Full Length Research Paper

A geometrical approach for fuzzy production possibility set in data envelopment analysis (DEA) with fuzzy input - output levels

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Data envelopment analysis (DEA) is a managerial tool used to measure the relative efficiency of decision making units (DMU). Classic DEA models estimates a production frontier using efficient DMUs. This frontier bounds all feasible production plans named production possibility set. Traditional DEA models require crisp input and output data. However, in real-world problems inputs and outputs are often imprecise like fuzzy numbers. When the inputs and outputs of the DMUs are fuzzy numbers the exact location of production frontier cannot be determined precisely, therefore production possibility set is an imprecise set. This paper considers production possibility set as a fuzzy set that all production plans are considered as its member with different degrees of membership and a membership function is derived under a geometrical approach in a two dimensional space for the case when the DMUs have only one fuzzy input or output and finally this membership function is generalized to the models with multiple fuzzy input and output.

Key words: Data envelopment analysis, production possibility set, fuzzy set, T-norm.

INTRODUCTION

Data envelopment analysis (DEA) is a mathematical method for organizations to evaluate their plans by measuring the relative efficiency of decision making units (DMU). DEA measure a ratio of the weighted sum of outputs to the weighted sum of inputs with the weights as variables to evaluate the relative efficiency of the DMUs.

Classic DEA models (Cooper et al., 2000) estimate a nonparametric linear piecewise frontier called production frontier which is determined by efficient DMUs to evaluate the relative efficiency of the DMUs.

Charnes, Cooper and Rhodes (CCR) and Banker, Charnes and Cooper (BCC) models (Charnes et al., 1978; Banker et al., 1984) are two frequently used DEA models. CCR and BCC models accept precise inputs and outputs to evaluate the relative efficiency since they focus on an estimated production frontier. A few changes in data may change the production frontier significantly.

Production process in real world problems often deals with qualitative data or imprecise data instead of accurate data. Using Fuzzy set theory, established by Zadeh, is one of the most common way to enter the imprecision and vagueness in calculations. DEA models with fuzzy data can better than conventional DEA models represent the real world problems.

A comprehensive literature review on DEA models with imprecise data (IDEA) is presented in Zhu (2003). Zhu classifies the imprecise data into three different groups: interval data, ordinal data and interval data ratio.

The IDEA model was applied by Despotis and Smirlis (2002) to deal with interval data and ordinal data. As a result, the boundaries of the efficiency of each DMU are obtained.

Biondi et al. (2009) used a geometrical approach in order to build a fuzzy efficient frontier set when one of inputs or outputs of the DMUs have imprecise values.

Lertworasirikul et al. (2003) used fuzzy theory to enter the imprecision of data into DEA models (fuzzy DEA). Consequently, linear fuzzy programming was applied to

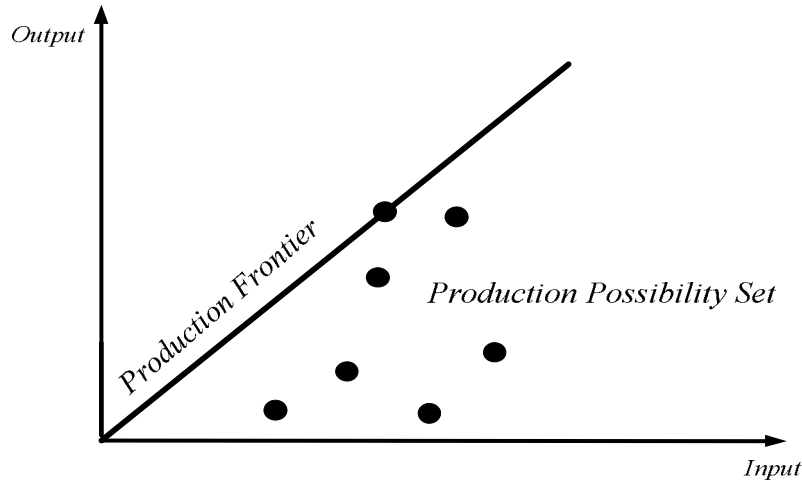---

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Figure 1. Production frontier and T_C in a CCR model.

compute the efficiency score.

The existing approaches for evaluating DMUs with fuzzy inputs and outputs are classified by Angiz et al. (2010) into four groups: the fuzzy ranking approach; the defuzzification approach; the tolerance approach; and the α-level based approach. The fuzzy ranking approach was developed by Guo and Tanaka (2001). In this approach ranking models are used to define fuzzy inequalities and fuzzy equalities in the fuzzy CCR model, hence the resulting model is based on a bi-level linear programming.

The defuzzification approach was developed by Lertworasirikul et al. (2001). In this approach the fuzzy numbers are changed into crisp values so this approach ignores the ambiguity of the information. The α-level based approach was proposed by Meada et al. (1998), and then improved by Mohtadi et al. (2002). In this approach a fuzzy DEA model is solved by parametric programming using α-cuts and the resulted efficiency score of the DMU under evaluation has interval format. Sengupta (1992) proposed the tolerance approach. Kahraman et al. (1998) extended the idea of Sengupta. In this approach the tolerance levels on constraint violations transfer imprecision into DEA models.

When the inputs and outputs of the DMUs are fuzzy numbers, the location of the production frontier cannot be easily determined; this frontier may be placed in a bounded region. Imprecision in the location of production frontier causes the imprecision of production possibility set.

This paper considers the production possibility set as a fuzzy set that any production plan belongs to it with a specific degree of membership. In other words, this paper appropriates a membership degree to any production plan included in production possibility set and this membership degree can be considered as the possibility of production for production plans.

CLASSIC DEA MODELS

Assume that there are n DMUs to be evaluated, indexed by $j = 1,..., n$ and each DMU is assumed to consume m different inputs to produce s different outputs. Let $X_j = (x_{1j}, x_{2j},..., x_{mj})$ and $Y_j = (y_{1j}, y_{2j},..., y_{mj})$, respectively be the inputs and outputs vectors of DMUj that all components of these vectors have non-negative value and each DMU has at least one strictly positive input and output. If the vector (X, Y) indicates a production plan then the production possibility set in a CCR model is defined as follows:

$$T_C = \left\{ (X,Y) \middle| \quad X \geq \sum_{j=1}^{n} \lambda_j X_j \quad , \quad Y \leq \sum_{j=1}^{n} \lambda_j Y_j \quad , \quad \lambda_j \geq 0, j = 1,...,n \right\} \tag{1}$$

In other words, T_C includes all feasible production plans. The CCR model creates its production frontier using linear combination of the existing production plans, whereas the BCC model has its production frontier spanned by convex hull of the existing production plans. The production possibility set of a BCC model is presented as follows:

$$T_{BCC} = \left\{ (X,Y) \middle| \quad X \geq \sum_{j=1}^{n} \lambda_j X_j \quad , \quad Y \leq \sum_{j=1}^{n} \lambda_j Y_j \quad , \quad \sum_{j=1}^{n} \lambda_j = 1 \quad , \quad \lambda_j \geq 0, j = 1,...,n \right\} \tag{2}$$

Especially when the DMUs have only one input and output, the production possibility set and production frontier in a CCR and BCC model can be shown as shown Figures 1 and 2 .

The relative efficiency of a DMU falls in the range of (0, 1). A DMU is efficient, if there no other DMU which uses fewer inputs to produce more or equal outputs exist.

In Figures 1 and 2, all the DMUs that their production plans locate on the production frontier are efficient.

FUZZY SETS AND NOTATIONS

A classical set is normally defined as a group of objects $x \in X$ that any element can either belong to or not belong to a set A, $A \subseteq X$. But for a fuzzy set, an
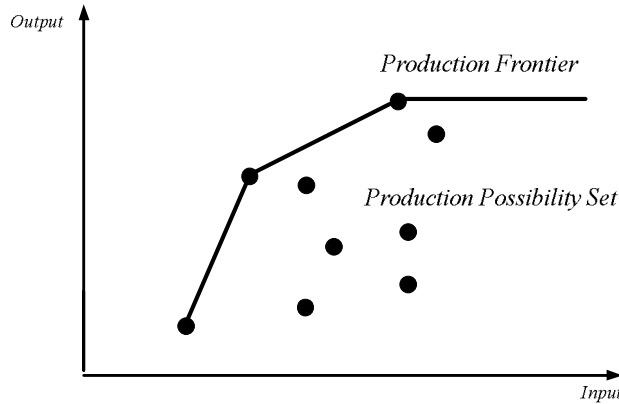
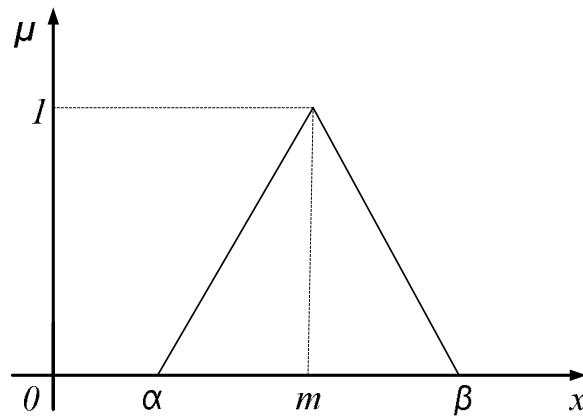**Figure 2.** Production frontier and T_{BCC} in a BCC model.

Figure 3. Triangular fuzzy number.

belong to a set A, $A \subseteq X$. But for a fuzzy set, an element belongs to set A by a degree of membership.

Definition 1

If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs (Zimmerman, 1996):

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X \}$$

$\mu_{\tilde{A}}(x)$ is called the membership function of x in $\tilde{A}$ and it is a real number in the range of [0, 1].

Definition 2

A fuzzy number $\tilde{M}$ is of LR-type if there is a reference function L (for left), R (for right) and scalars $\alpha > 0, \beta > 0$ with (Zimmerman, 1996):

$$\mu_{\tilde{M}}(x) = \begin{cases} L(\dfrac{m - x}{\alpha}) & x \leq m \\ R(\dfrac{x - m}{\beta}) & x \geq m \end{cases}$$

m, called the min value of $\tilde{M}$, is a real number, and α and β are called the left and right spreads, respectively. Symbolically, $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Specially, suppose

$$L(x) = R(x) = \begin{cases} 1 - x & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

then $\tilde{M} = (m, \alpha, \beta)_{LR} = (m, \alpha, \beta)$ is called a triangular fuzzy number. Figure 3 represents a fuzzy triangular number.
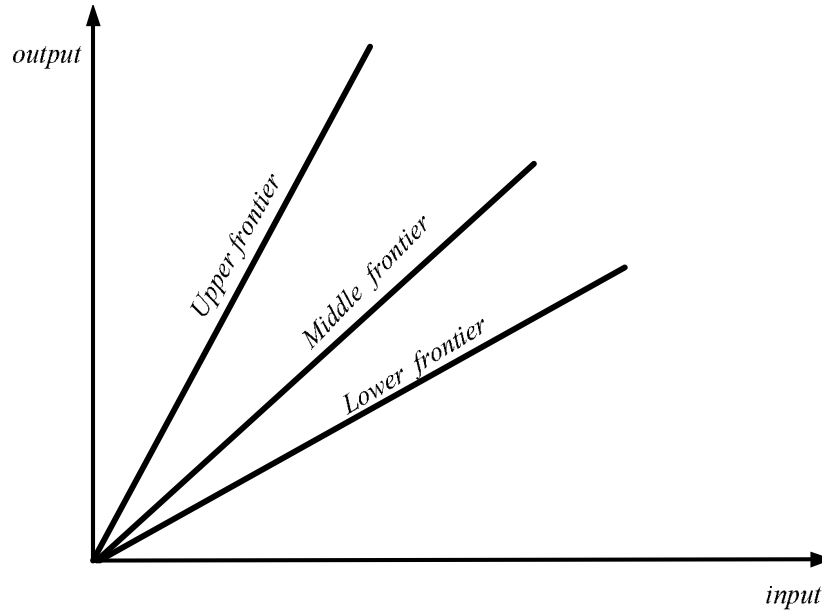
Figure 4. Upper, middle and lower frontiers in a CCR model.

Definition 3

T-norms are two-valued functions from $[0,1] \times [0,1]$ that satisfy the following conditions (Dubois and Prade, 1980):

1. $t(0,0) = 0$; $\quad t(\mu_{\tilde{A}}(x),1) = t(1,\mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x), \qquad x \in X$
2. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   $\quad$ if $\quad \mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x) \quad$ and $\quad \mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
3. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
4. $t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$

The functions t, defines a general class of intersection operators for fuzzy sets.

Definition 4

The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$ is the crisp set of all $x \in X$ such that (Zimmerman, 1996):

$$\mu_{\tilde{A}}(x) > 0$$

FUZZY PRODUCTION POSSIBILITY SET WITH ONE IMPRECISE OUTPUT

Firstly, we assume that each DMU has only one input and output and the output of each DMU is given by a triangular fuzzy number. To determine production possibility set, we need to present some basic concepts (Figure 4).

Upper frontier

It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the maximum value in the support of the imprecise output for each DMU.

Middle frontier

It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the value with unitary membership degree of the imprecise output for each DMU.

Lower frontier

It is the frontier obtained by a classic DEA model (CCR or BCC), that considers the minimum value in the support of the imprecise output for each DMU.

Production plan

We assume the ordered pair (x, y) that x is the value of input and y is a value in the support of fuzzy output as a production plan. In other words a production plan is a point in production possibility set.

According to definition of production possibility set and middle frontier, all the production plans below or on the middle frontier can be definitely produced because this frontier is created by the values of output with unitary membership degree. Hence, all the production plans below or on the middle frontier must have unitary membership degree regarding the production possibility set.
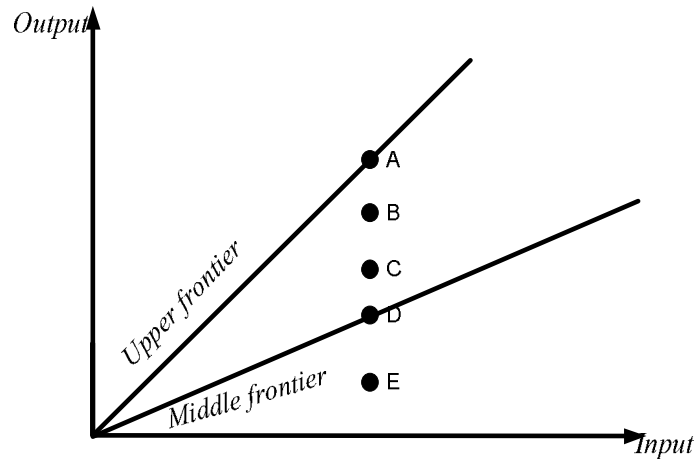
Figure 5. All possible location of production plans in a CCR model-fuzzy output.

To determine the production plan's membership degree generally, we need to consider all possible locations of a production plan related to the frontiers. Figure 5 illustrates these locations, considering the CCR model.

1. Figure 5 shows that production plan E is inside the region defining the production possibility set by middle frontier. Such production plans must have unitary membership degree to the fuzzy production possibility set.
2. Production plan D is on the middle frontier, which is made by production plans with the unitary membership of output. Hence the membership degree of such production plans must be one.
3. The point that represents production plan A locates on the upper frontier, this frontier is made by production plans with zero membership degree of fuzzy output. Hence the membership degree of such production plans is considered zero.
4. Between those extreme situations, production plans B and C would have intermediate membership degrees. Production plan C is closer to the middle frontier than B, so the membership degree of C must be greater than the membership degree of B. Let d^u and d^m be successively, the distance of such production plans from the upper and the middle frontier. The ratio $\dfrac{d^u}{d^u + d^m}$ can be an appropriate value for their membership degree (Figure 6).

The ratio $\mu_1 = \dfrac{d^u}{d^u + d^m}$ has two important properties:

a) $\dfrac{\partial \mu_1}{\partial d^m} < 0$ \hfill (3)

b) $\dfrac{\partial \mu_1}{\partial d^u} > 0$ \hfill (4)

Property (a) proves that when a production plan in the frontier region gets closer to the middle frontier the membership degree or possibility of its production, increases. Property (b) shows that, for two production plans that have the same distance from the middle frontier, the membership degree of the production plan that is closer to upper frontier is less than the other one.

ALGEBRAIC CALCULATION OF MEMBERSHIP DEGREE

The previous calculations are based on a geometrical definition, which is feasible only for very simple models. In order to obtain an expression that might be used for multidimensional general models, in which multiple outputs are fuzzy numbers, it is essential to change the geometric terms, d^u and d^m, into variables that might be derived from the classic DEA models.

For the case of one imprecise output, considering the classic DEA definitions for output oriented models, we need to introduce two variables:

(i) Eff_m is the efficiency score related to the middle frontier.
(ii) Eff_u is the efficiency score related to the upper frontier.

Assuming that (x, y) is a production plan and y_1 and y_2 are, respectively the projected output on the middle and upper frontier; considering Figure 6, the values of Eff_m and Eff_u are measured as follows:
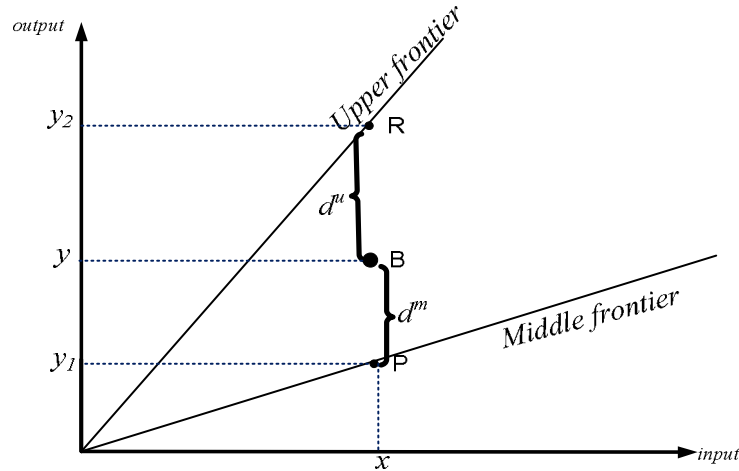
$$Eff_m = \frac{y_1}{y}$$ \hfill (5)

Figure 6. Distance of a production plan from the middle and upper frontiers.

And;

$$Eff_u = \frac{y_2}{y} \tag{6}$$

With the purpose of avoiding misunderstandings, Eff_u should not be named upper efficiency, as there is no guarantee that $Eff_m \leq Eff_u$.

From the geometrical representation we can easily obtain algebraic terms that measure d^m and d^u as follows:

$$d^m = y - y_1 = y - y.Eff_m \tag{7}$$

$$d^u = y_2 - y = y.Eff_u - y \tag{8}$$

Therefore;

$$\frac{d^u}{d^u + d^m} = \frac{y.Eff_u - y}{y.Eff_u - y + y - y.Eff_m} = \frac{Eff_u - 1}{Eff_u - Eff_m} \tag{9}$$

From the previous relationships, it is possible to derive an expression that represents algebraically the membership degree:

$$\mu_1 = \begin{cases} 1 & if & Eff_m \geq 1 \\ 0 & if & Eff_m < 1 \quad , \quad Eff_u = 1 \\ \dfrac{Eff_u - 1}{Eff_u - Eff_m} & if & Eff_m < 1 \quad , \quad Eff_u > 1 \end{cases} \tag{10}$$

Numerical example

Table 1 shows the data of 4 DMUs with single input and single fuzzy output. Table 2 details the algebraic calculation of membership degree of 11 different production plans. It should be noticed that, due to the output orientation model, the inefficient production plans produce an efficient score greater than one.

FUZZY PRODUCTION POSSIBILITY SET WITH ONE IMPRECISE INPUT

The case of one fuzzy input may be analyzed in a way similar to that of one fuzzy output. In this case, the upper frontier is obtained by replacing the smallest value of fuzzy input in a classic DEA model and the middle frontier is created by replacing the value of input with unitary membership degree in a classic DEA model.

Figure 7 depicts the middle and the upper frontiers for the case of one imprecise input. In this figure, the points $R(x_1, y)$ and $P(x_2, y)$ are respectively the projection of production plan $B(x, y)$ on the lower and upper frontiers.

Considering the classic DEA definitions for input oriented models the values of Eff_m and Eff_u can be obtained from the following relations:

$$Eff_m = \frac{x_2}{x} \tag{11}$$

$$Eff_u = \frac{x_1}{x} \tag{12}$$

Expression 12 presents the membership degree of a production plan to the fuzzy production possibility set.

Table 1. Data of 4 DMUs with single input and single fuzzy output.

Variable	DMU1	DMU2	DMU3	DMU4
x (Input)	1	2	2.5	1.2
y (Output)	(6, 5,10)	(12,10, 20)	(13, 15, 20)	(6, 5, 9)

Table 2. Membership degrees of different production plans.

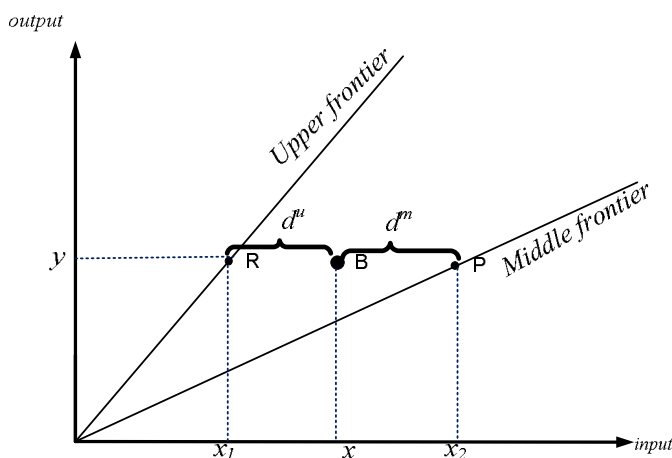| Production plan	Eff$_m$	Eff$_u$	μ_1
(1, 10)	0.6	1	0
(1, 8)	0.75	1.25	0.5
(1, 7)	0.86	1.43	0.75
(1, 6)	1	1.67	1
(1, 5)	1.2	2	1
(1.2, 8.5)	0.84	1.42	0.72
(1.2, 6.5)	1.1	1.85	1
(2, 11)	1.09	1.81	1
(2, 19)	0.63	1.05	0.11
(2.5, 18)	0.83	1.39	0.7
(2.5, 14)	1.07	1.78	1

Figure 7. Distance of a production plan from the frontiers.

$$\mu_2 = \begin{cases} 1 & if & Eff_m < 1 \\ 0 & if & Eff_u = 1 \\ \dfrac{Eff_u - 1}{Eff_u - Eff_m} & if & Eff_m > 1 \quad , \quad Eff_u < 1 \end{cases} \quad (13)$$

Extension of the idea

In order to extend the idea to the models with multiple fuzzy inputs and multiple fuzzy outputs we need to combine the membership degrees μ_1 and μ_2. To do this, we use the concept T-Norm.

Here, we use the T-norms minimum (13) and geometric mean (14) to derive the membership degree of production plans regarding the production possibility set when the DMUs have multiple fuzzy inputs and multiple fuzzy outputs.

$$\mu_M = Min\{\mu_1, \mu_2\} \quad (14)$$

$$\mu_G = \sqrt{\mu_1 . \mu_2} \quad (15)$$
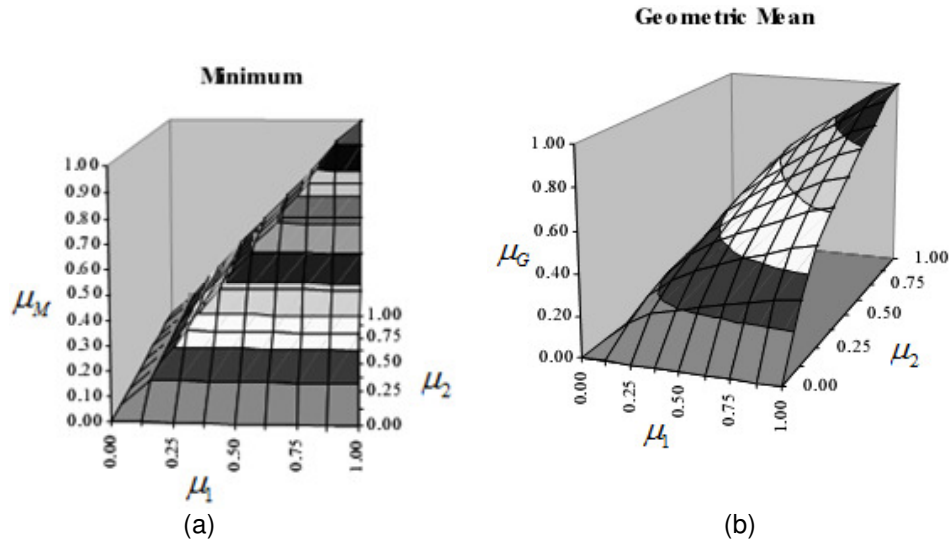
Figure 8 indicates the geometrical representation of

Minimum

Geometric Mean

(a)   (b)

Figure 8. Representation of T-norms: (a) minimum; (b) geometric mean.

presented T-norms.

Conclusion

The approach proposed in this paper, in order to incorporate uncertainties in classic DEA models, has the advantages of simplicity in algebraic calculations and transparency in geometrical consideration.

When the data of decision making units are fuzzy numbers the location of the efficiency frontier cannot be determined precisely, therefore the boundary of the production possibility set is imprecise. In this paper the production possibility set is assumed as a fuzzy set that all production plans are considered as its member with different degrees of membership.

This paper firstly uses a geometrical approach in the case of only one fuzzy input or output but not both to derive the membership function of fuzzy production possibility set and then converts the geometrical terms to the algebraic expression using some basic concepts of DEA models. Finally the membership function is generalized to the models with multiple fuzzy inputs and outputs using the concept of T-norms.

This approach can be developed by introducing other membership functions for the production plans to the production possibility set.

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