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Full Length Research Paper

Group geometric consistency index of analytic hierarchy process (AHP)

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The analytic hierarchy process (AHP) is a structured technique for dealing with complex decision-making and is already used to solve many group decision problems. This paper used the Cauchy-Schwarz inequality to improve the AHP algorithm that was developed by Escobar et al. (2004). They provided an upper bound for the group geometric consistency index in order to prove that the group geometric consistency index is less than the maximum of each individual geometric consistency indexes. Although they proposed a useful and novel AHP method, the upper bound estimation still could be improved to provide better group consistency estimation accuracy. This paper proposed a new upper bound estimation that would be able to function in a situation where there are some individual decision makers, whose geometric consistency indexes are greater than the threshold that is proposed by Aguarón and Moreno-Jiménez. The experiment results showed that this paper provided a robust and better estimation. The purposes of this study are as follow; first, this study used Cauchy-Schwarz inequality to improve the synthesized method in AHP method, and to achieve better upper bound estimation. Second, numerical examples are provided to illustrate the findings. Our relative error is 7% of that by Escobar and others to indicate the accuracy. Third, this paper showed that even if the weights changed, the proposed method is still robust with different combinations for decision makers. This study modified one entry of the comparison matrix. The results showed that our estimation performed well on most cases (16 of 17, about 94%) by the sensitivity analysis. Finally, two existing papers with group decision problem were examined with our findings to indicate 6 of 9 are predictable by our upper bound. We also provide a reasonable explanation why the other 3 of 9 cannot be predicted by ours.

Key words: Analytic hierarchy process, group decisions, geometric consistency index.

INTRODUCTION

The analytic hierarchy process (AHP) was designed by Saaty (1980) as a flexible and easily understandable method, assisting decision-makers to solve multi-criteria decision-making problems in a reasonable and practical manner. The AHP model is widely and successfully used today in many fields; for example, Zanakis et al. (1995) studied over 100 applications of AHP within service and government sectors. In spite of all this, some researchers still question its appropriateness and completeness. To mention a few examples, Apostolou and Hassell (1993) considered that comparison matrices with consistency ratio >0.1 are acceptable. Bernhard and Canada (1990) suggested that the incremental benefit/cost ratios should be compared with a cutoff ratio instead of just the benefit/cost ratios found in Saaty (1980, 1995). Finan and Hurley (1996) made a diagonal procedure that constructed a rank-order consistent matrix.

Some researchers have tried to revise those improvements. For instance, Chu and Liu (2002)

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illustrated problems within Apostolou and Hassell (1993). Yang et al. (2004) demonstrated that the methods used in Bernhard and Canada (1990) was incomplete and later modified it. Chao et al. (2004) explained that the diagonal procedure of Finan and Hurley (1996) did not pass the consistency test found by Saaty (1980). Lin et al. (2008b) showed that the proof of the proposition shown in Finan and Hurley (2002) was false. Moreover, their counter example not only failed to satisfy the 1 to 9 scale bound of Saaty (1980), but after making revisions to meet Saaty's condition, the rank reversal disappeared.

The AHP model was originally designed for one expert. As the managerial situation becomes multi-dimensional, several experts need to be considered in the example of a group situation where there is more knowledge and experience to solve the decision-making problem. Early observations and suggestions for using AHP in group decision making were given in the research publication by Saaty (1989).

Some of the published papers have focused on how groups construct the hierarchy, compare elements in the hierarchy, and aggregate weights. Four basic approaches can be used to set the weights of elements in a hierarchy: 1. Consensus, 2. Vote or compromise, 3. Geometric mean of the individual judgments, and 4. Weighted arithmetic mean.

Aczel and Saaty (1983) have shown that the geometric mean preserves the reciprocal property in the combined pairwise comparison matrix. The geometric mean is the approach most commonly used by groups to set priorities. For example, the geometric mean has been incorporated into the popular Expert Choice 2000 Team software (2001).

Xu (2000) claimed that if the consistency index of all individuals passed the test proposed by Saaty (1980), then the group's consistency index would also pass the test. Lin et al. (2008a) pointed out that Xu's proof was dependent on a false relation among the individual priority vectors and the group priority vector that is implicitly assumed by Xu (2000).

Earlier research based on the consistency measure of Aguarón and Moreno-Jiménez (2003), was later used by, Escobar et al. (2004) which also provided an upper bound for the group consistency index in order to show that if all individual comparison matrices passed the consistent test, then the group comparison matrix will also pass the consistent test.

AN UNSOLVED PHENOMENON IN GROUP DECISION OF AHP

However, according to the upper bound of Escobar et al. (2004), researchers can not answer the question that is proposed by Aull-Hyde et al. (2006): While it is known that the weighted geometric mean comparison matrix is of acceptable consistency if all individual comparison matrices are of acceptable consistency, this paper attempts to address the following question: Under what conditions would an aggregated geometric mean comparison matrix be of acceptable consistency if some (or all) of the individual comparison matrices are not of acceptable consistency?

The main purpose of this technical note is to provide an acceptable answer to the open question asked in the paper by Aull-Hyde et al. (2006). Here we present an improved upper bound estimate such that if some individual comparison matrices are not of acceptable consistency, then our upper bound would still provide a valuable estimation to insure that the group comparison matrix will pass the consistency test. In attempting to answer the previous question, we found that Aull-Hyde et al. (2006) used different approaches in applying Monte Carlo's simulation to show that a sufficiently large group size is enough to insure that the group comparison matrix would pass the consistency test.

We try to improve the findings from the paper of Escobar et al. (2004) by considering the consistency for group decision-making in the analytic hierarchy process. Escobar et al. (2004) extended the results for the eigenvector priorization method (EM) (Xu, 2000) and for its associated consistency index (Saaty, 1980). They used the weighted geometric mean method (WGMM) as the aggregation method, the row geometric mean method (RGMM) as the priorization procedure, and the geometric consistency index (GCI), proposed by Crawford and Williams (1985), as the inconsistency measure. They then derived that the group geometric consistency index is less than the maximum of the individual geometric consistency index.

María et al. (2005) and Lin et al. (2008a) had both referred to Escobar et al. (2004) in their references; however, in the paper by María et al. (2005), they did not consider the following discussions and revisions that we propose here. On the other hand, Lin et al. (2008a) have focused to improve the work done by Xu (2000).

We will show how the methods found in Escobar et al. (2004) are invalid and we attempt to revise their estimation in order to derive our new estimation for the group geometric consistency index. In using the same numerical example, we demonstrate that our new estimation is better applied than the results shown previously.

According to our hypothetical example, which is a small modification of the example of Xu (2000), there are at times with some decision-makers, whose geometric consistency index is greater than the threshold that is proposed by Aguarón and Moreno-Jiménez (2003) and Escobar et al. (2004). Our new estimation helps to yield meaningful results and provides an easy method to check the group consistency.

REVIEW OF PREVIOUS RESULTS

Let us suppose that for an analytic hierarchy process problem, there are m decision-makers and the comparison matrices for the alternatives A_1, A_2, $\cdots, A_n$, corresponding to a criterion for the k-th decision maker, is denoted by $\left(a_{ij}^{[k]}\right)_{n\times n}$. By the RGMM, it implies the priority vector, $w^{[k]} = \left(w_1^{[k]}, \cdots, w_n^{[k]}\right)$ with $w_i^{[k]} = \left(\prod_{j=1}^{n} a_{ij}^{[k]}\right)^{\frac{1}{n}}$ for $i = 1,2,\cdots,n$. The error matrix for the k-th decision maker, $E^{[k]} = \left(e_{ij}^{[k]}\right)_{n\times n}$ where $e_{ij}^{[k]} = a_{ij}^{[k]} \dfrac{w_j^{[k]}}{w_i^{[k]}}$ for $1 \le i, j \le n$.

The geometric consistency index

The geometric consistency index for the k-th decision maker, ($GCI^{[k]}$), is assumed as:

$$GCI^{[k]} = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\log e_{ij}^{[k]}\right)^2 . \tag{1}$$

Aguarón and Moreno-Jiménez (2003) suggested that for comparison matrices with size $n = 3$, the threshold is 0.31, for $n = 4$, the threshold is 0.35, and when $n > 4$ the threshold is 0.37. If $GCI^{[k]}$ is less than the corresponding threshold, then the comparison matrix of the *k*-th decision maker will pass the consistency test.

According to the WGMM, with the weight β_k for the *k*-th decision maker and $\beta_k > 0$, $k = 1,2,\cdots,m$, $\sum_{k=1}^{m} \beta_k = 1$, Escobar et al. (2004) defined the group consistency index as follows:

$$GCI^{G} = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\sum_{k=1}^{m} \beta_k \log e_{ij}^{[k]} \right)^2 . \tag{2}$$

They applied the Schwarz inequality in the next lemma.

Lemma 1 of Escobar et al. (2004): For $a, b \in R^n$, it holds that $\sum_{i=1}^{n} a_i b_i \le max\left\{ \sum_{i=1}^{n} a_i^2, \sum_{i=1}^{n} b_i^2 \right\}$.

Proof of Lemma 1 in Escobar et al. (2004). They assumed that when there is without loss of generality, $\max\left\{ \sum_{i=1}^{n} a_i^2, \sum_{i=1}^{n} b_i^2 \right\} = \sum_{i=1}^{n} a_i^2$, in order to derive that;

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} = \sum_{i=1}^{n} a_i^2 . \tag{3}$$

In the next step, we quote their theorem 1.

Theorem 1 of Escobar et al. (2004): Using the WGMM as the aggregation procedure, the RGMM as the priorization procedure, and the GCI to measure the inconsistency, it holds that $GCI^{G} \le \max_{k=1,\cdots,m} \left\{ GCI^{[k]} \right\}$.

They used the numerical example of Xu (2000) with four decision makers (I, II, III, and IV) and four alternatives (A, B, C, and D) such that the comparison matrices are;

$$A^{I} = \begin{bmatrix} 1 & 4 & 6 & 7 \\ 1/4 & 1 & 3 & 4 \\ 1/6 & 1/3 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{bmatrix}, \tag{4}$$

$$A^{II} = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 1/5 & 1 & 4 & 6 \\ 1/7 & 1/4 & 1 & 2 \\ 1/9 & 1/6 & 1/2 & 1 \end{bmatrix}, \tag{5}$$

$$A^{III} = \begin{bmatrix} 1 & 3 & 5 & 8 \\ 1/3 & 1 & 4 & 5 \\ 1/5 & 1/4 & 1 & 2 \\ 1/8 & 1/5 & 1/2 & 1 \end{bmatrix}, \tag{6}$$

$$A^{IV} = \begin{bmatrix} 1 & 4 & 5 & 6 \\ 1/4 & 1 & 3 & 3 \\ 1/5 & 1/3 & 1 & 2 \\ 1/6 & 1/3 & 1/2 & 1 \end{bmatrix}. \tag{7}$$

They provided two different weights where $(\beta_1, \beta_2, \beta_3, \beta_4)$ is $(0.25, 0.25, 0.25, 0.25)$, denoted as G1, and $(0.1, 0.2, 0.3, 0.4)$ denoted as G2. We quote their computation and showed the results in Table 1 of Escobar et al. (2004).

In Table 1 of Escobar et al. (2004), the typing error has been corrected for GCI^{I} from 0.134987 to 0.134897. Aguarón and Moreno-Jiménez (2003) found that for 4 x 4 comparison matrix if $GCI < 0.35$, the comparison matrix passes the consistency test. From the last row of Table 1,

Table 1. Priorities and GCIs for the individual and group (Escobar et al., 2004).

Alternatives	I	II	III	IV	G1	G2
A	0.614455	0.646125	0.569339	0.596672	0.607838	0.601506
B	0.224617	0.227012	0.276410	0.220793	0.236901	0.238691
C	0.098538	0.079288	0.096733	0.108937	0.095543	0.097987
D	0.062390	0.047575	0.057518	0.073598	0.059717	0.061816
GCI	0.134897	0.235805	0.119358	0.165691	0.155377	0.154707

$$\max_{k=1,\cdots,4}\left\{GCI^{[k]}\right\}=GCI^{II}=0.2358<0.35. \qquad (8)$$

Moreover, $GCI^{G1}=0.1554$ and $GCI^{G2}=0.1547$ are both less than 0.35, so that in the paper by Escobar et al. (2004), they claimed that their estimation for the group geometric consistency index is valid. Their findings were highlighted after each individual GCI had passed the consistency test, so that their Theorem 1 guaranteed that the group geometric consistency index would also pass the consistency test.

The inherent problem of this study

Here, we will attempt to point out why the results from published literature are questionable. In trying to explain our idea, let us recall Table 1, in which all individual GCIs are less than 0.35. For a well-defined group geometric consistency index, say GCI^G, it is logical to expect that a $GCI^G<0.35$ can pass the consistency test.

In our point of view, the purpose of Theorem 1 in Escobar et al. (2004) is to insure that their definition, Equation (2), for the group geometric consistency index is well-defined. If we hypothetically changed some entries of the comparison matrix, A^{II}, we can construct a new comparison matrix, say A^V:

$$A^V=\begin{bmatrix}1 & 5 & 7 & 3\\ 1/5 & 1 & 4 & 6\\ 1/7 & 1/4 & 1 & 2\\ 1/3 & 1/6 & 1/2 & 1\end{bmatrix}, \qquad (9)$$

so that $GCI^V=0.738313>0.35$. Given this new situation, Theorem 1 of Escobar et al. (2004) cannot help us to determine whether or not the group geometric consistency index will pass the consistency test. This would indicate that the estimation of the group geometric consistency index should be able to handle, for instance, the problem when some decision makers have the GCI beyond the threshold as proposed by Aguarón and Moreno-Jiménez (2003).

REVISIONS OF THIS STUDY

First, we mention an improved approach to using the Schwarz inequality.

The Schwarz inequality: For $a,b\in R^n$, it holds that :

$$\sum_{i=1}^{n}a_ib_i\le\sqrt{\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2}.$$

If we adopt the upper bound $\sqrt{\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2}$, proposed by the Schwarz inequality, is small, than the upper bound, $\max\left\{\sum_{i=1}^{n}a_i^2,\sum_{i=1}^{n}b_i^2\right\}$, of Escobar et al. (2004). It will help us derive better results than that of Escobar et al. (2004).

Proposition 1: $\frac{2}{(n-1)(n-2)}\sum_{i<j}\log e_{ij}^{[k]}\log e_{ij}^{[l]}\le\sqrt{GCI^{[k]}}\sqrt{GCI^{[l]}}$.

Proof **:** By the Schwarz inequality, we derive that:

$$\frac{2}{(n-1)(n-2)}\sum_{i<j}\log e_{ij}^{[k]}\log e_{ij}^{[l]}\le\sqrt{\frac{2}{(n-1)(n-2)}}\sqrt{\sum_{i<j}\left(\log e_{ij}^{[k]}\right)^2}\sqrt{\frac{2}{(n-1)(n-2)}}\sqrt{\sum_{i<j}\left(\log e_{ij}^{[l]}\right)^2}$$
$$=\sqrt{GCI^{[k]}}\sqrt{GCI^{[l]}}. \qquad (10)$$

Next, we revise theorem 1 of Escobar et al. (2004).

Proposition 2: Using the WGMM, with the weight $(\beta_1,\beta_2,\cdots,\beta_m)$, as the aggregation procedure, the RGMM as the priorization procedure, the GCI to measure the inconsistency, it holds that,

$$GCI^G\le\left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2.$$

Proof**:** From Equation (2) and Schwarz inequality, we know that:

$$GCI^G = \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\sum_{k=1}^{m}\beta_k \log e_{ij}^{[k]}\right)^2$$
$$= \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\sum_{k=1}^{m}(\beta_k \log e_{ij}^{[k]})^2 + 2\sum_{1\le k<l\le m}\beta_k\beta_l \log e_{ij}^{[k]} \log e_{ij}^{[l]}\right)$$
$$= \sum_{k=1}^{m}\beta_k^2 \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\log e_{ij}^{[k]}\right)^2 + 2\sum_{k<l}\beta_k\beta_l \frac{2}{(n-1)(n-2)}\sum_{i<j}\log e_{ij}^{[k]} \log e_{ij}^{[l]}$$
$$\le \sum_{k=1}^{m}\beta_k^2 GCI^{[k]} + 2\sum_{k<l}\beta_k\beta_l\sqrt{GCI^{[k]}}\sqrt{GCI^{[l]}}$$
$$= \left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2. \quad (11)$$

Here, we begin to compare our revisions with that of Escobar et al. (2004).

Proposition 3: Our estimation, $\left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2$ is superior to $\max_{k=1,\cdots,m}\left\{GCI^{[k]}\right\}$ of Escobar et al. (2004), that is $\left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2 \le \max_{k=1,\cdots,m}\left\{GCI^{[k]}\right\}$, and $\left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2 = \max_{k=1,\cdots,m}\left\{GCI^{[k]}\right\}$ if and only if $GCI^{[1]} = GCI^{[2]} = \cdots = GCI^{[m]}$.

Proof: Owing to $\max_{k=1,\cdots,m}\left\{\sqrt{GCI^{[k]}}\right\} = \sum_{k=1}^{m}\left(\beta_k \max_{k=1,\cdots,m}\sqrt{GCI^{[k]}}\right) \ge \sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}$, we finish the first part of the proof. Since β_k denotes the weight for the *k*-th decision maker, in a group decision making environment, every expert opinion should be considered, otherwise we would not invite the expert in the decision group, such is that we can assume that $\beta_k > 0$, for $k = 1,2,,...,m$. It implies that only when $GCI^{[1]} = GCI^{[2]} = \cdots = GCI^{[m]}$, all the individual decision makers have the same personal geometric consistency index, then the estimation of Escobar et al. (2004), $\max_{k=1,\cdots,m}\left\{GCI^{[k]}\right\}$ will be equal to our estimation, $\left(\sum_{k=1}^{m}\beta_k\sqrt{GCI^{[k]}}\right)^2$.

However, under the condition that $GCI^{[1]} = GCI^{[2]} = \cdots = GCI^{[m]}$, any reasonable estimation will imply that the evaluation of the group geometric consistency index is the same as $GCI^{[1]}$. Hence, when $GCI^{[1]} = GCI^{[2]} = \cdots = GCI^{[m]}$, our estimation should have the same value as that of Escobar et al. (2004). It points out that both methods satisfy the idempotency property; that is $f(x,...,x) = x$. A desired property is that proposed by Xu and Da (2003) and Xu (2004) for an aggregation operator. Moreover, in other cases, our estimation is smaller than that of Escobar et al. (2004). Therefore, we claim that we derived an improved upper bound for the group geometric consistency index.

Numerical examples

Our estimation

We first consider the numerical example in Xu (2000) and Escobar et al. (2004). By Proposition 2, we use the results in Table 1 to derive that:

For G1, $\left(\sum_{k=1}^{4}\beta_k\sqrt{GCI^{[k]}}\right)^2 = 0.161109 < 0.35$, an (12)

For G2, $\left(\sum_{k=1}^{4}\beta_k\sqrt{GCI^{[k]}}\right)^2 = 0.160260 < 0.35$. (13)

In comparing the results of Escobar et al. (2004), they found that:

For G1 and G2, $\max_{k=1,\cdots,4}\left\{GCI^{[k]}\right\} = 0.235805 < 0.35$. (14)

Our estimations for the group geometric consistency index are 0.1611 and 0.1603, both being smaller than 0.2358, found in Escobar et al. (2004). If we compute the relative error, then:

$$\frac{0.1611 - 0.1554}{0.2358 - 0.1554} = 7.2\% \quad (15)$$

and

$$\frac{0.1603 - 0.1547}{0.2358 - 0.1547} = 6.9\%. \quad (16)$$

From Equations (15) and (16), our estimation, $\left(\sum_{k=1}^{4}\beta_k\sqrt{GCI^{[k]}}\right)^2$, has a relatively small margin of error, measuring an average of 7% as compared to the estimation, $\max_{k=1,\cdots,4}\left\{GCI^{[k]}\right\}$, of Escobar et al. (2004).

Table 2. For different combination of decision makers' weights.

β_1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2
β_2	0.2	0.2	0.3	0.3	0.4	0.4	0.1	0.1
β_3	0.3	0.4	0.2	0.4	0.2	0.3	0.3	0.4
β_4	0.4	0.3	0.4	0.2	0.3	0.2	0.4	0.3
G2	0.207	0.201	0.254	0.241	0.301	0.294	0.170	0.165
Our estimation	0.226	0.220	0.277	0.264	0.327	0.320	0.181	0.176
β_1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3
β_2	0.3	0.3	0.4	0.4	0.1	0.1	0.2	0.2
β_3	0.1	0.4	0.1	0.3	0.2	0.4	0.1	0.4
β_4	0.4	0.1	0.3	0.1	0.4	0.2	0.4	0.1
G2	0.258	0.239	0.307	0.292	0.172	0.162	0.214	0.196
Our estimation	0.279	0.260	0.329	0.315	0.183	0.173	0.230	0.212
β_1	0.3	0.3	0.4	0.4	0.4	0.4	0.4	0.4
β_2	0.4	0.4	0.1	0.1	0.2	0.2	0.3	0.3
β_3	0.1	0.2	0.2	0.3	0.1	0.3	0.1	0.2
β_4	0.2	0.1	0.3	0.2	0.3	0.1	0.2	0.1
G2	0.305	0.298	0.170	0.164	0.212	0.200	0.255	0.248
Our estimation	0.325	0.318	0.180	0.175	0.226	0.214	0.271	0.265

An individual consistency index is beyond the threshold value

Moreover, we consider the hypothetical example in "reviews of previous results". It follows that we replace decision-maker II with decision-maker V. When $(\beta_1, \beta_2, \beta_3, \beta_4)$ is $(0.25, 0.25, 0.25, 0.25)$ for G1, $GCI^{G1} = 0.225938$ and our estimation.

$$\left(\sum_{k=1, k\neq 2}^{5} \beta_k \sqrt{GCI^{[k]}} \right)^2 = 0.244795 < 0.35. \quad (17)$$

When $(\beta_1, \beta_2, \beta_3, \beta_4)$ is $(0.1, 0.2, 0.3, 0.4)$ for (G2), $GCI^{G2} = 0.206802$ and our estimation,

$$\left(\sum_{k=1, k\neq 2}^{5} \beta_k \sqrt{GCI^{[k]}} \right)^2 = 0.225667 < 0.35. \quad (18)$$

From Equations (17) and (18), we provide examples where some decision-makers whose $GCI > 0.35$, such that the estimation of Escobar et al. (2004) fails to offer a prediction to whether or not the group GCI will be less than 0.35. By Proposition 2, our approach still provides a good estimation for the group GCI that will pass the consistency test.

Variation of relative weights of decision makers

Next, we consider the problem as to whether or not our results are strongly dependent on some particular weights of decision makers. We propose using the permutation of the previous example where $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.1, 0.2, 0.3, 0.4)$ to demonstrate that our approach is stable for different combinations of decision-makers' weights in Table 2.

Table 2 illustrates that our estimation is reliable for all different possible combinations. In this situation, when the decision-maker A^V is with $GCI > 0.35$, our approach provides a good estimation for group GCI.

Variation of one entry in the comparison matrix

Next, we will study the shortcoming of our approach. In this note, our main goal is to improve the method of finding an upper bound of Escobar et al. (2004). Unfortunately, their estimation can only apply in cases when all individual decision makers pass the consistency test. By Proposition 3, we are able to offer an improved upper bound. However, it is still an upper bound with limitations that will sometimes show our estimation to be an overestimate of the group GCI, such that the group $GCI < 0.35$ but our estimation > 0.35.

We run the sensitivity analysis for $a_{13}^{[5]}$ with

Table 3. The sensitivity analysis for $a_{13}^{[5]}$.

$a_{13}^{[5]}$	group GCI	Our estimation	Relative error (%)	GCI of A^V
9	0.1547	0.1603	3.58	0.2358
8	0.1595	0.1658	3.94	0.2704
7	0.1652	0.1726	4.50	0.3153
6	0.1721	0.1811	5.26	0.3744
5	0.1860	0.1919	6.26	0.4546
4	0.1916	0.2060	7.53	0.5678
3	0.2068	0.2257	9.12	0.7383
2	0.2301	0.2558	11.15	1.0254
1	0.2750	0.3131	13.86	1.6431
a=0.6662	b=0.3043	0.3500	15.03	2.0795
1/2	0.3263	0.3775	15.69	2.4210
c=0.3726	0.3500	d=0.4068	16.24	2.7995
1/3	0.3593	0.4182	16.42	2.9503
1/4	0.3840	0.4486	16.81	3.3590
1/5	0.4039	0.4729	17.06	3.6951
1/6	0.4207	0.4932	17.23	3.9820
1/7	0.4353	0.5108	17.35	4.2332
1/8	0.4481	0.5263	17.44	4.4572
1/9	0.4596	0.5401	17.51	4.6597

$a_{13}^{[5]} = 9,8,...,1,1/2,...,1/9$ to list the results in Table 3. In examining Table 3 sensitivity analysis, we may say that the group comparison matrix has an average effect to moderate higher GCI of an individual decision-maker. For example, when $a_{13}^{[5]}$ changes from 9 to $1/9$, the GCI of A^V varies from 0.2358 to 4.6597, which is 20 times greater. On the other hand, the group GCI varies from 0.1547 to 0.4596, three times greater.

As shown in Table 3, a majority of our estimates, 16 out of 17, can predict whether or not group GCI will pass the consistency test. This indicates that our estimation provides a reliable prediction for the group comparison matrix. Only when our estimation happens to be slightly higher than 0.35, for example, 0.37, does our approach require further examination for group GCI.

ANALYSIS OF RESULTS

Moreover, we consider the monotonic property for the relative error between our estimation and the exact group GCI. Then we may claim that the relative error is an increasing function of group GCI. This is an advantage for our estimation method, since we are only concerned about those groups with $GCI < 0.35$.

To be more precise, we assumed a,b,c and d in the following. When $a_{13}^{[5]} = a$, then group $GCI = b$ and our estimation is 0.35. On the other hand, when $a_{13}^{[5]} = c$, then group $GCI = 0.35$ and our estimation is d. It follows that $a = 0.6662$, $b = 0.3043$, $c = 0.3726$, and $d = 0.4068$.

Under the increasing property, we only need to study those groups of GCI of $(b, 0.35]$ such that the group GCI passes the consistency test, but our estimation as $(0.35, d]$ fails the consistency test.

When we execute the sensitivity analysis, we discover that when $c \le a_{13}^{[5]} < a$, then group GCI are $[a, 0.35)$ and our estimations are $(0.35, d]$. A case in point is that only when $a_{13}^{[5]} = 1/2$ does it satisfy the condition $c = 0.3726 \le a_{13}^{[5]} < a = 0.6662$. Our estimation is too high and it implies a false estimation for the group GCI.
The conclusion we draw from this analysis is that for the remaining 16 cases for the Saaty 1 to 9 scales bound (Saaty, 1980), our estimation can provide accurate inference for the group GCI.

Practical application

Finally, we provide some practical applications of our proposed estimation for the group consistency index. We reviewed some articles and searched for papers with

Table 4. The individual expert and group consistency index, before reducing the dispersion.

Index	Weight	Weight	Weight
G (1)=0.9424	1/5	1/15	5/15
G (2)=0.4121	1/5	2/15	4/15
G (3)=0.7023	1/5	3/15	3/15
G (4)=0.9470	1/5	4/15	2/15
G (5)=0.7113	1/5	5/15	1/15
Our proposed estimation	0.7284	0.7371	0.7197
Group consistency index	0.3149	0.3656	0.3926

Table 5. The individual expert and group consistency index, after reducing the dispersion.

Index	Weight	Weight	Weight
G (1)=0.5899	1/5	1/15	5/15
G (2)=0.1466	1/5	2/15	4/15
G (3)=0.4091	1/5	3/15	3/15
G (4)=0.5792	1/5	4/15	2/15
G (5)=0.8011	1/5	5/15	1/15
Our proposed estimation	0.3908	0.5042	0.2918
Group consistency index	0.2033	0.3339	0.1405

examples of group decision-making, where the comparison matrices are documented in detail. There are two papers that fulfilled our requirement: Saaty and Vargas (2007) and Altuzarra et al. (2007). The published matrices can potentially form a data set for analysis. We cite the original comparison matrices for a group of five experts from Saaty and Vargas (2007) in the following,

$$\begin{pmatrix} 1 & (2,3,4,5,6) & \left(\frac{1}{2},2,1,\frac{1}{3},4\right) & \left(3,4,\frac{1}{2},2,8\right) \\ & 1 & (1,2,3,4,5) & (5,4,3,2,1) \\ & & 1 & \left(\frac{1}{4},\frac{1}{3},1,2,5\right) \\ & & & 1 \end{pmatrix} \quad (19)$$

In Saaty and Vargas (2007), they did not provide the weights for those five decision-makers. Therefore, to start, we can consider the following cases: (a) uniformly distributed, (b) increasing order, and (c) decreasing order, and then find the consistency index for individuals and group (Table 4).

Since all five experts are inconsistent, our proposed upper bound will imply that there is a relatively high estimation for the group consistency index. We would say that in two-thirds of our overestimated upper bound, we are able to predict that the group consistency index is also greater than the threshold.

We also record the comparison matrices after reducing the dispersion in Saaty and Vargas (2007),

$$\begin{pmatrix} 1 & (2,3,4,5,6) & (2,2,1,1,2) & (3,4,3,2,8) \\ & 1 & (1,2,3,4,5) & (5,4,3,2,1) \\ & & 1 & (1,2,1,2,5) \\ & & & 1 \end{pmatrix}, \quad (20)$$

and then we find the consistency index for individual experts and the group (Table 5).

After reducing dispersion, four of them are still inconsistent having a relatively lower inconsistency index. Consequently, the group consistency index and our estimation both decrease. Our proposed upper limit can only provide accurate predictions for cases where the group consistency index has a very small value. Otherwise, our estimations are too high to provide a meaningful prediction. This is a phenomenon that has been discussed by Aull-Hyde et al. (2006). They claimed that for group decision-making problems, if the number of experts is large enough, then the synthesized group comparison matrix will pass the consistency test. Their experiment data showed that for 4 by 4 comparison matrices, if the group of experts is more than 40 persons from their 10,000 randomly generated experiments, then all group comparison matrices pass the consistency test. This example may be a special case, where in only 5 experts with diverse opinions and after synthesizing can their opinions be a better trade-off for a consistent group comparison matrix.

Table 6. The individual expert and group consistency index in Altuzarra et al. (2007).

Index	Weight	Weight	Weight	Weight
G (1)=0.1042	1/6	1/21	6/21	2/21
G (2)=0.3608	1/6	2/21	5/21	3/21
G (3)=0.7616	1/6	3/21	4/21	6/21
G (4)=0.4972	1/6	4/21	3/21	5/21
G (5)=0.4305	1/6	5/21	2/21	4/21
G (6)=0.0196	1/6	6/21	1/21	1/21
Our proposed estimation	0.3020	0.2786	0.3264	0.4428
Group consistency index	0.0583	0.0579	0.0671	0.0823

Application of Altuzarra et al. (2007) and Wang and Xu (1990)

In Altuzarra et al. (2007), they used a group decision problem that was proposed by Wang and Xu (1990). We adopt the abbreviated expression from Saaty and Vargas (2007), and list the comparison matrices of six experts as shown in the following:

$$\begin{pmatrix} 1 & \left(3,4,\frac{1}{2},3,2,2\right) & (5,3,3,5,6,5) & (4,5,2,2,3,4) & (7,8,5,6,3,9) \\ & 1 & (3,4,5,1.2,3) & (2,3,1,3,5,2) & (5,6,2,2,4,6) \\ & & 1 & \left(\frac{1}{2},1,2,4,\frac{1}{2},1\right) & \left(3,5,\frac{1}{2},5,1,2\right) \\ & & & 1 & \left(3,7,5,\frac{1}{2},5,3\right) \\ & & & & 1 \end{pmatrix} \quad (21)$$

Then we find the consistency index in Table 6.

In Altuzarra et al. (2007), they did not provide the weights for those six decision-makers. Therefore, to start, we consider the following cases: (a) uniformly distributed, (b) increasing order, and (c) decreasing order.

Only $G(1)$, $G(2)$ and $G(6)$ are less than the threshold 0.37, which indicates that the comparison matrices of these three experts are consistent. The remaining three comparison matrices are inconsistent.

According to our proposed method, our upper bound can provide an accurate prediction for the three assigned cases. On the other hand, we create a special case that is related to the worst condition for our proposed method with relatively high weights for those inconsistent experts. Our estimation is 0.4428, but as the group consistency index is acceptable as 0.0823 to indicate some special situations, our upper bound may be too big to apply.

DISCUSSION AND CONCLUSION

We derived a new estimation for the group geometric consistency index. By the same numerical example of Escobar et al. (2004), our estimation is more accurate than their results given, since our estimation error is 7% of theirs. When some decision makers have comparison matrices that are not consistent, the procedure of Escobar et al. (2004) cannot provide estimation. However, our new estimation is still applicable to indicate whether or not the group geometric consistency index is acceptable. With one individual, decision-maker, A^V, whose consistency index is $GCI^V = 0.74$ for 24 different cases of weights for decision-makers, we show that our upper bound can correctly predict the group consistency index. Using a sensitivity analysis for one entry of the comparison matrix A^V, we demonstrated that 94% of our upper bound can provide the right prediction for the group consistency index.

Our new estimation implies a simple procedure to predict the group consistency index, such that our revisions will undoubtedly result in helping the development of theoretical analysis and practical applications.

REFERENCES

Aczel J, Saaty TL (1983). Procedures for synthesizing ratio judgments. J. Math. Psychol., 27: 93-102.

Aguarón J, Moreno-Jiménez JM (2003). The geometric consistency index: Approximated thresholds. Eur. J. Oper. Res., 147: 137-145.

Altuzarra A, Moreno-Jiménez LM, Salvador M (2007). A Bayesian priorization procedure for AHP-group decision making. Eur. J. Oper. Res., 182: 367-382.

Apostolou B, Hassell JM (1993). An empirical examination of the sensitivity of the analytic hierarchy process to departures from recommended consistency ratios. Math. Comp. Model., 17: 163-170.

Aull-Hyde R, Erdogan S, Joshua M, Duke JM (2006). An experiment on the consistency of aggregated comparison matrices in AHP. Eur. J. Oper. Res., 171: 290-295.

Bernhard RH, Canada JR (1990). Some problems in using benefit/cost ratios with the analytic hierarchy process. Eng. Econ., 36: 56-65.

Chao HCJ, Yang KL, Chu P (2004). Note on diagonal procedure in analytic hierarchy process. Math. Comp. Model, 40: 1089-1092.

Chu P, Liu JKH (2002). Note on consistent ratio. Math. Comp. Model, 35: 1077-1080.

Crawford G, Williams C (1985). A note on the analysis of subjective judgment matrices. J. Math. Psychol., 29: 387-405.

Escobar MT, Aguarón J, Moreno-Jiménez JM (2004). A note on AHP group consistency for the row geometric mean priorization procedure.

Eur. J. Oper. Res., 153(2): 318-322.
Expert Choice 2000 Team (2001). Pittsburgh, Expert Choice, Inc.
Finan JS, Hurley WJ (1996). A note on a method to ensure rank-order consistency in the analytic hierarchy process. Int. Tran. Opns. Res., 3(1): 99-103.
Finan JS, Hurley WJ (2002). The analytic hierarchy process: Can wash criteria be ignored. Comp. Opns. Res., 29: 1025-1030.
Lin R, Lin J, Chang J, Tang D, Chao H, Julian P (2008a). Note on group consistency in analytic hierarchy process. Eur. J. Oper. Res., 190: 672-678.
Lin J, Chou SY, Chouhuang WT, Hsu CP (2008b). Note on Wash criterion in analytic hierarchy process. Eur. J. Oper. Res., 185: 444-447.
María J, Jiménez M, Joven JA, Pirla AR, Lanuza AT (2005). A spreadsheet module for consistent consensus building in AHP-group decision making. Group Decis. Negot., 14: 89-108.
Saaty TL (1980). The analytic hierarchy process, McGraw-Hill, NY.
Saaty TL (1989). Group decision making and the AHP, In: Golden BL, Wasil EA, Harker PT, editors. The analytic hierarchy process: applications and studies. Heidelberg: Springer , pp. 59-67.
Saaty TL (1995). Decision Making for Leaders, RWS, Pittsburgh, PA.
Saaty TL, Vargas LG (2007). Dispersion of group judgments. Math. Comp. Model, 46: 918-925.
Wang YM, Xu NR (1990). Linear programming method for solving the group-AHP. J. South. Univ., 20(6): 58-63.
Xu Z (2000). On consistency of the weighted geometric mean complex judgment matrix in AHP. Eur. J. Oper. Res., 126: 683-687.
Xu Z, Da QL (2003). An overview of operators for aggregating information. Int. J. Intell. Syst., 18: 953-969.
Xu Z (2004). A method based on Linguistic aggregation operators for group decision making with linguistic preference relations. Inf. Sci., 166: 19-30.
Yang KL, Chu P, Chouhuang WT (2004). Note on incremental benefit/cost ratios in analytic hierarchy process. Math. Comp. Model, 39: 279-286.
Zanakis SHA, Mandakovic T, Gupta SKA, Sahay S, Hong S (1995). A review of program evaluation and fund allocation methods within the service and government sectors. Socio-Econ. Plan. Sci., 29(1): 59-79.