African Journal of Business Management Vol. 5(6), pp. 2336-2344, 18 March, 2011
Available online at http://www.academicjournals.org/AJBM
DOI: 10.5897/AJBM10.1023
ISSN 1993-8233

Full Length Research Paper

Discussion of arithmetic defuzzifications for fuzzy production inventory models

Gino K. Yang

Department of Computer Science and Information Management, Hungkuang University, Taiwan.
E-mail: yangklung@sunrise.hk.edu.tw. Fax: +886-4-26521921.

Accepted 16 December, 2010

Inventory management is often very unreliable because of the variability of the demand and the uncertainty of the forecast. Taking human subjective into consideration, the collection of historical data and the inaccuracy of linguistic hedges, recently fuzzy theory has been applied to construct the uncertain factors in an inventory which was hard to describe before. This paper is an extension of the paper by Hsieh, published in Information Sciences 146 (2002) 29–40 which examined a production inventory model under a fuzzy environment. This paper purposes three major points. Firstly, we provided a patchwork to improve Hsieh's approach to show that the application of Taha's algorithm of the extended Lagrangean method results in a tedious iterative computation. Secondly, we generalize the Graded Mean Integration Representation method to a weighted average operation. Thirdly, we studied the consistency between two arithmetic defuzzifications to obtain the final minimum crisp estimation under a fuzzy environment. Numerical examples are provided to illustrate our findings.

Key words: Fuzzy production inventory, function principle, graded mean integration representation, fuzzy optimization.

INTRODUCTION

A common problem that arises in the management of an inventory is demand uncertainty. It is difficult to predict the future demand of a new seasonal product because of insufficient historical data. To cope with these kinds of problems, managers use linguistics to express the uncertain demand, such as very high, approximate...etc. Similarly, human factors will also make the production rate uncertain in the manufacturing procedure. Fuzzy theory can effectively resolve the problems that come up in cases where uncertain linguistics is present. The decision makers develop a fuzzy model with proper membership functions and then defuzzy the fuzzy objective function to a deterministic objective function to obtain the optimal solution.

Inventory models considering fuzzy conditions have been studied extensively in recent papers. Campos et al. (2006) proposed two different criteria to obtain robust solutions for linear optimization problems when the objective function coefficients are modeled with possibility distributions. Xie et al. (2006) presented a new hierarchical two-level approach for inventory management and control in supply chains. Li et al. (2006) considered two defuzzifying approaches (Graded Mean Integration Representation and the Median rule) and two fuzzy numbers (the triangle and trapezoidal fuzzy numbers) for depicting the fuzzy inventory model with backorder. Zhu (2006) applied an ant colony optimization algorithm to solve a continuous optimization problem of fuzzy inventory model. Dutta et al. (2007) considered a continuous review inventory system where fuzziness and randomness appear simultaneously in an optimization setting. Yao et al. (2007) constructed three different intervals to include the average demand per unit of time, the relative duration of setup, and the unit cost of production, respectively, which are fuzzified by triangular fuzzy numbers to derive fuzzy total cost. They applied the signed distance and centroid method for defuzzification to obtain the final fuzzy total cost. Panda et al. (2008) developed a mathematical model for a single period multi-product manufacturing system of stochastically imperfect items with continuous stochastic demand under budget and shortage constraints. Chen and Chang (2008) considered fuzzy production inventory model with defective productions that are not repairable. They used the Function

Principle and Graded Mean Integration Representation method to find the most economically advantageous production quantity with fuzzy inventory model. Vijayan and Kumaran (2009) studied an inventory models in which the time period of the sales was a decision variable considered in fuzzy environments. Dutta and Chakraborty (2010) analyzed a fuzzy inventory model based on single-period productions when the opportunity for product substitution was taken into consideration. We continued research on this subject with inventory models in fuzzy conditions. In this article will point out that Hsieh (2002) did not fully apply the convex property of objective functions, causing his approach to become lengthy and tedious.

REVIEW OF HSIEH'S METHOD

Hsieh (2002) considered how to extend a crisp production inventory model to a fuzzy production inventory model, using two different approaches. For the first extension, he assumed that all parameters are generalized as trapezoidal fuzzy numbers, but the variable, production quantity, was still a crisp number. In his second extension, the variable, production quantity, also became a trapezoidal fuzzy number. He adopted the Function Principle (Chen, 1985, 1985a) to construct his fuzzy production inventory model, and then applied the Graded Mean Integration Representation method (Chen, 1999) to defuzzify a fuzzy problem into a classical mathematical optimization problem. For a single variable problem, he used calculus to find the optimal solution. As for a multi-variable problem, he considered the Taha's algorithm of the extended Lagrangean method (Taha, 1997) to solve the optimization problem with multiple variables with inequality constraints.

Hsieh (2002) first considered a production inventory model so that the average cost , $C\left(Q_p\right)$, is expressed as

$$C(Q_p)=\frac{DT}{Q_p}+\frac{A}{2}Q_p\left(1-\frac{R}{P}\right), \tag{1}$$

where production quantity (Q_p), inventory cost (*A*), yearly demand (*D*), setup cost (*T*), daily demand rate (*R*), daily production rate (*P*), are all crisp values and the optimal production quantity is denoted as

$$Q_p^*=\sqrt{\frac{2DT}{A\left(1-\left(R/P\right)\right)}}\,. \tag{2}$$

Hsieh (2002) used the Function Principle (Chen, 1985, 1985a) to extend his proposed crisp production inventory model to a fuzzy production inventory model in two different approaches. In his first extension, the parameters are fuzzy numbers and the variable is still a crisp number. The inventory cost (*A*), yearly demand (*D*), setup cost (*T*), daily demand rate (*R*), daily production rate (*P*) are all generalized to trapezoidal fuzzy numbers. Hence, he considered the following fuzzy parameters: $\tilde{A}=(a_1,a_2,a_3,a_4)$, $\tilde{D}=(d_1,d_2,d_3,d_4)$, $\tilde{T}=(t_1,t_2,t_3,t_4)$, $\tilde{P}=(p_1,p_2,p_3,p_4)$ and $\tilde{R}=(r_1,r_2,r_3,r_4)$ to be nonnegative trapezoidal fuzzy numbers. Under the Function Principle (Chen, 1985, 1985a), he derived the trapezoidal fuzzy number for average cost, say $\tilde{C}_1$:

$$\tilde{C}_1=\left[\frac{d_1t_1}{Q_p}+\frac{a_1Q_p}{2}\left(1-\frac{r_4}{p_1}\right),\frac{d_2t_2}{Q_p}+\frac{a_2Q_p}{2}\left(1-\frac{r_3}{p_2}\right),\right.$$
$$\left.\frac{d_3t_3}{Q_p}+\frac{a_3Q_p}{2}\left(1-\frac{r_2}{p_3}\right),\frac{d_4t_4}{Q_p}+\frac{a_4Q_p}{2}\left(1-\frac{r_1}{p_4}\right)\right]. \tag{3}$$

To defuzzify the fuzzy average production cost of Equation (3) proposed by trapezoidal fuzzy number, Hsieh (2002) applied the Graded Mean Integration Representation method (Chen, 1999) to adopt a grade as the degree of importance of each point of support set of generalized fuzzy number. For a trapezoidal fuzzy number, $\tilde{B}=(b_1,b_2,b_3,b_4)$, the Graded Mean Integration Representation of $\tilde{B}$, say $P\left(\tilde{B}\right)$ is then represented as

$$P\left(\tilde{B}\right)=\frac{b_1+2b_2+2b_3+b_4}{6}\,. \tag{4}$$

Then Hsieh (2002) used the Graded Mean Integration Representation method (Chen, 1999) to defuzzify Equation (3) as follows

$$P\left(\tilde{C}_1\right)=\frac{1}{6}$$
$$\left[\frac{d_1t_1}{Q_p}+\frac{a_1Q_p}{2}\left(1-\frac{r_4}{p_1}\right)+\frac{2d_2t_2}{Q_p}+\frac{2a_2Q_p}{2}\left(1-\frac{r_3}{p_2}\right)\right.$$
$$\left.\frac{2d_3t_3}{Q_p}+\frac{2a_3Q_p}{2}\left(1-\frac{r_2}{p_3}\right)+\frac{d_4t_4}{Q_p}+\frac{a_4Q_p}{2}\left(1-\frac{r_1}{p_4}\right)\right]. \tag{5}$$

For Equation (5), he found $\frac{\partial P\left(\tilde{C}_1\right)}{\partial Q_p}$ and then solved

$\frac{\partial P(\tilde{C}_1)}{\partial Q_p}=0$ to derive the optimal production quantity

$$Q_p^*=\sqrt{\frac{2(d_1t_1+2d_2t_2+2d_3t_3+d_4t_4)}{a_1\left(1-\frac{r_4}{p_1}\right)+2a_2\left(1-\frac{r_3}{p_2}\right)+2a_3\left(1-\frac{r_2}{p_3}\right)+a_4\left(1-\frac{r_1}{p_4}\right)}}. \quad (6)$$

For his second extension, Hsieh (2002) considered the fuzzy inventory model with fuzzy production quantity, with a nonnegative trapezoidal fuzzy number, $\tilde{Q}_p=(q_{p_1},q_{p_2},q_{p_3},q_{p_4})$ under the condition $0<q_{p_1}\le q_{p_2}\le q_{p_3}\le q_{p_4}$. According to the Function Principle (Chen, 1985, 1985a), he obtained the trapezoidal fuzzy number for average cost $\tilde{C}_2$

$$\tilde{C}_2=\left[\frac{d_1t_1}{q_{p_4}}+\frac{a_1q_{p_1}}{2}\left(1-\frac{r_4}{p_1}\right),\frac{d_2t_2}{q_{p_3}}+\frac{a_2q_{p_2}}{2}\left(1-\frac{r_3}{p_2}\right)\right.$$

$$\left.\frac{d_3t_3}{q_{p_2}}+\frac{a_3q_{p_3}}{2}\left(1-\frac{r_2}{p_3}\right),\frac{d_4t_4}{q_{p_1}}+\frac{a_4q_{p_4}}{2}\left(1-\frac{r_1}{p_4}\right)\right]. \quad (7)$$

He applied the Graded Mean Integration Representation method (Chen, 1999) to defuzzify Equation (7) as follows

$$P(\tilde{C}_2)=\frac{1}{6}$$

$$\left[\frac{d_1t_1}{q_{p_4}}+\frac{a_1q_{p_1}}{2}\left(1-\frac{r_4}{p_1}\right)+\frac{2d_2t_2}{q_{p_3}}+\frac{2a_2q_{p_2}}{2}\left(1-\frac{r_3}{p_2}\right)\right.$$

$$\left.\frac{2d_3t_3}{q_{p_2}}+\frac{2a_3q_{p_3}}{2}\left(1-\frac{r_2}{p_3}\right)+\frac{d_4t_4}{q_{p_1}}+\frac{a_4q_{p_4}}{2}\left(1-\frac{r_1}{p_4}\right)\right], \quad (8)$$

under the constraint

$$0<q_{p_1}\le q_{p_2}\le q_{p_3}\le q_{p_4}. \quad (9)$$

Next, Hsieh (2002) decided to apply Taha's algorithm of the extended Lagrangean method (Taha, 1997) to solve fuzzy production quantity with inequality constraints. Now, we recall Taha's (1997) algorithm iteratively to convert inequality constraints into equality constraints. We also recall that the Lagrangean method was applied to solve the optimum solution of nonlinear programming problems with equality constraints. The extended Lagrangean method was used to find the optimum solution under inequality constraints.
Suppose that the problem is given by

Minimize $y=f(x)$

Subject to: $g_i(x)\ge 0$, $i=1,2,...,m$. (10)

The negative constraints $x\ge 0$, if any, are included in the m constraints. Then we briefly introduce the extended Lagrangean method of Taha (1997) that involves the following steps.
Step 1. Solve the unconstrained problem

Minimize $y=f(x)$. (11)

If the resulting optimum satisfies all the constraints, then we have derived the optimal solution. Otherwise, set $k=1$ and go to Step 2.
Step 2. Activate any k constraints (i.e. convert them into equality) and optimize $f(x)$ subject to the k active constraints by the Lagrangean method. If the resulting solution is feasible with respect to the remaining constraints, then it is a local optimum. Otherwise, activate another set of k constraints and repeat the step. If all sets of taking k active constraints are considered without encountering a feasible solution, then we will go to Step 3.
Step 3. If $k=m$, stop; no feasible solution exists. Otherwise, set $k=k+1$ and go to Step 2.
In the following, we briefly demonstrate that according to the extended Lagrangean method Hsieh (2002) solved the optimal problem of Equation (8), under constraint (9).

First, Hsieh (2002) solved $\frac{\partial P(\tilde{C}_2)}{\partial q_{p_1}}=0$ to imply that

$$q_{p_1}=\sqrt{\frac{2d_4t_4}{a_1(1-(r_4/p_1))}}. \quad (12)$$

Similarly, by considering that $\frac{\partial P(\tilde{C}_2)}{\partial q_{p_2}}=0$, $\frac{\partial P(\tilde{C}_2)}{\partial q_{p_3}}=0$, and $\frac{\partial P(\tilde{C}_2)}{\partial q_{p_4}}=0$ to derive that

$q_{p_2}=\sqrt{\frac{2d_3t_3}{a_2(1-(r_3/p_2))}}$, $q_{p_3}=\sqrt{\frac{2d_2t_2}{a_3(1-(r_2/p_3))}}$, and

$$q_{p_4}=\sqrt{\frac{2d_1t_1}{a_4(1-(r_1/p_4))}}, \quad (13)$$

respectively. However, Equations (12) and (13) yield that

$$q_{p_1} \ge q_{p_2} \ge q_{p_3} \ge q_{p_4} \quad (14)$$

In general, the results in Equation (14) will violate the constraint of Equation (9) unless

$d_1 = d_2 = d_3 = d_4$, $t_1 = t_2 = t_3 = t_4$, $a_1 = a_2 = a_3 = a_4$, $r_1 = r_2 = r_3 = r_4$,

and $p_1 = p_2 = p_3 = p_4$. (15)

It means that all trapezoidal fuzzy numbers are degenerated to crisp numbers. Hence, under the fuzzy environment, the results of Equations (12) and (13) cannot be accepted. Therefore, according to Step 2 of the extended Lagrangean method of Taha (1997), Hsieh (2002) considered changing one inequality, for example, from $q_{p_2} - q_{p_1} \ge 0$, into an equality as $q_{p_2} - q_{p_1} = 0$ so that he could solve the Lagrangean function

$$L\left(q_{p_1}, q_{p_2}, q_{p_3}, q_{p_4}, \lambda\right) = P\left(\tilde{C}_2\right) - \lambda\left(q_{p_2} - q_{p_1}\right) \quad (16)$$

to imply that

$$q_{p_1} = q_{p_2} = \sqrt{\frac{2\left(2d_3t_3 + d_4t_4\right)}{a_1\left(1-\left(r_4/p_1\right)\right) + 2a_2\left(1-\left(r_3/p_2\right)\right)}}, \quad (17)$$

$$q_{p_3} = \sqrt{\frac{2d_2t_2}{a_3\left(1-\left(r_2/p_3\right)\right)}}, \text{and } q_{p_4} = \sqrt{\frac{2d_1t_1}{a_4\left(1-\left(r_1/p_4\right)\right)}}. \quad (18)$$

However, Equations (17) and (18) yield that

$$q_{p_1} = q_{p_2} \ge q_{p_3} \ge q_{p_4} \quad (19)$$

In general, the results in Equation (19) still violate the constraint of Equation (9) unless Equation (15) is satisfied. It means that all trapezoidal fuzzy numbers degenerate to crisp numbers so that the meaning of fuzzy production inventory model is lost. There are two other ways to change an inequality into equality. If we consider $q_{p_2} = q_{p_3}$, then it will imply that $q_{p_1} \ge q_{p_2} = q_{p_3} \ge q_{p_4}$. On the other hand, if we consider $q_{p_3} = q_{p_4}$, then it will yield $q_{p_1} \ge q_{p_2} \ge q_{p_3} = q_{p_4}$. Those results will not satisfy Equation (9) under the fuzzy environment.

Following Taha's approach (1997) of the extended Lagrangean method, Hsieh (2002) considered changing two inequalities, fo r example, $q_{p_3} - q_{p_2} \ge 0$ and $q_{p_2} - q_{p_1} \ge 0$, into equalities as $q_{p_3} - q_{p_2} = 0$ and $q_{p_2} - q_{p_1} = 0$ so that he could try to solve the following Lagrangean function

$$L\left(q_{p_1}, q_{p_2}, q_{p_3}, q_{p_4}, \lambda_1, \lambda_2\right) = P\left(\tilde{C}_2\right) - \lambda_1\left(q_{p_2} - q_{p_1}\right) - \lambda_2\left(q_{p_3} - q_{p_2}\right). \quad (20)$$

to derive that

$$q_{p_1} = q_{p_2} = q_{p_3} = \sqrt{\frac{2\left(2d_2t_2 + 2d_3t_3 + d_4t_4\right)}{a_1\left(1-\left(r_4/p_1\right)\right) + 2a_2\left(1-\left(r_3/p_2\right)\right) + 2a_3\left(1-\left(r_2/p_3\right)\right)}}, \quad (21)$$

and

$$q_{p_4} = \sqrt{\frac{2d_1t_1}{a_4\left(1-\left(r_1/p_4\right)\right)}}. \quad (22)$$

However, from Equations (21) and (22), it yields that

$$q_{p_1} = q_{p_2} = q_{p_3} \ge q_{p_4}. \quad (23)$$

In general, the results in Equation (23) will not satisfy the constraint of the Equation (9) unless Equation (15) is satisfied.

There are two other choices to change two inequalities into equalities. If we consider $q_{p_2} = q_{p_3} = q_{p_4}$, then it will imply that $q_{p_1} \ge q_{p_2} = q_{p_3} = q_{p_4}$. On the other hand, if we consider $q_{p_1} = q_{p_2}$ and $q_{p_3} = q_{p_4}$, then it will yield $q_{p_1} = q_{p_2} \ge q_{p_3} = q_{p_4}$. Those results will not satisfy Equation (9) under the fuzzy environment. Therefore, Hsieh (2002) considered changing these three inequalities, for example, $q_{p_4} - q_{p_3} \ge 0$, $q_{p_3} - q_{p_2} \ge 0$ and $q_{p_2} - q_{p_1} \ge 0$, into equalities as $q_{p_4} - q_{p_3} = 0$, $q_{p_3} - q_{p_2} = 0$ and $q_{p_2} - q_{p_1} = 0$ so that he considered

$$L\left(q_{p_1}, q_{p_2}, q_{p_3}, q_{p_4}, \lambda_1, \lambda_2, \lambda_3\right) = P\left(\tilde{C}_2\right) - \lambda_1\left(q_{p_2} - q_{p_1}\right) - \lambda_2\left(q_{p_3} - q_{p_2}\right) - \lambda_3\left(q_{p_4} - q_{p_3}\right), \quad (24)$$

to imply that;

$$q_{p_1} = q_{p_2} = q_{p_3} = q_{p_4} = \sqrt{\frac{2\left(d_1t_1 + 2d_2t_2 + 2d_3t_3 + d_4t_4\right)}{a_1\left(1 - \frac{r_4}{p_1}\right) + 2a_2\left(1 - \frac{r_3}{p_2}\right) + 2a_3\left(1 - \frac{r_2}{p_3}\right) + a_4\left(1 - \frac{r_1}{p_4}\right)}}. \quad (25)$$

After iterative replacement from inequalities to equalities, Hsieh (2002) finally derived Equation (25).

In the next section, we will provide our proposed approach to directly prove the result of Equation (25). It will demonstrate that Hsieh (2002) applied Taha's approach (1997), which is a lengthy and tedious procedure.

OUR IMPROVEMENT FOR HSIEH'S APPROACH

We will show that to apply Taha's method (1997) is unnecessary for solving Equation (8) under the constraint (9). We rewrote Equation (8) as follows

$$P\left(\tilde{C}_2\right)=\frac{1}{6}\left(f_1\left(q_{p_1}\right)+f_2\left(q_{p_2}\right)+f_3\left(q_{p_3}\right)+f_4\left(q_{p_4}\right)\right),\quad(26)$$

where $f_1\left(q_{p_1}\right)=\dfrac{\alpha_1}{q_{p_1}}+\beta_1 q_{p_1}$,

$f_2\left(q_{p_2}\right)=2\left(\dfrac{\alpha_2}{q_{p_2}}+\beta_2 q_{p_2}\right)$, $f_3\left(q_{p_3}\right)=2\left(\dfrac{\alpha_3}{q_{p_3}}+\beta_3 q_{p_3}\right)$,

and

$f_4\left(q_{p_4}\right)=\dfrac{\alpha_4}{q_{p_4}}+\beta_4 q_{p_4}$, with $\alpha_1=d_4t_4$, $\alpha_2=d_3t_3$, $\alpha_3=d_2t_2$, $\alpha_4=d_1t_1$, $\beta_1=\dfrac{a_1}{2}\left(1-\dfrac{r_4}{p_1}\right)$, $\beta_2=\dfrac{a_2}{2}\left(1-\dfrac{r_3}{p_2}\right)$,

$\beta_3=\dfrac{a_3}{2}\left(1-\dfrac{r_2}{p_3}\right)$, and $\beta_4=\dfrac{a_4}{2}\left(1-\dfrac{r_1}{p_4}\right)$ such that the solution satisfies the constraint of Equation (9) .

Those $f_i(x)$ are the same type as

$$f(x)=\frac{\alpha}{x}+\beta x \quad (27)$$

that is a convex function so that the minimum point occurs at $x=\sqrt{\alpha/\beta}$, and this fact has already been discovered by Hsieh (2002). However, Hsieh (2002) did not use the full power of convex functions.

In the following, we will further examine the convex property of $f_i(x)$. We assume that the solutions of Equation (25) are $q_1^{\#}$, $q_2^{\#}$, $q_3^{\#}$ and $q_4^{\#}$ for functions $f_1(q_1)$, $f_2(q_2)$, $f_3(q_3)$ and $f_4(q_4)$, respectively, under the condition $q_1^{\#}\le q_2^{\#}\le q_3^{\#}\le q_4^{\#}$. Our goal is to verify that $q_1^{\#}=q_2^{\#}=q_3^{\#}=q_4^{\#}$. Before we prove the main result, we need the following two lemmas. Lemma 1 will provide a pair of lower bound and upper bound.

Lemma 1

$q_i^{\#}\le\sqrt{\alpha_1/\beta_1}$ and $\sqrt{\alpha_4/\beta_4}\le q_i^{\#}$, for $i=1,2,3,4.$

Proof of Lemma 1

We know for a fact that without constraint, $f_i(x)$ has a minimum at $\sqrt{\alpha_i/\beta_i}$ for $i=1,2,3,4,$ where $\sqrt{\alpha_4/\beta_4}\le\sqrt{\alpha_3/\beta_3}\le\sqrt{\alpha_2/\beta_2}\le\sqrt{\alpha_1/\beta_1}$, owing to $\alpha_4\le\alpha_3\le\alpha_2\le\alpha_1$ and $\beta_1\le\beta_2\le\beta_3\le\beta_4$.

If $\sqrt{\alpha_1/\beta_1}<q_1^{\#}$, then we evaluate $f_i(x)$ for $i=1,2,3,4,$ at $\sqrt{\alpha_1/\beta_1}$.

According to $q_1^{\#}\le q_2^{\#}\le q_3^{\#}\le q_4^{\#}$, it yields that:

$f_1\left(\sqrt{\alpha_1/\beta_1}\right)<f_1\left(q_1^{\#}\right)$, and for $i=2,3,4$

$f_i\left(\sqrt{\alpha_i/\beta_i}\right)\le f_i\left(\sqrt{\alpha_1/\beta_1}\right)<f_i\left(q_i^{\#}\right)$ then

$$f_1\left(\sqrt{\frac{\alpha_1}{\beta_1}}\right)+f_2\left(\sqrt{\frac{\alpha_1}{\beta_1}}\right)+f_3\left(\sqrt{\frac{\alpha_1}{\beta_1}}\right)+f_4\left(\sqrt{\frac{\alpha_1}{\beta_1}}\right)<\sum_{i=1}^{4}f_i\left(q_i^{\#}\right),\quad(28)$$

with $\sqrt{\dfrac{\alpha_1}{\beta_1}}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}$ satisfies Equation (9), so that $q_1^{\#}$, $q_2^{\#}$, $q_3^{\#}$ and $q_4^{\#}$ are not minimum solutions of Equation (8). It is a contradictive situation.

Hence, we have obtain $q_1^{\#}\le\sqrt{\alpha_1/\beta_1}$. After $q_1^{\#}\le\sqrt{\alpha_1/\beta_1}$, we begin to prove that $q_2^{\#}\le\sqrt{\alpha_1/\beta_1}$. If $\sqrt{\alpha_1/\beta_1}<q_2^{\#}$, then we know that:

$$f_1\left(q_1^{\#}\right)+\sum_{i=2}^{4}f_i\left(\sqrt{\alpha_1/\beta_1}\right)<\sum_{i=1}^{4}f_i\left(q_i^{\#}\right)\quad(29)$$

where:

$q_1^{\#}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}\le\sqrt{\dfrac{\alpha_1}{\beta_1}}$ satisfies Equation (9), to derive a contradiction, then it yields that $q_2^{\#}\le\sqrt{\alpha_1/\beta_1}$.

By the same argument, we can finish the rest o f the assertion of Lemma 1.

Lemma 2. $q_1^{\#}=q_2^{\#}$ and $q_3^{\#}=q_4^{\#}$.

Proof of Lemma 2. If $q_1^{\#}<q_2^{\#}$, since $q_1^{\#}<q_2^{\#}\le\sqrt{\alpha_1/\beta_1}$, it follows that

$$f_1\left(q_2^{\#}\right)+\sum_{i=2}^{4}f_i\left(q_i^{\#}\right)<\sum_{i=1}^{4}f_i\left(q_i^{\#}\right), \tag{30}$$

where $q_2^{\#}=q_2^{\#}\le q_3^{\#}\le q_4^{\#}$ satisfies Equation (9), to obtain a contradiction to imply that $q_1^{\#}=q_2^{\#}$. Similarly, we hold that $q_3^{\#}=q_4^{\#}$.

Theorem 1

$q_1^{\#}=q_2^{\#}=q_3^{\#}=q_4^{\#}$.

Proof of Theorem 1

We assumed that $q_1^{\#}=q_2^{\#}<q_3^{\#}=q_4^{\#}$, and so the problem was divided into two cases: (a) $\sqrt{\alpha_4/\beta_4}\le q_1^{\#}=q_2^{\#}\le\sqrt{\alpha_2/\beta_2}$ and (b) $\sqrt{\alpha_2/\beta_2}\le q_1^{\#}=q_2^{\#}\le\sqrt{\alpha_1/\beta_1}$. We further divide Case (a) into (a1) $q_3^{\#}=q_4^{\#}\le\sqrt{\alpha_2/\beta_2}$, and (a2) $\sqrt{\alpha_2/\beta_2}<q_3^{\#}=q_4^{\#}$.

For Case (a1), it yields that $\sum_{i=1}^{4}f_i\left(q_3^{\#}\right)$ has a smaller value, and for case (a2), $\sum_{i=1}^{2}f_i\left(q_1^{\#}\right)+\sum_{i=3}^{4}f_i\left(\sqrt{\alpha_2/\beta_2}\right)$ has a smaller value to imply that case (a) does not hold.

For case (b), it follows that $\sum_{i=1}^{4}f_i\left(q_1^{\#}\right)$ has a smaller value to imply that case (b) does not hold. Hence, both cases (a) and (b) do not hold.

Based on Theorem 1, we have showed that the constraint optimization problem $P\left(\tilde{C}_2\right)$ can be reduced to the unconstraint optimization problem, $P\left(\tilde{C}_1\right)$ such that it has the expression of Equation (27) so the results of Equation (25) can be obtained. By our proposed approach, we have provided an alternative method for the solution procedure.

NUMERICAL EXAMPLE

We recall another arithmetic defuzzification for trapezoidal fuzzy numbers, $\tilde{B}=\left(b_1,b_2,b_3,b_4\right)$, with the median rule (1987) then the result, say $m\left(\tilde{B}\right)$, is expressed as

$$m\left(\tilde{B}\right)=\frac{b_1+b_2+b_3+b_4}{4}. \tag{31}$$

The Graded Mean Integration Representation method (Chen, 1999) and the median rule (Park, 1987) are two popular arithmetic defuzzifications to defuzzify trapezoidal fuzzy numbers into crisp numbers.

Before we developed our generalized discussion, we tried to provide some numerical examples proposed by Hsieh (2002) that would explain the problem that we want to discuss in the following sections. In Hsieh's numerical example, Brown Manufacturing produces commercial refrigeration units in batches. The firm's estimated demand for the year is greater or less than 10,000 units. The setup cost is about \$100, and the inventory cost is about \$0.5 per unit per year. Once the production process has been set up, greater or less than 80 refrigeration units can be manufactured daily. The demand during the production period is about 60 units each day. Hence, the trapezoidal fuzzy number of yearly demand, $\tilde{D}=\left(d_1,d_2,d_3,d_4\right)=(9000,9500,10,500,11,000)$, the fuzzy setup cost, $\tilde{T}=\left(t_1,t_2,t_3,t_4\right)=(95,100,100,105)$, the fuzzy inventory cost, $\tilde{a}=\left(a_1,a_2,a_3,a_4\right)=(0.475,0.5,0.5,0.525)$, fuzzy production rate, $\tilde{P}=\left(p_1,p_2,p_3,p_4\right)=(72,76,84,88)$, and fuzzy production quantity $\tilde{Q}_p=\left(q_{p_1},q_{p_2},q_{p_3},q_{p_4}\right)$ with $0<q_{p_1}\le q_{p_2}\le q_{p_3}\le q_{p_4}$. Hsieh (2002) used the Graded Mean Integration Representation method (Chen, 1999) to consider the minimization of $P\left(\tilde{C}_2\right)$ then he derived that:

$$\tilde{Q}_p^{*}=\left(4028.77,4028.77,4028.77,4028.77\right), \tag{32}$$

and

$$\tilde{C}_2^{*}=\left(331.8277,447.8445,548.3947,659.2347\right). \tag{33}$$

After we find the trapezoidal fuzzy number for average cost in Equation (33), the next step is to convert this fuzzy number into a crisp number so that the derived crisp number will provide the estimation for the decision maker regarding how much of the budget should be used to support this production inventory model.

Based on Equation (33), if we use the Graded Mean Integration Representation method (Chen, 1999), it yields:

$$P\left(\tilde{C}_2^*\left(\tilde{Q}_p^*\right)\right)=497.2568. \quad (34)$$

On the other hand, if we apply the median rule (1987), then it implies that

$$m\left(\tilde{C}_2^*\left(\tilde{Q}_p^*\right)\right)=496.8254. \quad (35)$$

From Equations (34) and (35), we may observe that two different arithmetic defuzzifications will derive very close results. It may indicate that researchers can arbitrarily select any arithmetic defuzzification to estimate the average cost. However, in the following, we will provide a detailed examination to consider the best arithmetic defuzzification to select.

OUR GENERALIZATION TO A WEIGHTED AVERAGE OPERATION

We will generalize the Graded Mean Integration Representation method (Chen, 1999) to a weighted arithmetic method. For example, the Graded Mean Integration Representation method (Chen, 1999) is a special case with weight $(1/6, 2/6, 2/6, 1/6)$.

In fact, there are two defuzzifications in the solution's procedure for fuzzy production inventory models: (a) before the fuzzy average cost is obtained, for example, from Equation (7) to Equation (8), and (b) after the fuzzy average cost is obtained, for example, from Equation (33) to $m\left(\tilde{C}_2^*\left(\tilde{Q}_p^*\right)\right)=496.8247$. Hence, we need two arithmetic defuzzifications, respectively.

Assuming there are two arithmetic defuzzifications with two weights $(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)$ and (w_1,w_2,w_3,w_4) with $\sum_{i=1}^{4}\upsilon_i=1$, and $\upsilon_i\ge 0$ for $=1,2,3,4$, and $\sum_{i=1}^{4}w_i=1$, and $w_i\ge 0$ for $=1,2,3,4$ such that

$$\Lambda\left(\tilde{K}\right)=\upsilon_1k_1+\upsilon_2k_2+\upsilon_3k_3+\upsilon_4k_4, \quad (36)$$

and

$$W\left(\tilde{K}\right)=w_1k_1+w_2k_2+w_3k_3+w_4k_4, \quad (37)$$

with $\tilde{K}=(k_1,k_2,k_3,k_4)$ is a fuzzy trapezoid number. $\Lambda\left(\tilde{K}\right)$ and $W\left(\tilde{K}\right)$ are both used to transfer a fuzzy trapezoid cost into a crisp cost. $\Lambda\left(\tilde{K}\right)$ is used before finding the fuzzy average cost under condition of Equation (9), and $W\left(\tilde{K}\right)$ is applied after the fuzzy average cost is found.

Therefore, we use $\Lambda\left(\tilde{K}\right)$ to transfer the fuzzy objective function of Equation (7) into a classical analytical problem:

$$\Lambda\left(\tilde{C}_2\right)=\upsilon_1\left[\frac{d_1t_1}{q_{p_4}}+\frac{a_1q_{p_1}}{2}\left(1-\frac{r_4}{p_1}\right)\right]+\upsilon_2\left[\frac{d_2t_2}{q_{p_3}}+\frac{a_2q_{p_2}}{2}\left(1-\frac{r_3}{p_2}\right)\right]+\upsilon_3\left[\frac{d_3t_3}{q_{p_2}}+\frac{a_3q_{p_3}}{2}\left(1-\frac{r_2}{p_3}\right)\right]+\upsilon_4\left[\frac{d_4t_4}{q_{p_1}}+\frac{a_4q_{p_4}}{2}\left(1-\frac{r_1}{p_4}\right)\right]. \quad (38)$$

Similarly, based on the convexity property of the objective functions, we still obtain that the optimal solution for $\Lambda\left(\tilde{C}_2\right)$ which will occur at

$$q_{p_1}=q_{p_2}=q_{p_3}=q_{p_4} \quad (39)$$

such that

$$q_{p_1}=q_{p_2}=q_{p_3}=q_{p_4}=\sqrt{\frac{2(\upsilon_1d_1t_1+\upsilon_2d_2t_2+\upsilon_3d_3t_3+\upsilon_4d_4t_4)}{\upsilon_1a_1\left(1-\frac{r_4}{p_1}\right)+\upsilon_2a_2\left(1-\frac{r_3}{p_2}\right)+\upsilon_3a_3\left(1-\frac{r_2}{p_3}\right)+\upsilon_4a_4\left(1-\frac{r_1}{p_4}\right)}}. \quad (40)$$

To simplify the expression, we assumed that the optimal fuzzy average cost is $\tilde{C}_2^*(q^*)=\tilde{V}=(V_1,V_2,V_3,V_4)$, where $V_j=\frac{d_jt_j}{q^*}+\frac{a_jq^*}{2}\left(1-\frac{r_{5-j}}{p_j}\right)$ for $j=1,..,4$, and $q^*=q_{p_1}$ in Equation (40). We then applied $W\left(\tilde{V}\right)$ to transfer the optimal fuzzy average cost into a crisp cost,

$$W\left(\tilde{V}\right)=\sum_{i=1}^{4}w_iV_i. \quad (41)$$

Now, we will prepare an example for our abstract approach. When we consider the Graded Mean Integration Representation method (Chen, 1999), the solution is under the condition

$$(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)=\left(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6}\right), \quad (42)$$

for Equation (34) with the Graded Mean Integration Representation method (Chen, 1999),

$$\left(w_1, w_2, w_3, w_4\right)=\left(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6}\right), \tag{43}$$

and for Equation (35) with the median rule (Park, 1987),

$$\left(w_1, w_2, w_3, w_4\right)=\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right). \tag{44}$$

We point out that our previous approach implicitly raised the following minimization problem: given $\left(w_1, w_2, w_3, w_4\right)$, how is the best $\left(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4\right)$ selected. It was overlooked by previous researchers and so no one has ever analyzed the relation between two arithmetic defuzzifications. It means that the decision maker first decides how to defuzzy the final optimal fuzzy average cost into a crisp value, but then how does one select an arithmetic defuzzification for the original fuzzy inventory model to obtain the minimum final result ?

We express our problem in the following. For a predetermined weighting vector, say $\left(w_1, w_2, w_3, w_4\right)$ to defuzzify $\tilde{C}_2^*\left(q^*\right)=\tilde{V}$ where q^* is the feasible solution derived by defuzzifying $\tilde{C}_2^*(q)$ under the arithmetical operation of $\Lambda\left(\tilde{C}_2^*(q)\right)$ with the condition $\Lambda\left(k_1,k_2,k_3,k_4\right)=\sum_{i=1}^{4}\upsilon_i k_i$, then how is $\left(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4\right)$ selected as to minimize

$$\min_{(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)} W\left(\tilde{V}\right). \tag{45}$$

To solve the minimum of $\Lambda\left(\tilde{C}_2\right)$, similar to Equation (26), we derive:

$$q_{p_1}=q_{p_2}=q_{p_3}=q_{p_4}=\sqrt{\frac{\upsilon_1\alpha_4+\upsilon_2\alpha_3+\upsilon_3\alpha_2+\upsilon_4\alpha_1}{\upsilon_1\beta_1+\upsilon_2\beta_2+\upsilon_3\beta_3+\upsilon_4\beta_4}}, \tag{46}$$

with $\alpha_1=d_4t_4$, $\alpha_2=d_3t_3$, $\alpha_3=d_2t_2$, $\alpha_4=d_1t_1$,

$\beta_1=\frac{a_1}{2}\left(1-\frac{r_4}{p_1}\right)$, $\beta_2=\frac{a_2}{2}\left(1-\frac{r_3}{p_2}\right)$, $\beta_3=\frac{a_3}{2}\left(1-\frac{r_2}{p_3}\right)$,

and $\beta_4=\frac{a_4}{2}\left(1-\frac{r_1}{p_4}\right)$

.

On the other hand, we rewrite $W\left(\tilde{V}\right)$ as follows:

$$W\left(\tilde{V}\right)= w_1\left(\frac{\alpha_4}{q}+\beta_1 q\right)+w_2\left(\frac{\alpha_3}{q}+\beta_2 q\right)+w_3\left(\frac{\alpha_2}{q}+\beta_3 q\right)+w_4\left(\frac{\alpha_1}{q}+\beta_4 q\right) \tag{47}$$

and then it was further simplified as

$$W\left(\tilde{V}\right)=\frac{\alpha}{q}+\beta q, \tag{48}$$

with $\alpha=w_1\alpha_4+w_2\alpha_3+w_3\alpha_2+w_4\alpha_1$ and $\beta=w_1\beta_1+w_2\beta_2+w_3\beta_3+w_4\beta_4$. From the expression of Equation (48), it is apparent that the minimum will occur at

$$q=\sqrt{\frac{\alpha}{\beta}}=\sqrt{\frac{w_1\alpha_4+w_2\alpha_3+w_3\alpha_2+w_4\alpha_1}{w_1\beta_1+w_2\beta_2+w_3\beta_3+w_4\beta_4}}. \tag{49}$$

Hence, if we compare Equations (46) and (49), the minimum problem of Equation (45) will attain its minimum if

$$\sqrt{\frac{\upsilon_1\alpha_4+\upsilon_2\alpha_3+\upsilon_3\alpha_2+\upsilon_4\alpha_1}{\upsilon_1\beta_1+\upsilon_2\beta_2+\upsilon_3\beta_3+\upsilon_4\beta_4}}=\sqrt{\frac{w_1\alpha_4+w_2\alpha_3+w_3\alpha_2+w_4\alpha_1}{w_1\beta_1+w_2\beta_2+w_3\beta_3+w_4\beta_4}} \tag{50}$$

is satisfied. For a predetermined weight $\left(w_1, w_2, w_3, w_4\right)$, there may be many solutions for $\left(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4\right)$. For example, in a special case where $\alpha_1=\alpha_2=\alpha_3=\alpha_4$ and $\beta_1=\beta_2=\beta_3=\beta_4$, then

$$\sqrt{\frac{\upsilon_1\alpha_4+\upsilon_2\alpha_3+\upsilon_3\alpha_2+\upsilon_4\alpha_1}{\upsilon_1\beta_1+\upsilon_2\beta_2+\upsilon_3\beta_3+\upsilon_4\beta_4}}=$$

$$\sqrt{\frac{(\upsilon_1+\upsilon_2+\upsilon_3+\upsilon_4)\alpha_1}{(\upsilon_1+\upsilon_2+\upsilon_3+\upsilon_4)\beta_1}}=\sqrt{\frac{\alpha_1}{\beta_1}}, \qquad (51)$$

and

$$\sqrt{\frac{w_1\alpha_4+w_2\alpha_3+w_3\alpha_2+w_4\alpha_1}{w_1\beta_1+w_2\beta_2+w_3\beta_3+w_4\beta_4}}=$$
$$\sqrt{\frac{(w_1+w_2+w_3+w_4)\alpha_1}{(w_1+w_2+w_3+w_4)\beta_1}}=\sqrt{\frac{\alpha_1}{\beta_1}}. \qquad (52)$$

Therefore, any weighted vector will imply the same result. It points out that the solution of $(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)$ for Equation (50) is not unique. However, we may take a feasible solution

$$(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)=(w_1,w_2,w_3,w_4) \qquad (53)$$

so that Equation (50) is valid.

In the following, we will demonstrate our findings by an example. From the results of Equation (53), if we select $(w_1,w_2,w_3,w_4)=\left(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6}\right)$ in the beginning, then among all possible selections of $(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)$, the choice of

$$(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4)=\left(\frac{1}{6},\frac{2}{6},\frac{2}{6},\frac{1}{6}\right) \qquad (54)$$

will ensure that the minimum value for the final crisp estimation is attained.

We discovered that there is a consistent relationship between two arithmetic defuzzifications in the solution procedure for fuzzy production inventory models. It provides a reasonable managerial explanation that the two arithmetic defuzzifications, before and after, should be the same. Therefore, our findings may be the first step towards future studies of the selection of different weighted average operations.

Conclusion

We have studied the fuzzy production inventory to show that applying the convexity property can improve the lengthy solution procedure proposed by Hsieh (2002). We then generalize the Graded Mean Integration Representation method to a weighted average operation. We have pointed out that there are two arithmetic defuzzifications in the solution process and then discovered a consistent relation between the two operations to ensure the minimum of the final crisp estimation.

REFERENCES

Campos FA, Villar J, Jiménez M (2006). Robust solutions using fuzzy chance constraints. Eng. Optim., 38(6): 627-645.

Chen SH (1985). Operations on fuzzy numbers with function principle. Tamkang J. Manag. Sci., 6(1): 13–26.

Chen SH (1985a). Fuzzy linear combination of fuzzy linear functions under extension principle and second function principle. Tamsui Oxf. J. Manag. Sci., 1: 11–31.

Chen SH, Chang SH (2008). Optimization of fuzzy production inventory model with unrepairable defective products. Int. J. Prod. Econ., 113 : 887-894.

Chen SH, Hsieh CH (1999). Graded mean integration representation of generalized fuzzy number. J. Chin. Fuzzy Syst., 5 (2): 1–7.

Dutta P, Chakraborty D (2010). Incorporating one-way substitution policy into the newsboy problem with imprecise customer demand. Eur. J. Oper. Res., 200: 99-110.

Dutta P, Chakraborty D, Roy AR (2007). Continuous review inventory model in mixed fuzzy and stochastic environment. Appl. Math. Comput., 188 (1): 970-980.

Hsieh CH (2002). Optimization of fuzzy production inventory models. Inf. Sci., 146: 29–40.

Li Q, Zhang Q, Shen H (2006). Fuzzy inventory model with backorder under function principle. IEEE Int. Conference Serv. Oper. Logist.. Inf., SOLI '06, pp. 282-287.

Panda D, Kar S, Maity K, Maiti M (2008). A single period inventory model with imperfect production and stochastic demand under chance and imprecise constraints. Eur. J. Oper. Res., 188(1): 121-139.

Park KS (1987). Fuzzy set theoretic interpretation of economic order quantity. IEEE Trans. Syst. Man Cybern. SMC-17 (6): 1082–1084.

Taha HA (1997). Operations Research, Prentice-Hall. New Jersey, pp 753–777.

Vijayan T, Kumaran M (2009). Fuzzy economic order time models with random demand. Int. J. Approx. Reason, 50: 529-540.

Xie Y, Petrovic D, Burnham K (2006). A heuristic procedure for the two-level control of serial s supply chains under fuzzy customer demand. Int. J. Prod. Econ. 102 (1): 37-50.

Yao JS, Huang WT, Huang TT (2007). Fuzzy flexibility and product variety in lot-sizing. J. Inf. Sci. Eng. 23 (1): 49-70.

Zhu Y (2006). Solving fuzzy chance-constrained programming with ant colony optimization-based algorithms and application to fuzzy inventory model. Lect. Notes Artif. Intell. 4114: 977-983.