{0.1} \right), 0 \right) \right\}$,

$G_\delta = \left\{ G_\delta(e_1) = \left( \left( \frac{x_1}{0.4}, \frac{x_2}{0.1} \right), 0.2 \right), G_\delta(e_2) = \left( \left( \frac{x_1}{0.3}, \frac{x_2}{0.5} \right), 0.3 \right), G_\delta(e_3) = \left( \left( \frac{x_1}{0.6}, \frac{x_2}{0.1} \right), 0 \right) \right\}$.

We consider the GFS topology $T$ on $(X, E)$ as $T = \{ \emptyset, \{ x_1 \}, \{ x_1, x_2 \} \subseteq X \}$.

$T_Y = \{ \emptyset, \{ x_1 \}, \{ x_1, x_2 \} \subseteq Y \}$

(i) $\emptyset \cap Y_v = \emptyset_Y$.

(ii) $\{ x_1 \} \cap Y_v = \{ x_1 \}$.

(iii) $\{ x_1, x_2 \} \cap Y_v = \{ x_1, x_2 \}$.

(iv) $\{ x_1, x_2 \} \cap Y_v = \{ x_1, x_2 \}$.

$\mathcal{G}_\delta \cap Y_v = \{ \epsilon_r(e_1) = \left( \left( \frac{x_1}{0.3}, \frac{x_2}{0.5} \right), 0.2 \right), \epsilon_r(e_2) = \left( \left( \frac{x_1}{0.5}, \frac{x_2}{0.3} \right), 0.3 \right), \epsilon_r(e_3) = \left( \left( \frac{x_1}{0.6}, \frac{x_2}{0.1} \right), 0 \right) \}$.

$\mathcal{K}_\mu = \{ \epsilon_r(e_1) = \left( \left( \frac{x_1}{0.3}, \frac{x_2}{0.5} \right), 0.2 \right), \epsilon_r(e_2) = \left( \left( \frac{x_1}{0.5}, \frac{x_2}{0.3} \right), 0.3 \right), \epsilon_r(e_3) = \left( \left( \frac{x_1}{0.6}, \frac{x_2}{0.1} \right), 0 \right) \}$.

Thus, the collection $T_Y = \{ (\emptyset_Y)_Y, (\{ x_1 \})_Y, I_{\mathcal{G}_\delta}, I_{\mathcal{K}_\mu} \}$ is GFS a topology on $(Y, E)$.

**Theorem 1**

Let $(Y, T_Y, E)$ be a GFS subspace of $(X, T, E)$ and $F_\mu$ a GFSS over $(Y, E)$. Then

(i) $F_\mu$ is GFS closed in $(Y, E)$ if and only if $F_\mu = Y_v \cap \mathcal{G}_\delta$ for some GFS closed set $\mathcal{G}_\delta \in T_Y$.

(ii) $cl(F_\mu)_Y = \overline{F_\mu} \cap Y_v$ where $cl(F_\mu)_Y$ is the closure of $F_\mu$ in $(Y, E)$ with respective to $T_Y$.

**Proof 1**

(i) If $F_\mu$ is GFS closed in $(Y, E)$ then we have $F_\mu = Y_v \setminus \mathcal{G}_\delta$, for some $\mathcal{G}_\delta \in T_Y$.

Now, $\mathcal{G}_\delta = Y_v \cap M_\sigma$, for some $M_\sigma \in T$.

$F_\mu = Y_v \setminus (Y_v \cap M_\sigma) = Y_v \setminus (Y_v \cap M_\sigma^c) = Y_v \setminus (Y_v \cup M_\sigma^c)$.

$since$ $Y_v \cap Y_v^c = (\emptyset_Y)_Y$, see Definition 1) where $M_\sigma^c$ is GFS closed in $(X, E)$ as $M_\sigma \in T$. Conversely, assume that $F_\mu = Y_v \cap \mathcal{G}_\delta$ for some GFS closed set $\mathcal{G}_\delta$ in $(X, E)$. This means that $\mathcal{G}_\delta \in T$. Now, $Y_v \setminus F_\mu = Y_v \setminus \mathcal{G}_\delta = Y_v \setminus (Y_v \cap \mathcal{G}_\delta) = Y_v \cap (Y_v \cap \mathcal{G}_\delta) = Y_v \cap \mathcal{G}_\delta = Y_v \cap \mathcal{G}_\delta \in T_Y$ and hence $F_\mu$ is GFS closed in $(Y, E)$.

(ii) We have, $\overline{F_\mu}$ is a GFS closed set in $(X, E)$. Then $\overline{F_\mu} \cap Y_v$ is a GFS closed set in $(Y, E)$. Now $\overline{F_\mu} \subseteq \overline{F_\mu} \cap Y_v$, and GFS closure of $F_\mu$ in $(Y, E)$ is the smallest GFS closed set containing $F_\mu$.

so $cl(F_\mu)_Y \subseteq \overline{F_\mu} \cap Y_v$.

On other hand $\overline{F_\mu} = \overline{F_\mu} \cap Y_v$ where $\overline{F_\mu}$ is GFS closed in $(X, E)$. Then $\overline{F_\mu} \subseteq \overline{F_\mu} \cap Y_v = cl(F_\mu)_Y$.

It is useful to investigate the relationship between types of sets that are generalized fuzzy set.

**Example 1**

Let $X = \{ x_1, x_2, x_3 \}$, $E = \{ e_1, e_2, e_3, e_4 \}$. Consider $F_\mu$ and $G_\delta$ as follows:

$F_\mu \cap G_\delta$ for some GFS closed set $G_\delta$ in $(X, E)$. This means that $G_\delta \in T$. Now, $Y_v \setminus F_\mu = Y_v \setminus \mathcal{G}_\delta = Y_v \setminus (Y_v \cap \mathcal{G}_\delta) = Y_v \setminus (Y_v \cap \mathcal{G}_\delta) = Y_v \setminus \mathcal{G}_\delta \in T_Y$ and hence $F_\mu$ is GFS closed in $(Y, E)$.

(ii) We have, $\overline{F_\mu}$ is a GFS closed set in $(X, E)$. Then $\overline{F_\mu} \subseteq \overline{F_\mu} \cap Y_v$ is a GFS closed set in $(Y, E)$. Now $\overline{F_\mu} \subseteq \overline{F_\mu} \cap Y_v$, and GFS closure of $F_\mu$ in $(Y, E)$ is the smallest GFS closed set containing $F_\mu$.

so $cl(F_\mu)_Y \subseteq \overline{F_\mu} \cap Y_v$.

On other hand $\overline{F_\mu} = \overline{F_\mu} \cap Y_v$, so $cl(F_\mu)_Y$. Therefore, $\overline{F_\mu} \subseteq \overline{F_\mu} \cap Y_v = cl(F_\mu)_Y$.

It is useful to investigate the relationship between types of sets that are generalized fuzzy set.

**GENERALIZED FUZZY SOFT SET (GFSS) AND INTUITIONISTIC FUZZY SOFT SET(IFSS)**

**Lemma 1**

The relationships between the sets: FS, SS, FSS, IFSS, GFSS and GIFSS that generalized the crisp set (CS) notion are illustrated in Figure 1.

**Counter example 1**

Let $X = \{ x_1, x_2, x_3, x_4 \}$, and $E = \{ e_1, e_2, e_3 \}$.

(1) Let $t(F, E) = \{ (e_1, \{ x_1, 0.8, 0.1 \}, \{ x_2, 0.9, 0.1 \}, \{ x_3, 0.8, 0.1 \}, \{ x_4, 0.7, 0.2 \})$, $(e_2, \{ x_1, 0.6, 0.3 \}, \{ x_2, 0.65, 0.2 \}, \{ x_3, 0.7, 0.2 \}, \{ x_4, 0.65, 0.2 \})$, $(e_3, \{ x_1, 0.8, 0.2 \}, \{ x_2, 0.5, 0.3 \}, \{ x_3, 0.5, 0.4 \}, \{ x_4, 0.7, 0.2 \})$ is an IFSS but not GFSS.

(2) Let $t(F, E) = \{ (e_1, \{ x_1, 0.5 \}, \{ x_2, 0.7 \}, \{ x_3, 0.8 \}, \{ x_4, 0.1 \}) \}$, $(e_2, \{ x_1, 0.4 \}, \{ x_2, 0.5 \}, \{ x_3, 0.7 \}, \{ x_4, 0.2 \})$ is GFS but not IFSS.

(3) Let $t(F, E) = \{ (e_1, \{ x_1, 0.8 \}, \{ x_2, 0.3 \}, \{ x_3, 0.4 \}, \{ x_4, 0.7 \}) \}$ is GFS but not IFSS.
Let $X = x_1, x_2, x_3, x_4,$ and $E = e_1, e_2, e_3$. 
(1) Let $F, E = \{ e_1 = (x_1, 0.8, 0.1), (x_2, 0.9, 0.1), (x_3, 0.8, 0.1), \ldots, (x_3, 0.7, 0.2), (x_4, 0.65, 0.2), 0.6 \},$ 
$e_3 = (x_1, 0.8, 0.2), (x_2, 0.5, 0.3), (x_3, 0.5, 0.4), (x_4, 0.7, 0.2), 0.5 \}$ is GIFSS but neither IFSS nor GFSS. 

From the examples, we see that GFSS and IFSS are independent notions.

**Lemma 2**

Similarly, one can be deduce similar diagram of the relationship between analogues topologies.

**Conclusion**

In this paper, we have introduced generalized fuzzy soft point, generalized fuzzy soft open base and subbase. The generalized fuzzy soft topological subspaces is introduced. Finally, we concluded that GFSS and IFSS are independent notions, whereas each of them is GIFSS. So, one can try to introduce some special properties of compactness, some separation axioms, connected nessetc on generalized fuzzy soft topological spaces.

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests.

**REFERENCES**


