Review

Hesitant fuzzy prioritized operators and their application in multi-criteria group decision making

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Hesitant fuzzy set is a very useful technique in the situations where there are some difficulties in determining the membership of an element to a set. Some aggregation methods have been developed for hesitant fuzzy information. Current aggregation methods are under the assumption that the criteria are at the same priority level. However, in real decision making problems, criteria have different priority level commonly. In this paper, we investigated multi-criteria group decision making problems where a prioritization relationship existed over the criteria under hesitant fuzzy environment. Firstly, we propose the hesitant fuzzy prioritized weighted average (HFPWA) and the hesitant fuzzy prioritized weighted geometric (HFPWG) operators. Then, some of their desirable properties are studied in detail. Furthermore, the procedure of multi-criteria group decision making based on the proposed operators is given under hesitant fuzzy environment. Finally, a practical example about talent introduction is provided to illustrate the developed method.

Key words: Group decision making, hesitant fuzzy set, hesitant fuzzy prioritized weighted average operator, hesitant fuzzy prioritized weighted geometric operator.

INTRODUCTION

The fuzzy set (FS) theory introduced by Zadeh (1965) is a powerful technique to depict the uncertainty of the world. Based on the FS theory, many results have been made by scholars (Gil-Aluja, 1996; Gil-Aluja, 1998; Gil-Aluja, 1999; Gil-Aluja, 2003; Gil-Aluja and Gil-Lafuente, 2007; Gil-Aluja et al., 2009; Gil-Aluja et al., 2011; Kaufmann, 1975; Kaufmann, 1983; Kaufmann and Gil-Aluja, 1986; Kaufmann and Gil-Aluja, 1991; Merigó and Gil-Lafuente, 2008).

In order to better understand the uncertainty of the objective world and thus being able to explain it, the FS theory has been extended to many other forms. Such as interval-valued fuzzy set (Zadeh, 1975), type-2 fuzzy set (Dubois and Prade, 1980; Miyamoto, 2005), type-\(\frac{n}{n}\) fuzzy set (Dubois and Prade, 1980), fuzzy multiset (Yager, 1986; Miyamoto, 2000) and intutionistic fuzzy set (Atanassov, 1986). However, when defining the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possible values, but because we have a set of possible values (Torra, 2010). The above generalization forms of FS cannot express this kind of uncertainty effectively. To deal with this situation, Torra and Narukawa (2009) and Torra (2010) proposed another generalization form of FS, which is called the hesitant fuzzy set (HFS). The characteristic of HFS is that it allows membership degree having a set of possible values. For example, three ladies are to estimate the degrees that a gentleman satisfies the criterion of handsome. The first lady provides 0.7, the second lady provides 0.8 and the third provides 0.9; however, these three ladies cannot persuade each other, thus the degrees that the gentleman satisfy the criterion of handsome can be represented by a hesitant fuzzy set \(\{0.7, 0.8, 0.9\}\). Due to its powerfulness in expressing the uncertainty of real situation, HFS has achieved a lot of attention from researchers (Torra and Narukawa, 2009; Torra, 2010; Xu and Xia, 2011; Xia and Xu, 2011a). Xia and Xu (2011b) developed a series of
aggregation operators for hesitant fuzzy information and discussed the relationship between intuitionistic fuzzy set and hesitant fuzzy set. But the aggregation operators for hesitant fuzzy information are assuming that the criteria are at the same priority level. They are characterized by the ability to trade off between criteria. For example, if $C_i$ and $C_j$ are two criteria with the weight $w_i$ and $w_j$ respectively. By the above aggregation operators, we can compensate for a decrease of $q$ in satisfaction to criteria $C_i$ by gain $w_q$ in satisfaction to criteria $C_j$. However, in many real decisions making problems, this kind of compensation between criteria is not feasible. Consider the situation in which editor in chief is making a decision based on consideration of innovation and expression for a manuscript. He should not allow a benefit with respect to expression of the manuscript to compensate for a loss in innovation. This is a typical kind of prioritization of the criteria that is, innovation has a higher priority than expression. Yager (2008), Yager (2009) and Yager et al. (2011) have paid attention on this issue. In this paper, we research on the aggregation method for hesitant fuzzy information which has prioritization relationships between the criteria.

Some basic concepts

As a generalization of FS, HFS is characterized by the membership degree of an element to a set presented as several possible values between 0 and 1, so it is more powerful to deal with hesitancy and uncertainty in real applications than Zadeh’s FS. Torra and Narukawa (2009) and Torra (2010) first proposed the concept of hesitant fuzzy set (HFS).

Definition 1. (Torra and Narukawa, 2009; Torra, 2010) let $X$ be a fixed set, a HFS on $X$ is in terms of a function that when applied to $X$, returns a subset of $[0, 1]$ which can be represented as the following mathematical symbol:

$$ E = \{< x, f_E(x)> | x \hat{\in} X \} \tag{1} $$

where $f_E(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x$ to the set $E$. For convenience, we call $f_E(x)$ a hesitant fuzzy element (HFE). For three HFEs $f_1, f_2$, and $f_3$, Xia and Xu (2011b) gave the following operational laws on the HFEs as follows:

Definition 2. Let $f_1, f_2$, and $f_3$ be three HFEs, then

1) $f_1 = U_{x \in f_1} \{x^1 \}, l > 0$;
2) $f_2 = U_{x \in f_1} \{1 - (1 - x^1) \}, l > 0$;
3) $f_1 \hat{\&} f_2 = U_{x \in f_1, x \in f_2} \{x_1 + x_2 - x_1 x_2 \}$;
4) $f_1 \hat{\&} f_2 = U_{x \in f_1, x \in f_2} \{x_1 x_2 \}$.

In order to compare two HFEs, Xia and Xu (2011b) introduced the following method:

Definition 3. For a HFE $f$, the score function of $f$, denoted by $\# f$, is called the score function of $f$, where $\# f$ is the number of the elements in $f$. For two HFEs $f_1$ and $f_2$, if $S(f_1) > S(f_2)$, then $f_1 > f_2$; if $S(f_1) = S(f_2)$, then $f_1 = f_2$. Since $x \hat{\in} \hat{\emptyset}, \hat{\emptyset} \subseteq \hat{\emptyset}$, we have $S(f) = \hat{\emptyset}, \hat{\emptyset}$.

Yager (2008) first proposed the Prioritized Average (PA) operator, which was defined as follows:

Definition 4. (Yager, 2008) let $C = \{C_1, C_2, \ldots, C_n \}$ be a collection of criteria and there is a prioritization between the criteria expressed by the linear ordering $C_1 \rightarrow C_2 \rightarrow C_3 \ldots \rightarrow C_n$, indicating criteria $C_j$ has a higher priority than $C_k$ if $j < k$. The value $C_j(x)$ is the performance of any alternative $x$ under criteria $C_j$ and satisfies $C_j(x) \hat{\in} \hat{\emptyset}, \hat{\emptyset} \subseteq \hat{\emptyset}$. If

$$ PA(C_i(x)) = \hat{\sum}_{j=1}^{n} w_j C_j(x) \tag{2} $$
Hesitant fuzzy prioritized operators

When defining the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have a margin of error (as in intuitionistic fuzzy set or interval-valued fuzzy set), or some possibility distribution (as in type-2 fuzzy set) on the possible values, but because we have a set of possible values (Torra, 2010). HFS is a powerful technique to deal with this situation. In this section, we shall investigate the information aggregation operators which consider that there are prioritization relationships between criteria under hesitant fuzzy environment. Based on Definition 4, we give the definition of the HFPWA as follows:

Definition 5. Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs and let HFPWA: \( V^n \otimes V \), if

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{T_1}{\hat{a}^n} f_1 \frac{T_2}{\hat{a}^n} f_2 \ldots \frac{T_n}{\hat{a}^n} f_n
\]

then the function HFPWA is called an HFPWA operator, where \( T_j = \hat{O}^{j-1} S(f_k) \) \( (j = 2, \ldots, n) \), \( T_1 = 1 \) and \( S(f_k) \) is the score of HFE \( f_k \). We can drive the Theorem 1 based on Definitions 2 and 5.

Theorem 1. Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, then their aggregated value by using the HFPWA operator is also an HFE, and

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{1}{\hat{a}^n} \sum_{j=1}^{n} T_j \left( 1 - x_j \right)^{\frac{T_j}{\hat{a}^n}}
\]

where \( T_j = \hat{O}^{j-1} S(f_k) \) \( (j = 2, \ldots, n) \), \( T_1 = 1 \) and \( S(f_k) \) is the score of HFE \( f_k \).

Proof. The first result follows quickly from Definition 2 and Theorem 1. In the following, we prove HFPWA

\[
(f_1, f_2, \ldots, f_n) = \frac{T_1}{\hat{a}^n} f_1 \frac{T_2}{\hat{a}^n} f_2 \ldots \frac{T_n}{\hat{a}^n} f_n
\]

By using mathematical induction on \( n \):

1) For \( n = 2 \), since

\[
\frac{T_1}{\hat{a}^n} f_1 = U_{x_1} \left( 1 - \left( 1 - x_1 \right)^{\frac{T_1}{\hat{a}^n}} \right)
\]

\[
\frac{T_2}{\hat{a}^n} f_2 = U_{x_2} \left( 1 - \left( 1 - x_2 \right)^{\frac{T_2}{\hat{a}^n}} \right)
\]

We have

\[
\text{HFPWA}(f_1, f_2) = U_{x_1, x_2} \left( 1 - \left( 1 - x_1 \right) \left( 1 - x_2 \right) \right)^{\frac{T_1}{\hat{a}^n}}
\]

That is, for \( n = 2 \), the Equation 5 holds. Suppose \( n = k \), the Equation 5 holds. That is,
Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, where \( S(f_k) \) is the score of HFE \( f_k \). If for all \( j \), \( x_j = x \), where \( x_j \) are elements of hesitant fuzzy set \( f_j \), \( x \) is the element of hesitant fuzzy set \( f \), then

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{T_j}{\hat{a}_j} \tag{9}
\]

Hence, when \( n = k + 1 \), by the operational laws described in Definition 2, we have,

\[
\text{HFPWA}(f_1, f_2, \ldots, f_{k+1}) = \frac{T_j}{\hat{a}_j} \tag{10}
\]

That is Equation 5 holds for \( n = k + 1 \). Thus, Equation 5 holds for all \( n \). Then

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{T_j}{\hat{a}_j} \tag{11}
\]

Now, we look at some desirable properties of the HFPWA operator.

**Theorem 2.** Let \( f_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, where

\[
T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \quad (j = 2, \ldots, n), \quad T_1 = 1
\]

and \( S(f_k) \) is the score of HFE \( f_k \). If for all \( j \), \( x_j = x \), where \( x_j \) are elements of hesitant fuzzy set \( f_j \), \( x \) is the element of hesitant fuzzy set \( f \), then

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{T_j}{\hat{a}_j} \tag{12}
\]

Proof. By Theorem 1, we have

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \frac{T_j}{\hat{a}_j} \tag{13}
\]

which completes the proof of Theorem 2.

**Corollary 1.** If \( f_j (j = 1, 2, \ldots, n) \) be a collection of the largest HFEs, that is, \( f_j = f^* = \{1\} \), for all \( j \), then

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \text{HFPWA}(f^*, f^*, \ldots, f^*) = \{1\} \tag{14}
\]

which is also the largest HFE.

Proof. Similar to the proof of Theorem 2, we can get Corollary 1 easily.

**Corollary 2.** (Non-compensatory) If \( f_1 \) is the smallest HFE, that is, \( f_1 = f_1 = \{0\} \), then

\[
\text{HFPWA}(f_1, f_2, \ldots, f_n) = \text{HFPWA}(f_1, f_2, \ldots, f_n) = \{0\} \tag{15}
\]

which is also the smallest HFE.

Proof. Since \( f_1 = \{0\} \), then by the Definition 3, we have,
According to Theorem 1, we have

$$\text{HFPWA} (f_1 \, \hat{A} \, h, f_2 \, \hat{A} \, h, \ldots, f_n \, \hat{A} \, h) = \text{U}_{j=1}^{n} f_{j, g} (1 - g)$$

(23)

According to the operational laws of Definition 2, we can get

$$\text{HFPWA} (f_1, f_2, \ldots, f_n) \, \hat{A} \, h =$$

$$\text{U}_{j=1}^{n} (1 - x_j) \, \hat{A} \, g$$

(24)

Thus

$$\text{HFPWA} (f_1 \, \hat{A} \, h, f_2 \, \hat{A} \, h, \ldots, f_n \, \hat{A} \, h) \, \hat{A} \, h$$

(25)

which completes the proof of Theorem 3.

Theorem 4. Let $f_j \ (j = 1, 2, \ldots, n)$ be a collection of HFEs, where $T_j = \tilde{S}_{k=1}^{j} S(f_k) \ (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_k)$ is the score of HFE $f_k$. If $h$ is an HFE, $g$ is elements of hesitant fuzzy set $h$, $x_j$ is elements of hesitant fuzzy set $f_j$, then

$$\text{HFPWA} (f_1 \, \hat{A} \, h, f_2 \, \hat{A} \, h, \ldots, f_n \, \hat{A} \, h) =$$

$$\text{HFPWA} (f_1, f_2, \ldots, f_n) \, \hat{A} \, h$$

(21)

Proof. Since for any $j$,

$$f_j \, \hat{A} \, h = \text{U}_{j=1}^{n} f_{j, g} (x_j + g - x_j g) = \text{U}_{j=1}^{n} f_{j, g} (1 - (1 - x_j)(1 - g))$$

(22)
\[ l f_j = U_{x_j l f_j} \left\{ 1 - (1 - x_j)^y \right\} \]  \quad (27)

According to Theorem 1, we have

\[
\text{HFPWA}(r f_1, r f_2, K, r f_n) = U_{x_j l f_j} \left( 1 - \frac{\sum_{j=1}^n \left( 1 - x_j \right)^y}{\sum_{j=1}^n \left( 1 - x_j \right)^y} \right)^{t_j} \quad (28)
\]

\[
r \text{HFPWA}(f_1, f_2, K, f_n) = U_{x_j l f_j} \left( 1 - \frac{\sum_{j=1}^n \left( 1 - x_j \right)^y}{\sum_{j=1}^n \left( 1 - x_j \right)^y} \right)^{t_j} \quad (29)
\]

Thus

\[
\text{HFPWA}(r f_1, r f_2, K, r f_n) = r \text{HFPWA}(f_1, f_2, K, f_n) \quad (30)
\]

According to Theorems 3 and 4, we can get Theorem 5 easily.

**Theorem 5.** Let \( f_j (j = 1, 2, L, n) \) be a collection of HFES, \( T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \) \( (j = 2, \ldots, n) \), \( T_1 = 1 \) and \( S(f_k) \) is the score of HFE \( f_k \), if \( r > 0 \), \( h \) is an hesitant fuzzy element, then

\[
\text{HFPWA}(r f_1 \tilde{A} h, r f_2 \tilde{A} h, K, r f_n \tilde{A} h) = r \text{HFPWA}(f_1, f_2, K, f_n) \tilde{A} h \quad (31)
\]

\[
\text{Theorem 6.} \quad \text{Let} \quad f_j (j = 1, 2, L, n) \quad \text{and} \quad h_j (j = 1, 2, L, n) \quad \text{be two collections of HFES,} \quad T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \quad (j = 2, \ldots, n), \quad T_1 = 1 \quad \text{and} \quad S(f_k) \quad \text{is the score of HFES} \, f_k, \quad \text{then}
\]

\[
\text{HFPWA}(f_1 \tilde{A} h_1, f_2 \tilde{A} h_2, K, f_n \tilde{A} h_n) = \text{HFPWA}(f_1, f_2, K, f_n) \tilde{A} \text{HFPWA}(h_1, h_2, K, h_n) \quad (32)
\]

Proof. According to the operational laws of Definition 2, we have

\[
f_j \tilde{A} h_j = U_{x_j l f_j, x_j l h_j} \left\{ x_j + g_j - x_j g_j \right\} =
\]

\[
U_{x_j l f_j, x_j l h_j} \left\{ 1 - (1 - x_j)(1 - g_j) \right\} \quad (33)
\]

According to Theorem 1, we have

\[
\text{HFPWA}(f_1 \tilde{A} h_1, f_2 \tilde{A} h_2, K, f_n \tilde{A} h_n) = \text{HFPWA}(f_1, f_2, K, f_n) \tilde{A} \text{HFPWA}(h_1, h_2, K, h_n) \quad (34)
\]

Thus

\[
\text{HFPWA}(r f_1 \tilde{A} h, r f_2 \tilde{A} h, K, r f_n \tilde{A} h) = r \text{HFPWA}(f_1, f_2, K, f_n) \tilde{A} h \quad (31)
\]
where $T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \ (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_k)$ is the score of HFE $f_k$.

Proof. The prove of Theorem 7 is similar to Theorem 1.

**Theorem 8.** Let $f_j (j = 1, 2, L, n)$ be a collection of HFes, where $T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \ (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_k)$ is the score of HFE $f_k$. If for all $j$, $x^*_j = x^*$, where $x^*_j$ are elements of hesitant fuzzy set $f_j$, $x^*$ is the element of hesitant fuzzy set $f$, then

$$HFPWG(f_1, f_2, \ldots, f_n) = f$$

(39)

Proof. The prove of Theorem 8 is similar to Theorem 2.

**Theorem 9.** Let $f_j (j = 1, 2, L, n)$ be a collection of HFes, where $T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \ (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_k)$ is the score of HF2E $f_k$. If $h$ is an HFE, $g$ are elements of hesitant fuzzy set $h$, $x^*_j$ are elements of hesitant fuzzy set $f_j$, then

$$HFPWG(f_1, f_2, \ldots, f_n) \tilde{A} h$$

(40)

Proof. The prove of Theorem 9 is similar to Theorem 3.

**Theorem 10.** Let $f_j (j = 1, 2, L, n)$ be a collection of HFes, where $T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \ (j = 2, \ldots, n)$, $T_1 = 1$ and $S(f_k)$ is the score of HF2E $f_k$. If $r > 0$, then

$$\tilde{\alpha}(f_1) \tilde{\alpha}(f_2) \tilde{\alpha}(f_n) \tilde{A} \tilde{O}^{\gamma}$$

(41)

Proof. The prove of Theorem 10 is similar to Theorem 4. According to Theorems 9 and 10, we can get Theorem 11 easily.
Theorem 11. Let \( f_j (j = 1, 2, L, n) \) be a collection of HFEs, \( T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \) (\( j = 2, \ldots, n \)), \( T_1 = 1 \) and \( S(f_k) \) is the score of HFE \( f_k \), if \( r > 0 \), \( h \) is an hesitant fuzzy element, then

\[
\text{HFPWA} \left( f_j, h_j; K, f_n, h_n \right) = \tilde{O}_{k=1}^{n} S(f_k), \quad \tilde{O}_{k=1}^{n} \frac{S(f_k)}{h_k}, \quad (\text{HFPWG}(f_1, f_2, K, f_n))
\]

(42)

Theorem 12. Let \( f_j (j = 1, 2, L, n) \) and \( h_j (j = 1, 2, L, n) \) be two collections of HFEs, \( T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \) (\( j = 2, \ldots, n \)), \( T_1 = 1 \) and \( S(f_k) \) is the score of IFV \( f_k \), then

\[
\text{HFPWG} \left( f_1, h_1, f_2, h_2, K, f_n, h_n \right) = \text{HFPWG} \left( (f_1, f_2, K, f_n) \text{ HFPWG} \right) \left( h_1, h_2, K, h_n \right)
\]

(43)

Proof. The prove of Theorem 12 is similar to Theorem 6. In order to investigate the relationship between the HFPWA and HFPWG operators, we introduce the following Lemma.

Lemma 1. (Xu, 2000) let \( x_j > 0 \), \( l_j > 0 \), \( j = 1, 2, L, n \), and \( \tilde{a}_{j=1}^{n} l_j = 1 \), then

\[
\tilde{a}_{j=1}^{n} x_j \leq \tilde{a}_{j=1}^{n} l_j x_j
\]

with equality if and only if \( x_1 = x_2 = L = x_n \).

Theorem 13. Let \( f_j (j = 1, 2, L, n) \) be a collection of HFEs, \( T_j = \tilde{O}_{k=1}^{j-1} S(f_k) \) (\( j = 2, \ldots, n \)), \( T_1 = 1 \) and \( S(f_k) \) is the score of HFE \( f_k \), then

\[
\text{HFPWG} \left( f_1, f_2, L, f_n \right) \text{ HFPWA} \left( f_1, f_2, L, f_n \right)
\]

(45)

Proof. For any \( x_j \), \( \tilde{a} f_j (j = 1, 2, \ldots, n) \), based on Lemma 1, we have

\[
\tilde{a}_{j=1}^{n} x_j \leq \tilde{a}_{j=1}^{n} l_j x_j
\]

which implies that and completes the proof of Theorem 13.

From Theorem 13 we find that the values obtained by the HFPWA operator are greater than the ones obtained by the HFPWG operator.

Approach to multi-criteria group decision making under intuitionistic fuzzy environment

As is well known, group decision making (GDM) can obtain more accurate and objective results, which has attracted more and more attentions (Jin and Liu, 2010; Yang et al., 2010; Merigó, 2011; Li et al., 2010; Chen et al., 2011). Consider a group decision making problem under uncertainty. Let \( X = \{ x_1, x_2, \ldots, x_m \} \) be the set of alternatives, \( C = \{ C_1, C_2, \ldots, C_n \} \) the set of criteria and that there is a prioritization between the criteria expressed by the linear ordering \( C_1 f C_2 f C_3 \ldots f C_n \), indicating criteria \( C_j \) has a higher priority than \( C_i \) if \( j < i \), \( E = \{ e_1, e_2, \ldots, e_p \} \) the set of decision makers.

The decision maker \( e_k \) evaluate the alternative \( x_i \) under criteria \( C_j \) anonymously so as to protect the decision makers’ privacy or avoid psychic contagion. Then, the evaluation for alternative \( x_i \) under criteria \( C_j \) provided by the decision makers \( e_k \) \( (k = 1, 2, \ldots, p) \) are expressed by several values. IF two decision makers provide the same value, then the value emerges only once, then the evaluation can be represented by HFEs, based on which the hesitant fuzzy group decision matrix \( F = (f_{ij}) \) is constructed, and \( f_{ij} \) is the attribute value provided by the
decision makers \( e_k (k = 1, 2, \ldots, p) \), which is expressed in an HFE.

Based on the above analysis, the main steps of the multi-criteria group decision making method are as follows:

Step 1. All the decision makers provide their evaluations about the alternative \( x_i \) under the attribute \( C_j \), denoted by the hesitant fuzzy element \( f_{ij} \) \((i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\), and construct the group decision matrix \( F = \left( f_{ij} \right)_{m \times n} \).

Step 2. Calculate the values of \( T_{ij} \) \((i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\) based on following equations.

\[
T_{ij} = \bar{O}^{-1}_{k=1} S(r_{ik}) \quad (i = 1, 2, \ldots, m, j = 2, \ldots, n)
\]

\[
T_{i1} = 1 \quad i = 1, 2, \ldots, m
\]

Step 3. Aggregate the hesitant fuzzy values \( f_{ij} \) for each alternative \( x_i \) by the HFPWA (or HFPWG) operator.

\[
f_i = \left[ \begin{array}{c}
U_{x_j f_{ij}}^{x_j f_{ij}} \left( \frac{1}{\# f^{-1}_i} \tilde{a}_{x_j f_{ij} x_j} \right) \sum_{j=1}^{n} \frac{r_{ij}}{\tilde{a}_{x_j f_{ij} x_j}} \end{array} \right]_{i = 1, 2, \ldots, m}
\]

HFPWA

\[
f_i = \left[ \begin{array}{c}
U_{x_j f_{ij}}^{x_j f_{ij}} \left( \frac{1}{\# f^{-1}_i} \tilde{a}_{x_j f_{ij} x_j} \right) \sum_{j=1}^{n} \frac{r_{ij}}{\tilde{a}_{x_j f_{ij} x_j}} \end{array} \right]_{i = 1, 2, \ldots, m}
\]

HFPWG

Step 4. Rank all the alternatives by the score function defined by Definition 3.

\[
S(f_i) = \frac{1}{\# f^{-1}_i} \tilde{a}_{x_i f_{ij} x_i} \quad i = 1, 2, \ldots, m
\]

then the bigger the value of \( S(f_i) \), the larger the overall HFE \( \tilde{f}_i \) and thus the alternative \( x_i \) \((i = 1, 2, \ldots, m)\).

Example 1. Work to strengthen academic education, promote the building of teaching body, the school of management in a Chinese university wants to introduce overseas outstanding teachers. Five professors coming from the school of management set up the panel of decision makers which will take the whole responsibility for this introduction. They made strict evaluation for five candidates \( x_i \) \((i = 1, 2, \ldots, 5)\) from four aspects, namely morality \( C_1 \), research capability \( C_2 \), teaching skill \( C_3 \), education background \( C_4 \). This introduction will be in strict accordance with the principle of combine ability with political integrity. The prioritization relationship for the criteria is as follow, \( C_1 \prec C_2 \prec C_3 \prec C_4 \). Five professor \( e_k \) \((k = 1, 2, \ldots, 5)\) evaluate the candidates \( x_i \) \((i = 1, 2, 3, 4, 5)\) with respect to the attributes \( C_j \) \((j = 1, 2, \ldots, 4)\) anonymously. In order to sort all the candidates and select the best one, the main steps are given as follows:

Step 1. Five professors provide their evaluations about the five candidates under morality, research capability, teaching skill, education background respectively.

Denoted by the hesitant fuzzy element \( f_{ij} \) \((i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 4)\) and construct the following hesitant fuzzy group decision \( F = (f_{ij})_{5 \times 4} \) (Table 1).

Step 2. Calculate the values of \( T_{ij} \) \((i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 4)\), based on Equation 47 and 48.

\[
T_{ij} = \begin{pmatrix}
1.0000 & 0.3000 & 0.1275 & 0.0701 \\
1.0000 & 0.4333 & 0.2311 & 0.1248 \\
1.0000 & 0.5750 & 0.3019 & 0.1660 \\
1.0000 & 0.4667 & 0.1400 & 0.0840 \\
1.0000 & 0.3000 & 0.1200 & 0.0480
\end{pmatrix}
\]
Step 3. Utilize the HFPWA operator (Equation 49) to aggregate all the preference values \( f_{ij} \) \((i = 1, 2, 3, 4, 5)\) in the \( i \) th line of \( F \), and get the overall preference values \( f_i \). Here, we will not list them for data are much, just take \( f_4 \) for example.

\[
f_4 = \{ 0.3363, 0.3588, 0.3485, 0.3705, 0.3853, 0.3870, 0.4077, 0.3982, 0.4186, 0.4124, 0.4323, 0.4561, 0.4745, 0.4667, 0.4797, 0.3847, 0.3441 \}
\]

Step 4. Calculate the scores of \( f_i \) \((i = 1, 2, 3, 4, 5)\) respectively.

\[
\begin{align*}
S_1 &= 0.3847, \\
S_2 &= 0.5198, \\
S_3 &= 0.6037, \\
S_4 &= 0.4797, \\
S_5 &= 0.3441
\end{align*}
\]

Since

\[
S_3 > S_2 > S_4 > S_1 > S_5
\]

We have

\[
x_3 f x_2 f x_4 f x_1 f x_5
\]

The best option is candidate \( x_3 \).

**CONCLUDING REMARKS**

In this paper, we have studied the hesitant fuzzy information aggregation problems where there is a prioritization relationship over the criteria, and have proposed the hesitant fuzzy prioritized weighted average operator and the hesitant fuzzy prioritized weighted geometric operator on the basis of the idea of prioritized average. The significant feature of the proposed operators is that they consider prioritization among the criteria. Some of their desirable properties are investigated in detail. Then, we have applied our operators to develop a method of multi-criteria group decision making under hesitant fuzzy environment. Finally, an example is given to illustrate the given method. The proposed multi-criteria group decision making method considers prioritization relationship among these criteria, which allows our method to have more wide practical application potentials.

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**REFERENCES**


