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A study of discrete - time multi server retrial queue with finite population and fuzzy parameters

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Our aim in this work is to deal with the discrete-time multi server retrial queue with finite population and fuzzy parameters. First, we describe this system and mentioned its effective characteristics in a crisp case. Then, we apply the concepts of α cuts and extension principle to construct membership functions of the system characteristics using paired NLP models in the fuzzy case.

Key words: Retrial queue, fuzzy, discrete-time.

INTRODUCTION

In many queuing situations, finding all services busy customers leave the service area commonly and repeat their request after some random time. During trials, the blocked customers join a pool of unsatisfied customers called orbit. The most obvious application of retrial queues arise in telephony where customers receiving a busy signal are not allowed to queue and have to try again at some time later. Many other applications include communication protocols, local area networks and queues arising in daily life situations. Retrial literature was initially focused on continuous-time systems (Artalejo, 1999, 2003). Yang and Li (1998, 1999) were the first to study a discrete-time model of Geo/G/1 type with geometric retrial times. Since the publication of this pioneering paper, several authors have investigated a variety of single-server discrete - time retrial queues (Artalejo et al., 2004; Atencia and Moreno, 2004; Choi

and Kim, 1997; Li and Yang, 1999, 1998; Takahashi et al., 1999). Recently Artalejo and Lopez (2007) computed the steady-state distribution of a discrete-time multi server retrial queue with finite population in which the primary retrial times are required to follow certain probability distributions with fixed parameters. However, in many real-world applications, the parameter distribution may be fuzzy. Thus, fuzzy retrial queues would be potentially much more useful and realistic than the commonly used crisp retrial queues.

Jau and Hsin (2007) developed FM/FM/1/1-(FR) fuzzy system, where F represents fuzzy time and FM represents fuzzified exponential distribution and FR represents the fuzzified exponential retrial time. In this work, we develop an approach that provides system characteristics for the discrete-time multi server retrial queue with finite population and fuzzy parameters.

Through α cuts and extension principle, we transform the fuzzy retrial queues to a family of crisp retrial queues. As α varies, the NLP solutions completely and successfully yield the membership functions of the system characteristics.

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Fuzzy retrial queues

In this study, we consider an FGeo/ FGeo/c/N-(FR) queuing system in which customers arrive at a service facility from outside at rate a geometric independent random variables with fuzzy parameter \tilde{p} . An arriving customer enters the service facility if the facility is not occupied; otherwise he/she enters the orbit and attempts service after an uncertain amount of time, called retrial time. The number of servers is denoted by c. we deal with a finite population of size N ($N > c$) where each individual customer generates requests independently of the rest of population. Retrials and service times made by each blocked customer follow geometric independent random variables with fuzzy parameters \tilde{p}, \tilde{s} , and \tilde{q} in addition, various stochastic processes involved in the system are independent of each other.

In this model the arrival \tilde{p}, \tilde{s} , and \tilde{q} are approximately known and can be represented by convex fuzzy sets. Let $\mu_{\tilde{p}}(x), \mu_{\tilde{s}}(v)$, and $\mu_{\tilde{q}}(y)$ denote the membership function of \tilde{p}, \tilde{s} , and \tilde{q} respectively. Then, we have the following fuzzy sets:

$$\tilde{p} = \{(x, \mu_{\tilde{p}}(x)) \mid x \in X\} \tag{1a}$$

$$\tilde{s} = \{(v, \mu_{\tilde{s}}(v)) \mid v \in V\} \tag{1b}$$

$$\tilde{q} = \{(y, \mu_{\tilde{q}}(y)) \mid y \in Y\} \tag{1c}$$

Where X; V; Y are the crisp universal sets of the arrival, retrial and service rates, respectively. Let $f(x, v, y)$ denote the system characteristic of interest, since \tilde{p}, \tilde{s} and \tilde{q} are fuzzy numbers, $f(\tilde{p}, \tilde{s}, \tilde{q})$ is also a fuzzy number. Following Zadeh's (1978) extension principle (Artalejo, Lopez-Herrero (2007) and Li, Lee (1989)), the membership function of the system characteristic $f(\tilde{p}, \tilde{s}, \tilde{q})$ is defined as

$$\mu_{f(\tilde{p}, \tilde{s}, \tilde{q})}(z) = \sup_{x \in X, v \in V, y \in Y} \text{Min}\{\mu_{\tilde{p}}(x), \mu_{\tilde{s}}(v), \mu_{\tilde{q}}(y) \mid z = f(x, v, y)\}$$

Assume that the system characteristic of interest is the expected number of busy servers and the expected numbers of customers in the orbit. In this work, at first, consider the system characteristic for a crisp retrial queuing system. Then solve the problem for fuzzy parameters by using them.

System characteristics in the crisp case

We consider a multi server discrete-time retrial queue where the time axis is divided into equal intervals, of

width one, called slots. It is assumed that all queuing activities occur around the slot boundaries. For mathematical convenience, we suppose departures occur in the interval (h, h) while primary arrivals and retrials occur in the interval (h, h^+) , Events occurring in (h, h^+) , that is, primary arrivals, service times and retrials made by each blocked customer follow geometric independent random variables with parameters x, v , and y respectively.

The system state at time h can be described by the process $X_h = (C_h, O_h)$, where C_h represents the number of fuzzy servers and O_h denotes the numbers of customers in orbit. Note that the process $\{X_h; h \geq 0\}$ is a Markov chain with state space $S = \{0, \dots, c\} \times \{0, \dots, N - c\}$. Our main objective is to compute the steady-state probabilities $\pi_{ij} = \text{Lim}_{h \rightarrow \infty} P\{(C_h, O_h) = (i, j)\} \forall (i, j) \in S$. The key point for computing $\{\pi_{ij}, (i, j) \in S\}$ is to obtain the one step transition probabilities $P_{(i,j)(m,n)}$ which describe the evaluation of the Markov chain. At this point we observe that given the initial state (i, j) the following events can occur during the next slot:

- (i) N- i - j primary arrivals.
- (ii) i departures
- (iii) j retrials.

Depending on how many of these events occur, we have transition from (i, j) to any state (m, n) in the subset

$$S_{(i,j)} = H_j \cup \left(\bigcup_{k=0}^{c-1} D_j^k \right)$$

Where;

$$H_j = \{(m, n) \in S; j \leq n \leq N - c, m = c\}$$

$$D_j^k = \{(m, n) \in S; m + n = j + k, 0 \leq n \leq j\}$$

We also notice that the cardinality of these subsets is

$$\neq H_j = N - c - j + 1$$

$$\neq D_j^k = \text{Min}\{c - k, j\} + 1; 0 \leq j \leq N - c, 0 \leq k \leq c - k$$

Thus we find that

$$\neq S_{(i,j)} = \begin{cases} N + 1 + \frac{(2c - 1 - j)j}{2} & \text{if } 0 \leq j \leq c \\ N + 1 - j + \frac{(1 + c)c}{2} & \text{if } j \geq 0 \end{cases}$$

We notice that $S_{(i,j)}$ represents the set of accessible states in one step from the initial state $(i; j)$.

Theorem: The transition probabilities $P_{(i,j)(m,n)}$ are as follows:

(a) If $(m, n) \in H_j$ then

$$P_{(i,j)(m,n)} = \sum_{l=0}^{\min\{i, N-c-n\}} \binom{N-i-j}{l+c+n-i-j} x^{l+c+n-i-j} (1-x)^{N-l-c-n} \binom{i}{l} y^l (1-y)^{i-l}$$

For $0 \leq i \leq c, 0 \leq j \leq n \leq N-c$

(b) If $(m, n) \in D_j^k$, then

$$P_{(i,j)(m,n)} = \begin{cases} P\{d-a=i-k, r=j-n\} & \text{if } 0 \leq m \leq c-1 \\ P\{d-a=i-k, j-n \leq r \leq j\} & \text{if } m=c \end{cases}$$

Where;

$$P\{d-a=i-k, r=j-n\} = P\{d-a=i-k\} \binom{j}{j-n} v^{j-n} (1-v)^n$$

$$P\{d-a=i-k, j-n \leq r \leq j\} = P\{d-a=i-k\} \sum_{r=j-n}^j \binom{j}{r-n} v^r (1-v)^{j-r}$$

$$P\{d-a=i-k\} = \sum_{l=\max\{0, j-k\}}^{\min\{i, N-k-j\}} \binom{N-i-j}{l+k-i} x^{l+k-i} (1-x)^{N-j-l+k} \binom{i}{l} y^l (1-y)^{i-l}$$

The steady-state probabilities satisfy the system

$$\prod P = \prod, \sum_{(i,j)} \pi_{ij} = 1, \pi_{ij} \geq 0$$

Where p is the one step transition probabilities matrix and a vector \prod is $\prod = (\pi_{ij})$. For instance, we have

(i) The expected number of busy servers

$$E[c] = \sum_{(i,j) \in S} i \pi_{ij} \tag{3a}$$

(ii) The expected number of customers in orbit

$$E[N] = \sum_{(i,j) \in S} j \pi_{ij} \tag{3b}$$

Fuzzy retrial queues and solution procedure

Following equations (2), (3a) and (3b), the membership

function for the expected number of fuzzy servers $E[\tilde{c}]$ and the membership function for the expected number of customers in the orbit $E[\tilde{N}]$ can be obtained from

$$\mu_{E[\tilde{c}]}(z) = \text{Sup}_{x \in X, y \in Y, v \in V} \text{Min} \left\{ \mu_{\tilde{p}}(x), \mu_{\tilde{s}}(v), \mu_{\tilde{q}}(y) \mid z = \sum_{(i,j) \in S} i \pi_{ij} \right\} \tag{4a}$$

$$\mu_{E[\tilde{N}]}(z) = \text{Sup}_{x \in X, y \in Y, v \in V} \text{Min} \left\{ \mu_{\tilde{p}}(x), \mu_{\tilde{s}}(v), \mu_{\tilde{q}}(y) \mid z = \sum_{(i,j) \in S} j \pi_{ij} \right\} \tag{4b}$$

From now on we represent the subject about $E[\tilde{c}]$ and the same for $E[\tilde{N}]$. To re express the membership function $\mu_{E[\tilde{c}]}(z)$ of $E[\tilde{c}]$ in an understandable and usable form we adopt Zadeh's approach which relies on α -cuts of $E[\tilde{c}]$. Definitions for the α -cuts of \tilde{p}, \tilde{s} and \tilde{q} as crisp intervals are as follows:

$$p(\alpha) = [x_{\alpha}^l, x_{\alpha}^u] = [\text{Min}\{x \mid \mu_{\tilde{p}}(x) \geq \alpha\}, \text{Max}\{x \mid \mu_{\tilde{p}}(x) \geq \alpha\}]$$

$$s(\alpha) = [v_{\alpha}^l, v_{\alpha}^u] = [\text{Min}\{v \mid \mu_{\tilde{s}}(v) \geq \alpha\}, \text{Max}\{v \mid \mu_{\tilde{s}}(v) \geq \alpha\}]$$

$$q(\alpha) = [y_{\alpha}^l, y_{\alpha}^u] = [\text{Min}\{y \mid \mu_{\tilde{q}}(y) \geq \alpha\}, \text{Max}\{y \mid \mu_{\tilde{q}}(y) \geq \alpha\}]$$

As a result, the bounds of these intervals can be described as functions of α and can be obtained as

$$x_{\alpha}^l = \text{Min} \mu_{\tilde{p}}^{-1}(\alpha), \quad x_{\alpha}^u = \text{Max} \mu_{\tilde{p}}^{-1}(\alpha)$$

$$v_{\alpha}^l = \text{Min} \mu_{\tilde{s}}^{-1}(\alpha), \quad v_{\alpha}^u = \text{Max} \mu_{\tilde{s}}^{-1}(\alpha)$$

$$y_{\alpha}^l = \text{Min} \mu_{\tilde{q}}^{-1}(\alpha), \quad y_{\alpha}^u = \text{Max} \mu_{\tilde{q}}^{-1}(\alpha)$$

Thus, $\forall \alpha \in [0, 1]$

$$x \in [x_{\alpha}^l, x_{\alpha}^u], \quad v \in [v_{\alpha}^l, v_{\alpha}^u], \quad y \in [y_{\alpha}^l, y_{\alpha}^u]$$

Now, we can use the α -cuts of $E[\tilde{c}]$ to construct its membership function since the membership function $E[\tilde{c}]$ depends on x, v , and y , in addition these variables depends on α parameters too, thus the membership function $E[\tilde{c}]$ depends on α parameter. If

$$(E[c]_{\alpha}) = [(E[c])_{\alpha}^l, (E[c])_{\alpha}^u] \quad \alpha \in [0, 1]$$

Our objective is finding $(E[c]_{\alpha})^l, (E[c]_{\alpha})^u$ that α and z satisfies this conditions:

$$\mu_{E[\tilde{c}]}(z) = \alpha, \quad z = \sum_{(i,j) \in S} i \pi_{ij}$$

Therefore we must solve the following programming problems:

$$(E[c])_{\alpha}^l = \text{Min} \sum_{(i,j) \in s} i \pi_{ij}$$

s.t

$$\mu_{E[\tilde{c}]}(z) = \alpha$$

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S \quad 5a$$

Since $\mu_{E[\tilde{c}]}(z)$ is the minimum of $\mu_{\tilde{p}}(x)$, $\mu_{\tilde{q}}(y)$, and $\mu_{\tilde{s}}(v)$ thus to satisfy $\mu_{E[\tilde{c}]}(z) = \alpha$ at least we need one of the following cases to hold

- (i) $(\mu_{\tilde{p}}(x) = \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) \geq \alpha)$
- (ii) $(\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) = \alpha, \mu_{\tilde{q}}(y) \geq \alpha)$
- (iii) $(\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) = \alpha)$

With using of these cases instead of $\mu_{E[\tilde{c}]}(z) = \alpha$, problems (5a) and (5b) are transformed as follow:

$$(E[c])_{\alpha}^l = \text{Min} \sum_{(i,j) \in s} i \pi_{ij} \quad (6a)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) = \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) \geq \alpha$$

$$(E[c])_{\alpha}^{u_1} = \text{Max} \sum_{(i,j) \in s} i \pi_{ij} \quad (6b)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) = \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) \geq \alpha$$

$$(E[c])_{\alpha}^l = \text{Min} \sum_{(i,j) \in s} i \pi_{ij} \quad (6c)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) = \alpha, \mu_{\tilde{q}}(y) \geq \alpha$$

$$(E[c])_{\alpha}^{u_2} = \text{Max} \sum_{(i,j) \in s} i \pi_{ij} \quad (6d)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) = \alpha, \mu_{\tilde{q}}(y) \geq \alpha$$

$$(E[c])_{\alpha}^l = \text{Min} \sum_{(i,j) \in s} i \pi_{ij} \quad (6e)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) = \alpha$$

$$(E[c])_{\alpha}^{u_3} = \text{Max} \sum_{(i,j) \in s} i \pi_{ij} \quad (6f)$$

s.t

$$\prod P = \prod$$

$$\sum_{(i,j) \in s} \pi_{ij} = 1$$

$$\pi_{ij} \geq 0 \quad \forall (i, j) \in S$$

$$\mu_{\tilde{p}}(x) \geq \alpha, \mu_{\tilde{s}}(v) \geq \alpha, \mu_{\tilde{q}}(y) = \alpha$$

The α -cuts form a nested structure with respect to α given $0 < \alpha_2 < \alpha_1 \leq 1$ we have:

$$\begin{aligned}
 [x_{\alpha_1}^l, x_{\alpha_1}^u] &\subseteq [x_{\alpha_2}^l, x_{\alpha_2}^u] \\
 [v_{\alpha_1}^l, v_{\alpha_1}^u] &\subseteq [v_{\alpha_2}^l, v_{\alpha_2}^u] \\
 [y_{\alpha_1}^l, y_{\alpha_1}^u] &\subseteq [y_{\alpha_2}^l, y_{\alpha_2}^u]
 \end{aligned}$$

Therefore equations (6a), (6c) and (6e) have the same optimum value and the problem (6b), (6d), (6f) have a same optimum value too. Therefore; instead of solving six problems, it suffices to solve the set of following problems:

$$(E[c])_{\alpha}^l = \text{Min} \sum_{(i,j) \in s} i\pi_{ij} \tag{7a}$$

s.t

$$\begin{aligned}
 \prod P &= \prod \\
 \sum_{(i,j) \in s} \pi_{ij} &= 1 \\
 \pi_{ij} &\geq 0 \quad \forall (i,j) \in S \\
 x_{\alpha}^l \leq x \leq x_{\alpha}^u, \quad y_{\alpha}^l \leq y \leq y_{\alpha}^u, \quad v_{\alpha}^l \leq v \leq v_{\alpha}^u
 \end{aligned}$$

$$(E[c])_{\alpha}^u = \text{Max} \sum_{(i,j) \in s} i\pi_{ij} \tag{7b}$$

s.t

$$\begin{aligned}
 \prod P &= \prod \\
 \sum_{(i,j) \in s} \pi_{ij} &= 1 \\
 \pi_{ij} &\geq 0 \quad \forall (i,j) \in S \\
 x_{\alpha}^l \leq x \leq x_{\alpha}^u, \quad y_{\alpha}^l \leq y \leq y_{\alpha}^u, \quad v_{\alpha}^l \leq v \leq v_{\alpha}^u
 \end{aligned}$$

Optimum value in equations (7a) and (7b) obtained with respect to α . This model is a set of mathematical programs with boundary constraints and lends itself to the systematic study of how optimal solution changes with $x_{\alpha}^l, x_{\alpha}^u, v_{\alpha}^l, v_{\alpha}^u, y_{\alpha}^l, y_{\alpha}^u$ as α varies over (0, 1]. The model is a special case of parametric NLPS (None linear programming). The result interval $[(E[c])_{\alpha}^l, (E[c])_{\alpha}^u]$ obtained from equations (7a) and (7b) represents the α -cuts of $E[\tilde{c}]$. We know that $(E[c])_{\alpha_1}^l \geq (E[c])_{\alpha_2}^l, (E[c])_{\alpha_1}^u \leq (E[c])_{\alpha_2}^u$ where $0 < \alpha_2 < \alpha_1 \leq 1$ in other words, $(E[c])_{\alpha}^l$ increases and $(E[c])_{\alpha}^u$ decreases as α increases. Consequently, the membership function $\mu_{E[\tilde{c}]}(z)$ can be found from

equation (8). If both $((E[c])_{\alpha}^l, (E[c])_{\alpha}^u)$ in equation (8) are invertible with respect to α , then, a left shape function $L(z) = [(E[c])_{\alpha}^l]^{-1}$ and a right shape function $R(z) = [(E[c])_{\alpha}^u]^{-1}$ can be derived, from which the membership function $\mu_{E[\tilde{c}]}(z)$ is constructed:

$$\mu_{E[\tilde{c}]}(z) = \begin{cases} L(z) & (E[c])_{\alpha=0}^l \leq z \leq (E[c])_{\alpha=1}^l \\ 1 & (E[c])_{\alpha=1}^l \leq z \leq (E[c])_{\alpha=1}^u \\ R(z) & (E[c])_{\alpha=1}^u \leq z \leq (E[c])_{\alpha=0}^u \end{cases} \tag{8}$$

Since the Yager's ranking index method possesses the property of area compensation, we adopt this method for transforming the fuzzy values of system characteristic into a crisp one to provide suitable values for system characteristics.

$$O(E[\tilde{c}]) = \int_0^1 \frac{1}{2} \{ (E[c])_{\alpha}^l + (E[c])_{\alpha}^u \} d\alpha$$

We can use all steps for $E[\tilde{N}]$ too.

Conclusion

In this work, we have applied concepts of α -cuts and Zadeh's extension principle to construct membership functions of the expected number of customers in the orbit and the expected number of busy server by the use of paired NLP models. In this approach, α -cuts of the membership functions are determined and their interval limits inverted to get explicit closed-form expressions for the system characteristics. In case when the membership function intervals can not explicitly be inverted, one can also specify the system effective characteristics, and perform numerical results to examine the corresponding α -cuts, and finally uses this information to improve system processes.

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