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A new approach to robust modeling of the multi-period portfolio problem

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In this paper, we developed a new robust model of multi-period portfolio problem. One of the key concerns in any asset allocation problem is how to cope with uncertainty about future returns. There are some approaches in the literature for this purpose including stochastic programming and robust optimization. Applying these techniques to multi-period portfolio problem may increase the problem size in a way that the resulting model is intractable. In this paper, we proposed a novel approach to formulate multi-period portfolio problem as an uncertain linear program assuming that asset return follows the single-index factor model. We also used robust optimization technique to solve the resulted problem. In order to evaluate the performance of the proposed model, we applied a numerical example using simulated data.

Key words: robust optimization, multi-period portfolio problem, asset pricing model, parameter uncertainty.

INTRODUCTION

The capital asset pricing model (CAPM) plays an important role in finance theory. The idea of CAPM was introduced by three researchers who worked independently based on Markowitz's portfolio optimization model (Sharpe, 1964; Lintner, 1965; Mossin, 1966). As reported by Graham and Harvey (2001), the CAPM is the most popular model to estimate the cost of equity capital. The uncertainty about future reward from investing in stock market is considered as risk that one has to bear. As the risk increases in a portfolio, obviously, there is higher potential return on asset. The CAPM considers the risk and the rates of the return and compares them to the overall stock market.

There are two important types of risk in stock market (Markowitz, 1959). First, the risk which is the consequence of market related factors also known as systematic risk. Second, the risk which is firm-specific and only embraces the company factors called unsystematic risk. This kind of risk can be easily removed from the overall portfolio through diversification and the market normally awards those who bear the systematic risk. The final goal of portfolio theory is to help the investors allocating their wealth among different financial securities in an optimal way.

In the classical Markowitz model, the selection is assigned by a quantitative criterion which considers a balance between the return of an investment and its corresponding risk (Markowitz, 1952). He formulates the portfolio problem as a decision of the mean and variance of a portfolio. He proves the basic theorem of mean variance portfolio theory, namely holding constant variance will maximize the expected portfolio return, and on the other hand, holding constant expected return will minimize the portfolio variance. These two principles lead to the formulation of an efficient frontier from which the investor could choose her preferred portfolio, depending on individual risk return preferences namely risk aversion.

Specifically, in the Markowitz approach, each asset is described by means of its return over a fixed period of time and the vector of asset returns is assumed to be normally distributed, with known mean vector μ and covariance matrix Σ. Keeping in mind the Markowitz assumptions, an optimal portfolio weights of assets are hence determined by minimizing the portfolio variance $w' \Sigma w$, subject to a given lower bound on the expected return. This leads to a quadratic programming problem which may be efficiently solved using a traditional solution procedures such as active set method (Best et al., 1985). However, a major issue on the Markowitz method is that it only considers the asset allocation for a single period investment.

One of the key issues facing any given investor is how to allocate her wealth among different alternative assets. Almost all financial institutions have the

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identical problem with the growing complication that they have to include the characteristics of their liabilities in the analysis explicitly. Despite the fact that the arrangement of these problems varies, they are similar enough in such a way that they could be classified both as portfolio theory. Portfolio theory is a well-developed paradigm among practitioners in finance. There are good reviews in more advanced level (Ingersoll, 1987; Huang and Litzenberger, 1988; Constantinides and Malliaris, 1995). There are also some meticulous mathematical treatments (Szego, 1980).

The important aspect of the theory is that assets could not be selected only on the basis of a single security. The investor might as well consider how each security co-move with all the other securities. Furthermore, taking these co-movements into account resulted in an ability to construct a portfolio that had the same expected return and less risk than a portfolio constructed by ignoring the interactions among various securities. So far, we have been concerned with how to construct an optimal portfolio. The final payoff from portfolio models is mainly a function of the quality of the data used. Portfolio evaluation is an important task of the portfolio managers. Evaluation models follow in a natural manner from the theories of portfolio management, the subject include when and how the investor should rebalance her portfolio (Elton and Gruber, 1997).

Merton (1971, 1973) is believed to be the first who introduced the concept of multi-stage portfolio optimization; where an approach based on continuous dynamic programming (DP) is proposed and it is still in use (Brennan et al., 1997; Lynch and Balduzzi, 2000; Ait-Sahalia et al., 2001). However, the dynamic programming formulation is impractical for actual numerical implementation, due to the curse of dimensionality. DP often becomes a complicated problem as we consider more stages in the problem formulation (Brandt, 1999; Brennan et al., 1997). On the other hand, a mean-variance discrete-time problem is reduced to a control problem with only one state variable in Infanger(2006), under the hypotheses of no transaction costs, no composition constraints and independent returns. It should be noted that the introduction of constraints on portfolio composition makes the problem harder from the computational point of view.

Dantzig and Infanger (1993) formulate multi-stage portfolio allocation as a linear programming problem and propose a classical stochastic programming method based on Bender's decomposition. Many of these models which include recursive decision problems in the presence of uncertainty are solved by multi-stage stochastic programming (Birge et al., 1997; Gulpinar et al., 2002; Ruszczynski et al., 2003). Although stochastic programming provides a sound conceptual framework for posing multi-stage decision problems, its results is computationally impervious to exact and efficient numerical solution (Shapiro et al.,

2005). The main issue in the stochastic programming formulation is the fact that the stage decisions are actually conditional rules defining the action which should be taken in response to past outcomes. To model the conditional nature of the problem in a tractable manner, a discretization of the decision space is typically introduced by constructing a scenario tree. This scenario tree and its corresponding computational time may grow exponentially if an accurate discretization is needed (Ermoliev et al., 1988). On the other hand, if branching is kept low the resulting discretization cannot be guaranteed to be a valid representation of real world conditions. Portfolio optimization normally requires good estimates of returns for risky assets as input. The quality of resulting optimal solution depends heavily on the quality of the estimated parameters. However, any estimation procedure for the return is based on some assumptions which may or may not hold. In other words, we live in an uncertain world where many parameters affecting the return of an asset are subject to uncertainty.

Linear programming (LP) is a mathematical model to determine the optimal outcome for a given mathematical model for some types of requirements represented as linear equations known as constraints. Actually, linear programming is a technique of the optimization of a linear function e.g. $f(x)= c'x$ with respect to linear equality and/or inequality constraints e.g. $Ax \geq b$, where is the vector of variables to be decided, and are vectors of known coefficients and is a deterministic matrix of constants which is also known as technical coefficients. The availability and certainty of technical coefficients is under question in many practical situations.

During the past two decades, the idea of robust optimization has become an interesting area of research. Soyster (1973) is the first one who shed light on this subject, but his idea turns to be very pessimistic which makes it unfavorable among practitioners. Ben-Tal et al., (1998) develop new robust methodology where the optimal solution is more optimistic. Their idea suggests solving a counterpart of the initial model to obtain the robust solution. They also apply their robust method on some portfolio optimization problems and show that the final optimal solution remains feasible against the uncertainty on different input parameters (Ben-Tal et al., 2000). The proposed parameter controls the probability of deviation from the nominal constraints. The implementation of the robust optimazation of Ben-Tal normally changed an ordinary linear programming problem into a convex nonlinear problem. This makes their method unpopular among many people who get used to using regular optimization techniques. Bertsimas et al. (2004) developed different robust optimization techniques in an attempt to keep the structure of the original problem. Although their method does not provide solutions which are as optimistic as the Ben-Tal's method, the structure of the original problem remains the same and in many cases,

this makes it more applicable. As a recently emerging modeling tool, robust optimization can incorporate the perturbations in the parameters of the problem into the decision making process. Generally speaking, robust optimization aims to find solutions to given optimization problems with uncertain problem parameters, that will achieve good objective values for all or most of realizations of the uncertain problem parameters (Gharakhani et al., 2010). Robust optimization approach has addressed portfolio selection problems in order to alleviate the sensitivity of optimal portfolios to statistical errors in the estimates of problem parameters. Goldfarb et al. (2003) consider a factor model for the random portfolio returns, and proposed some statistical procedures for constructing uncertainty sets for the model parameters. They propose a factor model for the asset return and demonstrate that, by using a robust optimization approach to deal with the parameter ambiguity, the mean–variance portfolio selection problem, the maximum Sharpe ratio portfolio selection problem and the Value-at-Risk (VaR) portfolio selection problem can be reformulated as Second Order Cone Programs (SOCP), provided that the parameter ambiguity sets are ellipsoidal. DeMiguel et al.,(2009) propose a generalized approach to portfolio optimization in the presence of estimation error. Their method relies on solving the traditional minimum-variance problem but subject to the additional constraint that the norm of the portfolio-weight vector be smaller than a given threshold.

Zhu et al. (2009) propose a robust portfolio optimization based on worst-case conditional Value-At-Risk. They consider the situation where only partial information on the underlying probability distribution is available. They investigate the minimization of the worst-case Conditional Value-At-Risk (CVaR) under different uncertainty structure including box and ellipsoidal type. Halldórsson et al. (2003) applied interior-point method for saddle-point problems to solve the robust mean-variance portfolio selection considering uncertainty as box model in the elements of the mean vector and the covariance matrix. Alternatively, Tütüncü et al., (2004) propose a robust asset allocation model in which they discuss techniques for generating uncertainty sets from historical data. They further, consider a box-type uncertainty structure for the mean and covariance matrix of the assets returns. For this uncertainty type, the authors show that the robust portfolio selection problems can be formulated and solved as smooth saddle-point problems involving semi definite constraints. Bertsimas et al. (2008) study the viability of different robust optimization approaches for multi-period portfolio selection. Robust models treat asset returns as uncertain coefficients in an optimization problem, and map the level of risk aversion of the investor to the level of tolerance of the total error in asset return forecasts. The resulting robust optimization formulations of the multi-period portfolio optimization problem are linear and computationally efficient. Since there are many available stocks for investment in a

market and the planning horizon is always long, the number of constraints embracing uncertain parameters in the model is high. Applying robust approach will introduce some new variables and constraints for each uncertain parameter. Adding up these new variables and constraints to that of initial model, the resulting robust counterpart is a mathematical model where its dimension is increased almost threefold.

In this paper, we propose a different way to address multi-stage portfolio allocation in order to obtain robust problem formulation of tractable size. We achieve this goal by using a simple factor model to formulate asset returns. In particular, we apply CAPM to estimate future returns, where the coefficients of the model is obtained through long historical data. Our proposed model has decisive advantages. First, we use a simple model for estimation purpose that reduces the need to estimate too many input parameters. Second, the resulted robust counterpart remains linear which can be solved using an ordinary optimization technique.

Multi-period portfolio problem

Dantzig et al. (1993) propose a standard framework for multi-period asset allocation problem in discrete time. They assume m risky assets in the market; n trading period, linear transaction costs for trading stock and one riskless asset e.g. cash account, with fixed and known minimum return. Naturally, in a multi-period investment problem the rational investor likes to gather her final wealth W_n at last planning period and her goal is to control the portfolio of these assets in a way to maximize some utility function of her final wealth $U(W_n)$. Based on above assumptions the multi-period portfolio problem with linear transaction costs can be formulated. The investor's dollar holdings at the start of time period $t, t = 0, 1, \ldots, n$ in asset i is denoted by x_i^t, $i = 0, 1, \ldots, m$, in which $i = 0$ represents cash account and $t = 0$ stands for initial holding states. If she sells an amount y_i^t or buys an amount z_i^t of stock i at the beginning of time period t, she incurs transaction costs of $\mu_i^t y_i^t$ and $v_i^t z_i^t$, respectively. The dynamics of the quantities x_i^t for non-cash assets are given as follows:

$$x_i^t = (1 + \tilde{r}_i^t)(x_i^{t-1} - y_i^{t-1} + z_i^{t-1}), \qquad t = 1, \ldots, n, \qquad i = 1, \ldots, m,$$

where uncertain coefficient $\tilde{r}_t^i$ denotes the return of i_{th} risky asset over time period $(t, t + 1]$. The dynamic of cash account comprises adding cash return of previous period, adding proceeds from the sales of the risky assets, subtracting the expenses from the purchasing risky assets and subtracting all the transaction costs of the trading within the period which is as follows:

$$x_0^t = (1 + r_0^{t-1})\left(x_0^{t-1} + \sum_{i=1}^{m}(1 - \mu_i^{t-1})y_i^{t-1} - \sum_{i=1}^{m}(1 + v_i^{t-1})z_i^{t-1}\right), t = 1, \ldots, n,$$

Where r_0^t denotes the cash account return over time period $(t, t + 1]$. Note that assets are measured by their dollar amounts, so in the case of costless trading,

selling amount y_i^t of asset i will be equal to cash amount y_i^t; as a matter of fact the transactions are not costless in real world situation, and the transaction cost μ_i^t is the percent we pay for the transaction at time period t. Hence, the resulting cash from selling y_i^t of asset i is equal to $(1 - \mu_i^t)y_i^t$.

On the other hand, $(1 + v_i^t)z_i^t$ is the cash amount we pay in order to buy amount of z_i^t of asset i and v_i^t is the corresponding transaction cost. When making decision at time period t we know all amounts x_i^{t-1}, r_i^{t-1}, $i = 0, \dots, m$. At time $t + 1$, the investor's holdings are updated according to the realized returns over $(t, t + 1]$. The decision at the beginning of time period t is to determine quantities $y_i^t, z_i^t, i = 0, \dots, m$, which in turn needs to satisfy all bound constraints as follows:

$$\underline{y}_i^t \le y_i^t \le \overline{y}_i^t, \qquad i = 0, \dots, m,$$

$$\underline{z}_i^t \le z_i^t \le \overline{z}_i^t, \qquad i = 0, \dots, m,$$

$$\underline{x}_i^t \le x_i^t \le \overline{x}_i^t, \qquad i = 0, \dots, m,$$

where $\underline{y}^t$, $\overline{y}^t$, $\underline{z}^t$, $\overline{z}^t$, $\underline{x}^t$ and $\overline{x}^t$ are some given vectors of bounds. For the sake of simplicity and without lose of generality we focus on simple bounds i.e. all the lower bounds are zero and all the upper bounds are assumed to be infinity.

In the classical literature on portfolio optimization, the utility function $U(W_n)$ is assumed to be concave to reflect aversion to risk. We consider a linear utility for objective function as follows:

$$U(\sum_{i=0}^m x_i^n) = \sum_{i=0}^m x_i^n.$$

If the investor could foresee the realizations of the uncertain returns $\tilde{R}_M^t, t = 0, \dots, n - 1, i = 1, \dots, m$, her optimal strategy would be given by the optimal solution to the following optimization problem:

$$\max \sum_{i=0}^m x_i^n$$

s.t.

$$x_i^t = (1 + \tilde{r}_i^{t-1})(x_i^{t-1} - y_i^{t-1} + z_i^{t-1}), \quad t = 1, \dots, n, \qquad i = 1, \dots, m,$$

$$x_0^t = (1 + r_o^{t-1})\left(x_0^{t-1} + \sum_{i=1}^m (1 - \mu_i^{t-1})y_i^{t-1} - \sum_{i=1}^m (1 + v_i^{t-1})z_i^{t-1}\right), \qquad t = 1, \dots, n,$$

$$x_i^t \ge 0, \qquad t = 1, \dots, n, \qquad i = 0, \dots, m,$$

$$y_i^t \ge 0, \qquad z_i^t \ge 0, \qquad t = 1, \dots, n, \qquad i = 1, \dots, m.$$

Here we impose non-negativity constraints on the investor's holdings at each time period. This can be interpreted as not allowing for borrowing or short selling. In fact, future returns are not known at time 0. In practice, the investor has to treat this portfolio optimization problem as a rolling horizon problem, that is, she has to act upon information available at time period

t, and rebalance her portfolio at time period $t + 1$ after obtaining additional information over time period $(t, t + 1]$. Actually in real world situations at each time period, the investor takes only the first step of the optimal allocation strategy computed with information up to that time period, that is, that she solves consecutive multi-period portfolio optimization problems with decreasing planning horizons.

Modeling input parameters using CAPM

The standard CAPM measures the risk of a security by it's covariance with the stock market return (Lintner, 1965; Sharpe, 1964; Mossin, 1966) which is the so-called market beta. The expected return of an individual security simply is equal to the risk-free rate plus the value of the market beta times the market risk premium. In other words, the expected equity premium also known as excess return is proportional to its corresponding market beta. It is believed that riskier assets are likely to earn a higher expected return to encourage the investors to hold them. The CAPM measures this relationship between risk and return. Because of simplicity of mathematical relationship between risk and return, the CAPM has been widely used in the financial industry. Based on the traditional static CAPM, it is well known that the expected return on an asset i, $E(\tilde{r}_i)$ is equal to

$$E(\tilde{r}_i) = r_f + \beta_i[E(\tilde{R}_M) - r_f]$$

where r_f is the risk-free interest rate, β_i is the market beta, a measure of the systematic risk of asset i which can be defined as below:

$$\beta_i = \frac{cov(\tilde{R}_M, \tilde{r}_i)}{var(\tilde{R}_M)}$$

In order to estimate $\hat{\beta}_i$, we can simply regress excess asset returns on the market risk premium with historical data which is as follows:

$$\tilde{r}_i - r_f = \hat{\beta}_i(\tilde{R}_M - r_f) + \tilde{\varepsilon}_i$$

Where $\tilde{\varepsilon}_i$ represent error parameter that is a firm specific random variable. By construction one can assume that mean of error is equal to zero ($E[\tilde{\varepsilon}_i] = 0$). Furthermore, we assume that market index is unrelated to a given unique return ($E[\tilde{\varepsilon}_i(\tilde{R}_M - \hat{R}_M)] = 0$) and securities are only related trough common response to market ($E(\tilde{\varepsilon}_i\tilde{\varepsilon}_j) = 0$) and they are not jointly related. We can define now variance of $\tilde{\varepsilon}_i$ as $E(\tilde{\varepsilon}_i)^2 = \sigma_{\varepsilon i}^2$ and Variance of $\tilde{R}_M$ as $E(\tilde{R}_M - \hat{R}_M)^2 = \sigma_M^2$.

Based on afore assumptions we can derive the expected return, standard deviation and covariance when the single-index model is adopted to represent the joint movement of securities as the covariance of returns between stocks i and j with $\sigma_{ij} = \beta_i\beta_j\sigma_m^2$. On

the other hand, there is a simple relationship between variance of a single asset, market return and its corresponding residual which can be calculated as follows:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon i}^2.$$

Based on this relationship, one can simply find the variance of residuals. Substituting for $\tilde{r}_i^t$ with its nominal values based on Equation (6) yields the following optimization model based on CAPM.

$$\max \sum_{i=0}^{m} x_i^n$$

$$\text{s.t } x_i^t = \left(1 + \beta_i(\tilde{R}_M^t - r_0^t) + r_0^t\right)\left(x_i^{t-1} - y_i^{t-1} + z_i^{t-1}\right) \leq 0, t = 1, \dots, n, \ i = 1, \dots, m,$$

$$x_0^t = (1 + r_o^{t-1})\left(x_0^{t-1} + \sum_{i=1}^{m}(1 - \mu)y_i^{t-1} - \sum_{i=1}^{m}(1 + vzit-1, \ t=1,\dots,n,\right.$$

$$x_i^t \geq 0, t = 1, \dots, n, i = 0, \dots, m,$$

$$y_i^t \geq 0, z_i^t \geq 0, t = 1, \dots, n, i = 1, \dots, m.$$

Assuming parameters beta as fixed and known parameters from long historical data, it can be observed from the model (10) that the number of uncertain parameters in proposed model decreases substantially respect to nominal model.

PROPOSED ROBUST FORMULATION

There are different approaches for handling uncertainty in mathematical program including stochastic programming and robust methodology. The stochastic programming methodology uses decision tree and considers all possible scenarios and this makes the approach hard to solve because the dimension of the resulting problem increases exponentially as size of the main problem increases. Robust optimization is not a very new method but recent advances in this methodology shed light on the issue of addressing uncertainty in optimization problems. As we know, the optimal solution of a linear programming problem is located on some extreme point that is, it is usually on the edge of an area of intersection of all constraint known as feasible region. Imposing a little change in the data would make previously optimal solution completely infeasible. The early work done by Soyster (1973) was too conservative to be applicable for real-world applications in the sense that the method protects against the worst-case scenario. He considered the worst possible conditions for the data which is not a case in practice. We cannot find any other work following Soyster until 1990s. In the last decade a number of researchers including Ben-Tal and Bertsimas proposed new approaches. These works address the issue of conservativeness by letting the data uncertainty in special forms such as ellipsoidal and convex that results the models that was computationally tractable. Ben-Tal et al. (1998) considered ellipsoidal uncertainty which transforms the main linear programming to conic quadratic program called robust counterpart. Bertsimas et al. (2008) propose a robust approach for multi-period portfolio problem which uses a change in variables similar to that in Ben-Tal et al. (2000) which allows to reduce the number of constraints with uncertain coefficients to n. The trick is to work with cumulative returns instead of the simple

return which is defined as follows:

$$R_i^0 = 1$$

$$\tilde{R}_i^t = (1 + \tilde{r}_i^0)(1 + \tilde{r}_i^1) \dots (1 + \tilde{r}_i^{t-1}), \qquad t = 1, \dots, n.$$

$$\xi_i^t = \frac{x_i^t}{\tilde{R}_i^t}, \eta_i^t = \frac{y_i^t}{\tilde{R}_i^t}, \zeta_i^t = \frac{z_i^t}{\tilde{R}_i^t}.$$

Based on the change in variables, the multi-period portfolio problem could be re-written as:

$$\max w$$

$$\text{s.t } w \leq \sum_{i=1}^{m} \tilde{R}_i^n \xi_i^n + \tilde{R}_0^n \xi_0^n$$

$$\xi_i^t = \xi_i^{t-1} - \eta_i^{t-1} + \zeta_i^{t-1}, \qquad t = 1, \dots, n, \qquad i = 1, \dots m,$$

$$\xi_0^t \leq \xi_0^{t-1} + \sum_{i=1}^{m}(1 - \mu)\frac{\tilde{R}_i^{t-1}}{R_0^{t-1}}\eta_i^{t-1} - \sum_{i=1}^{m}(1 - v)\frac{\tilde{R}_i^{t-1}}{R_0^{t-1}}\zeta_i^{t-1}, \qquad t = 1, \dots, n,$$

$$\xi_i^t \geq 0, \qquad t = 1, \dots, n, \qquad i = 0, \dots, m,$$

$$\eta_i^t \geq 0, \zeta_i^t \geq 0, \qquad t = 1, \dots, n - 1, \qquad i = 1, \dots, m.$$

If data on the covariance matrices of future cumulative returns are available, then it can impose restrictions on the movement of returns across assets using the uncertainty set as follows:

$$P = \left\{ \left\| \Sigma_t^{-1/2}(\tilde{R}_t - \hat{R}_t) \right\| \leq \Delta_t, t = 1, \dots, n \right\},$$

where Δ_t are constants decided by the user in advance. Low values for Δ_t can be interpreted as a low aversion to risk. When $\Delta_t = 0$, the investor decides solely based on expected values, and her strategy is equivalent to the nominal strategy produced by solving the nominal problem. The norm in the formulation of uncertainty set can be any norm. If we use the L_2 norm, it results in Ben-Tal et al.'s (2000) formulation. When the norm in the uncertainty sets is the D-norm, the robust counterpart of the problem is as follows (Bertsimas et al., 2008):

$$\max w$$

$$s.t \ w \leq \sum_{i=1}^{m} \tilde{R}_i^n \xi_i^n + \tilde{R}_0^n \xi_0^n$$

$$\xi_i^t = \xi_i^{t-1} - \eta_i^{t-1} + \zeta_i^{t-1}, \qquad t = 1, \dots, n, \qquad i = 1, \dots m,$$

$$\xi_0^t \leq \xi_0^{t-1} + \sum_{i=1}^{m}(1 - csell)\frac{\tilde{R}_i^{t-1}}{R_0^{t-1}}\eta_i^{t-1} - \sum_{i=1}^{m}(1 - cbuy)\frac{\tilde{R}_i^{t-1}}{R_0^{t-1}}\zeta_i^{t-1}, \qquad t = 1, \dots, n,$$

$$\xi_0^{t+1} - \xi_0^t - \begin{pmatrix} -\frac{(1-\mu)}{R_0^{t-1}}\eta_1^t + \frac{1+v}{R_0^t}\zeta_1^t \\ \dots \\ -\frac{(1-\mu)}{R_0^{t-1}}\eta_m^t + \frac{1+v}{R_0^t}\zeta_m^t \end{pmatrix} \hat{R}_t + \frac{\Delta_t}{d}(w^t)'e \leq 0, \qquad t = 1, \dots, n - 1,$$

$$(2p^n - w^n)' = -\xi_n' \Sigma_n^{1/2},$$

It can be observed from the model (14) that too many new constraints are imposed to the original linear programming problem. Bertsimas et al. (2004) propose another robust approach for linear programming that its robust counterpart is also linear. Consider a given linear programming problem in the following form:

Table 1. Comparison the proposed model (17) with Bertsimas' with respect to the number of variables and constraints.

Number of	Bertsimas's model	Proposed model
Constraints	8mn - 2m + 2n + 1	3mn + n
Variables	5mn - 2m + n	5mn - 2m + n

max $C'X$

s.t. $\tilde{A}X \leq b,$

$L \leq X \leq U.$

Based on Bertsimas' work assume that data uncertainty only affects the elements in matrix A. suppose that there are there are only J_i coefficient subject to uncertainty in a particular row i and each entry a_{ij}, $j \in J_i$ is modeled as a symmetric random variable $\tilde{a}_{ij}$ that only takes value in interval $[\hat{a}_{ij} - \delta_{ij}, +\hat{a}_{ij} + \delta_{ij}]$; in which $\hat{a}_{ij}$ and δ_{ij} is the nominal value and maximum deviation of element a_{ij}, respectively. Bertsimas et al. (2004) show that under these assumptions, the constraints of robust counterpart of model (10) can be rewritten as follows:

max $C'X$

s.t. $\sum_j \hat{a}_{ij} x_j + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \leq b_i, \forall i$

$z_i + p_{ij} \geq \delta_{ij} y_j, \quad \forall i, j \in J_i$

$-y_j \leq x_j \leq y_j, \quad \forall j$

$l_j \leq x_j \leq u_j, \quad \forall j$

$p_{ij} \geq 0, \quad y_j \geq 0, \quad z_i \geq 0, \quad \forall i, j \in J_i$

in which new set of variables including p_{ij}, z_i and y_j and new parameter Γ_i that is correspond to robust modeling, is imposed to the original linear programming problem. Based on Bertsimas' approach the robust counterpart of multi-period portfolio problem can be rewritten as follows:

max $\sum_{i=0}^m x_i^n$

s.t. $x_i^t - (1 + \beta_i(\hat{R}_M^{t-1} - r_0^{t-1}) + r_0^{t-1})(x_i^{t-1} - y_i^{t-1} + z_i^{t-1}) + q_i^t \Gamma_i^t + p_i^t \leq 0, \ t = 1, \dots, n, \ i = 1, \dots, m,$

$x_0^t = (1 + r_0^{t-1})\left(x_0^{t-1} + \sum_{i=1}^m (1-\mu)y_i^{t-1} - \sum_{i=1}^m (1+v)z_i^{t-1}\right), \quad t = 1, \dots, n,$

$q_i^t + p_i^t \geq \delta R_t^M \varphi^t, \ t = 1 \dots n, \ i = 1, \dots, m,$

$-\varphi^t \leq x_i^{t-1} - y_i^{t-1} + z_i^{t-1} \leq \varphi^t, \quad t = 1, \dots, n, \quad i = 0, \dots, m,$

$p_t^i \geq 0, \quad v_t^i \geq 0, \quad x_t^i \geq 0, \quad t = 1, \dots, n, \quad i = 0, \dots, m,$

$y_t^i \geq 0, \quad z_t^i \geq 0, \quad t = 1 \dots n, \quad i = 1, \dots, m,$

$\varphi^t \geq 0, \quad t = 1, \dots, n$

in which q_i^t, p_i^t and φ^t are new variables that applied robust modeling approach imposes to the nominal portfolio problem; furthermore, parameter Γ_i^t is some control parameter known as price of robustness that adjust the robustness of the proposed method over the level of conservatism of its final solution. In this formulation if $\tilde{R}_t^M$ changes by $\delta \Gamma_i^t$ time its nominal value $\hat{R}_t^M$ then

the robust solution will be feasible, deterministically. Furthermore, if it changes even more, then the resulted robust solution is feasible with a high probability; that is at least $1 - exp\left(-\frac{r_i^2}{2|J_i|}\right)$.

In Table 1, we compare the number of variables and constraints in proposed model with the model (14) proposed by Bertsimas. It can be observed from the proposed robust counterpart that both the number of variables and constraints are reduced, considerably. This makes the proposed model interesting when the number of asset available increases substantially. For instance, consider a market with 500 assets where there are 12 period of investment strategy. The robust optimization model proposed by Bertsimas has 47,025 constraints with 29,012 variables. On the other hand, considering proposed model the investor only has to solve a linear programming problem with 18,012 constraints and the same number of variables. In order to examine the performance of proposed model in the following section we solve a numerical example.

RESULTS

In this section, we use some sample simulated data to examine the performance of the proposed model of this paper and consider two types of the problems as follows;

1. Nominal multi-period portfolio optimization (henceforth abbreviated NPO)
2. Robust multi-period portfolio optimization (henceforth abbreviated RPO)

Since it is obvious that any multi-period modeling approach outperforms single-period mean-variance method in long term planning horizon, we do not study single period for our experiment. The NPO is the standard multi-period portfolio optimization technique in which the optimal solution is calculated based on estimation of all future returns of the assets. The RPO is proposed robust multi-period portfolio optimization approach in which its future assets' returns are subject to perturbation. In both cases the required future return are estimated based on CAPM while in the NPO we use the expected values. However, in the latter case we consider an interval for asset returns. In order to compare the results of two approaches we use simulated market data. Consider a market embracing a risk free asset and six risky assets ($n = 6$) where the covariance matrix is given in Table 2.

The portfolio manager has to handle her initial portfolio for next 5 periods. Her initial wealth is evenly allocated in all assets in a way that the portfolio consists of all seven assets and the initial value of each asset is 100 dollars. We assume that transaction costs are 1% of the amount traded in all time periods. Since

Table 2. The covariance matrix of six available stocks in the market.

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Stock 1	0.044	0.027	0.031	0.013	0.025	0.038
Stock 2	0.027	0.058	0.040	0.017	0.025	0.062
Stock 3	0.031	0.040	0.058	0.009	0.032	0.048
Stock 4	0.013	0.017	0.009	0.063	0.034	0.030
Stock 5	0.025	0.025	0.032	0.034	0.084	0.034
Stock 6	0.038	0.062	0.048	0.030	0.034	0.096

Table 3. The expected returns of available assets in different time periods.

Trading period	Expected returns					
	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
1	0.07	0.07	0.07	0.07	0.08	0.07
2	0.03	0.03	0.03	0.03	0.03	0.03
3	0.10	0.10	0.09	0.11	0.12	0.10
4	0.15	0.17	0.14	0.18	0.19	0.16

Table 4. The final solution of nominal model.

Trading period	Dollar holding						
	Cash	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
1	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2	0.00	109.65	110.35	109.16	110.84	218.18	109.79
3	0.00	120.23	121.77	119.16	122.86	243.81	120.54
4	0.00	131.83	134.37	130.07	136.17	272.46	132.34
5	0.00	144.56	148.28	141.99	150.93	304.48	145.30

we adopt single-index mode in order to estimate future returns, consider there is an index in the market where its expected return vector and volatility is $\hat{R}_M=$ [0.07 0.03 0.1 0.16 0.09] and 0.1732, respectively. The fixed riskless return is 0.03 for all periods. Furthermore, we assume a fixed beta for each assets in planning horizon which is $\beta' =$ [0.95 1.05 0.88 1.12 1.25 0.97]. The variance of error of each risky asset is calculated based on Equation (9) as:

$$\sigma_\varepsilon = [0.13, 0.16, 0.18, 0.15, 0.19, 0.26].$$

The expected return of all assets which are estimated based on CAPM in four planning periods is summarized in Table 3.

Table 4 summarizes the details of the implementation of NPO with nominal information given in Table 3. The first row of Table 4 represents the initial wealth which is distributed equally among a risk free asset and six other risky assets in the first period. Row 2 to row 5 show the details disposition among all assets. As we can observe from the table, in the second period, we do not consider any cash investment. The last row of the portfolio demonstrates the outcome of the investment

on all five risky assets based on the predicted returns given on Table 3. Note that there is no perturbation on the input data given on Table 3 so Table 4 represents only the nominal optimal solution. Next, we consider the optimal solution of the proposed method using different perturbations.

In order to solve RPO, we consider 50% perturbations in market return and we set the parameter Γ_i^t so that all constraints hold at least 95% of all possible cases. The optimal solutions of RPO are summarized in Table 5. As we can see from Table 5, in the first period all assets start with equal amount of money, 100.00. The asset holdings change from the second period to the fifth period ending with the optimal returns given in the last row of the table. When we compare the results of the nominal and robust solutions, we find out that the changes on robust solution are smooth.

Since we have used expected return to solve these two optimization problems, we cannot add up final holdings in each asset to locate the final wealth. In order to evaluate the performance of RPO and NPO we simulate the market based on the assumptions we have already stated. Table 6 summarizes the simulated returns of different assets for period one to five. For

Table 5. The final solution of robust model.

Trading period	Dollar holding						
	Cash	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
1	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2	6.24	119.64	120.44	119.07	121.01	121.01	119.80
3	0.94	125.85	126.70	125.26	127.29	128.40	126.02
4	0.00	131.81	132.70	131.19	133.32	134.47	131.99
5	0.00	138.09	139.02	136.42	139.67	140.88	138.28

Table 6. The actual return based on simulating the market using CAPM

Trading period	Simulated returns						
	Cash	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
1	0.03	0.04	-0.16	0.11	0.15	-0.10	0.41
2	0.03	0.25	0.10	0.15	0.14	0.08	0.29
3	0.03	0.02	0.45	0.07	0.13	0.32	0.11
4	0.03	0.08	-0.03	0.15	-0.10	0.26	0.52

Table 7. The actual holdings of NPO based on simulated returns.

Trading period	Actual dollar holding						
	Cash	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Initial holdings	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1	0.10	104.01	84.27	111.48	115.38	89.63	140.82
2	0.10	130.19	92.49	128.46	131.06	194.93	181.24
3	0.11	132.75	133.69	136.98	147.63	257.96	201.77
4	0.11	143.91	130.10	157.00	132.52	323.81	306.87
5	0.11	144.80	161.05	207.88	113.59	271.84	382.58

Table 8. The actual holdings of RPO based on simulated returns.

Trading period	Actual Dollar holding						
	Cash	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
Initial holdings	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1	103.00	104.01	84.27	111.48	115.38	89.63	140.82
2	8.32	145.87	108.17	144.13	146.74	111.66	196.91
3	3.19	150.12	156.92	155.65	165.28	147.76	220.44
4	6.87	164.33	152.70	179.43	147.26	182.31	336.55
5	7.08	165.35	189.03	237.58	126.22	153.05	419.57

instance, the first row represents the returns gained during the first period.

Tables 7 and 8 summarize the required adjustment to the investor's wealth based on the input information given in Table 6 for each period in both NPO and RPO models, respectively. Since these two adjustments are made based on the realization of the future returns, the last rows of the tables represent the final wealth of the investor. The final wealth of the investor can be calculated through adding the last row elements. It can be observed that the final wealth in RPO (1297.86) is higher than that of NPO (1281.84). It can be seen that incorporating market information using single factor model (CAPM) to robust portfolio problem (RPO) could increase the investor wealth compared to nominal formulation (NPO).

Conclusion

We have developed a novel model of multi-period portfolio problem when asset returns are subject to uncertainty. One of the most important shortcomings of different approaches to handle uncertainty is the computational costs. Robust modeling approaches naturally incorporate uncertainty in by introducing several variables and constraints. Therefore the resulted problem formulation becomes more complicated when there are several assets at the market and the planning horizon is long. In order to overcome this issue, we have modeled the uncertain asset returns using CAPM which is a popular tool to model asset return. It is obvious that the proposed modeling approach reduces the size of the resulted problem, significantly. The resulted uncertain problem has been solved using numerical examples. We have also evaluated the performance of proposed model through comparing the results obtained from the robust model (RPO) with its corresponding nominal model (NPO). Numerical example with simulated data shows the proposed robust model outperforms the nominal one both in dimension and in final results.

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