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Full Length Research Paper

A two-warehouse inventory model under trade credit in the supply chain management

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The traditional EOQ (Economic Ordering Quantity) inventory model has three basic assumptions (A), (B) and (C) to be summarized as follows: (A) The retailer must be paid for the items as soon as the items are received; (B) The replenishment rate is infinite; (C) The inventories are stored by a single warehouse with unlimited capacity. Few inventory models with generalizing assumptions (A), (B) and (C) together have been found in the literature. This paper tries to incorporate the above concepts to consider the inventory model with the trade credit, finite replenishment rate and limited storage capacity to relax assumptions (A), (B) and (C) simultaneously to establish a new economic production quantity model. The mathematical model and the solution procedure are developed and numerical examples are provided to illustrate them.

Key words: Economic ordering quantity, permissible delay in payments, trade credit, limited storage capacity, finite replenishment rate.

INTRODUCTION

The traditional EOQ (Economic Ordering Quantity) has three basic assumptions (A), (B) and (C) to be summarized as follows:

a) The retailer must be paid for the items as soon as the items are received;
b) The replenishment rate is infinite;
c) The inventories are stored by a single warehouse with unlimited capacity.

In practice, the above three assumptions are unrealistic. In general, the supplier will offer the retailer a trade credit period in paying for the amount of purchasing cost to promote their commodities. Secondly, the replenishment rate depends on the production rate is always not infinite. Finally, as we all know, the capacity of any warehouse is limited. In fact, there exist many practical cases that force inventory managers to hold more items than can be stored in their own warehouse. Chung and Huang (2006, 2007) relax assumptions (A) and (C) to consider an EOQ model when the delay in payment is permitted and the capacity of own warehouse is limited. Ker et al. (2001); Pakkala and Achary (1992) further generalize assumptions (B) and (C) to develop the two-warehouse model for deteriorating items with finite replenishment rate and shortages. Recently, Chung and Huang (2003), Huang (2007) extend assumptions (A) and (B) to discuss an economic production quantity (EPQ) model under finite replenishment rate and permissible delay in payments. However, few inventory models generalizing assumptions (A), (B) and (C) together have been found in the literature. To make the inventory system to be discussed closes to the real situation. Based on Chung and Huang (2003), this paper tries to present an inventory model to incorporate concepts of the trade credit, limited storage capacity and finite replenishment rate to relax assumptions (A), (B) and (C) simultaneously to establish a new

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economic production quantity model. Consequently, the inventory model in this paper is more close to a real world than the traditional EOQ model. Of course, this paper extends Chung and Huang (2003), Chung (1998) and Goyal (1985). Many related articles about the trade credit, finite replenishment rate and limited storage capacity can be found in Benkherouf (1997), Bhunia and Maiti (1998), Chu et al. (1998), Goswami and Chaudhuri (1992), Hariga (1998), Hartely (1976), Jamal et al. (2000), Liao (2000), Sarker et al. (2000), Sarma (1987), Yang (2004), Zhou and Yang (2005) and their references.

MODEL FORMULATION

Notation

D demand rate per year

P replenishment rate per year, $P > D$

A cost of placing one order

ρ $= 1 - \dfrac{D}{P} > 0$

s unit selling price per item,

c unit purchasing price per item, $s \geq c$

h unit stock-holding cost per item per-year in owned warehouse excluding interest charges

k unit stock-holding cost per item per year in rented warehouse, $k \geq h$

I_k interest charges per $ investment in inventory per year

I_e interest which can be earned per $ per year, $I_k \geq I_e$

M permissible delay period

T the cycle time

W storage capacity of owned warehouse

$TVC(T)$ the total relevant cost per unit time when $T > 0$

$tw1$ the point in time when the inventory level increases to W during the production period

$tw2$ the point in time when the inventory level decreases to W during the production cease period

$tw2 - tw1$ the time of rented warehouse

$$= \begin{cases} \dfrac{DT\rho - W}{P-D} + \dfrac{DT\rho - W}{D}, & \text{if } DT\rho > W \\ 0, & \text{if } DT\rho \leq W \end{cases}$$

T^* the optimal solution of $TVC(T)$

Assumptions

(1) Demand rate, D, is known and constant.
(2) Replenishment rate, P, is known and constant.
(3) Shortages are not allowed.
(4) Time period is infinite.
(5) $I_k \geq I_e,\ k \geq h,\ s \geq c.$
(6) During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. When $T \geq M$, the account is settled at M and we start paying for the interest charges on the items in stock. When $T \leq M$ the account is settled at M and we do not need to pay any interest charge.
(7) When the ordering quantity is more than the limited storage capacity of owned warehouse, the retailer will store the excess stock in a rented warehouse. When the demand produces, the stock of rented warehouse will be consumed first.

The annual total relevant cost

The annual total relevant cost consists of the following elements:

1. Annual ordering cost = $\dfrac{A}{T}$

2. Annual stock-holding cost (including owned warehouse and rented warehouse)

i) Two cases to occur in costs of rented warehouse

a) $DT\rho \leq W$, shown in Figures 1 or 2.

Annual stock-holding cost in rented warehouse $= 0$

b) $DT\rho > W$, shown in Figures 3 or 4.

Annual stock-holding cost in rented warehouse $= \dfrac{Pk(DT\rho - W)^2}{2DT(P-D)}$

ii) Two cases to occur in costs of owned warehouse

a) $DT\rho \leq W$, shown in Figures 1 or 2.

Annual stock-holding cost in owned warehouse $= \dfrac{DTh\rho}{2}$

b) $DT\rho > W$, shown in Figures 3 or 4.

Annual stock-holding cost in owned warehouse $= Wh - \dfrac{PW^2h}{2DT(P-D)}$

3. There are three cases to occur in costs of interest charges for the items kept in stock per year:

i) $M \leq \dfrac{PM}{D} \leq T$, shown in Figure 5.

Annual interest payable $= \dfrac{cI_k\rho\left(\dfrac{DT^2}{2} - \dfrac{PM^2}{2}\right)}{T}$

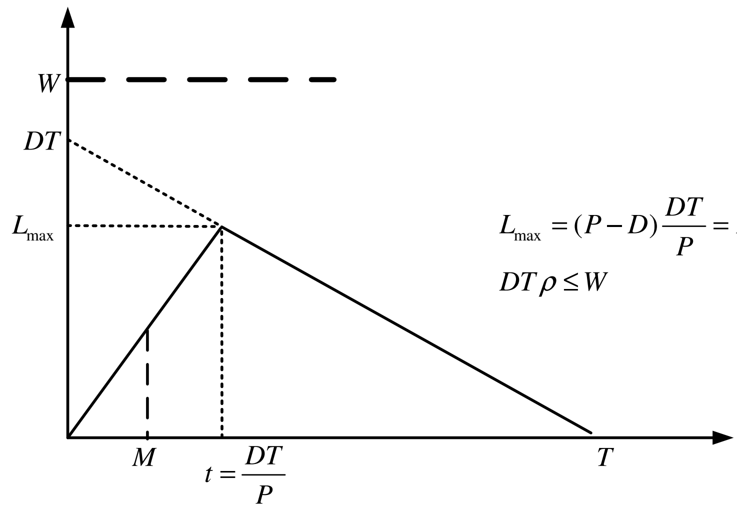

Figure 1. $DT\rho \le W$ and $M \le \frac{PM}{D} \le T$.

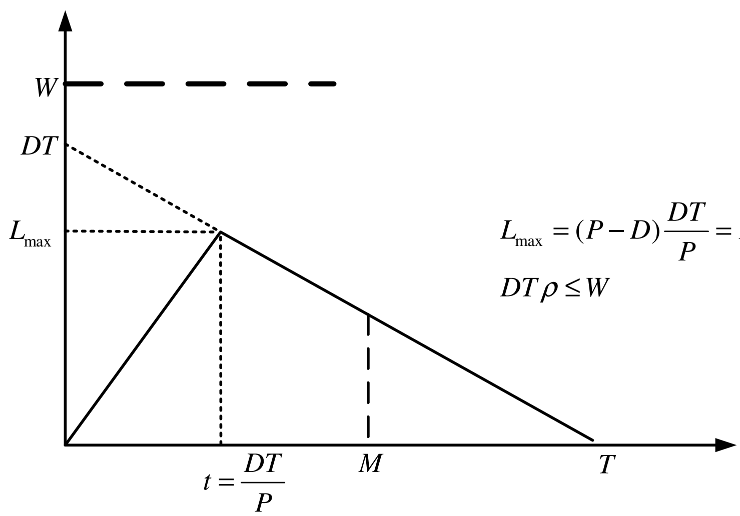

Figure 2. $DT\rho \le W$ and $M \le T \le \frac{PM}{D}$.

ii) $M \le T \le \frac{PM}{D}$, shown in Figure 6.

Annual interest payable $= \frac{cI_k\left(\frac{D(T-M)^2}{2}\right)}{T}$

iii) $T \le M$.
Annual interest payable $= 0$

4. There are three cases to occur in interest earned per year:

i) $M \le \frac{PM}{D} \le T$, shown in Figure 5.

Annual interest earned $= \frac{sI_e\left(\frac{DM^2}{2}\right)}{T}$

ii) $M \le T \le \frac{PM}{D}$, shown in Figure 6.

Annual interest earned $= \frac{sI_e\left(\frac{DM^2}{2}\right)}{T}$

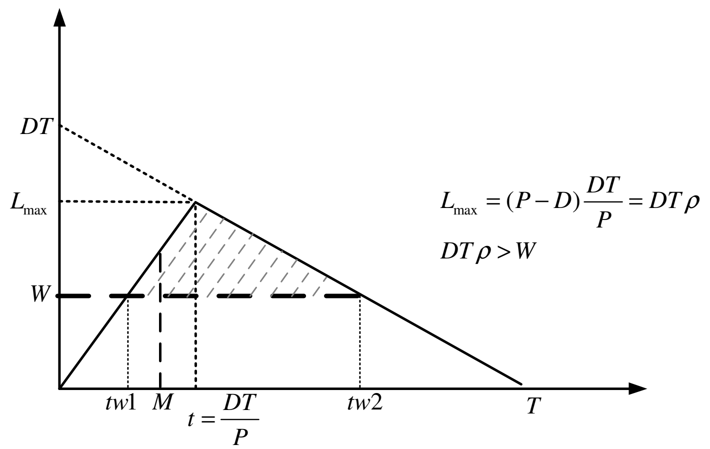

Figure 3. $DT\rho > W$ and $M \le \frac{PM}{D} \le T$.

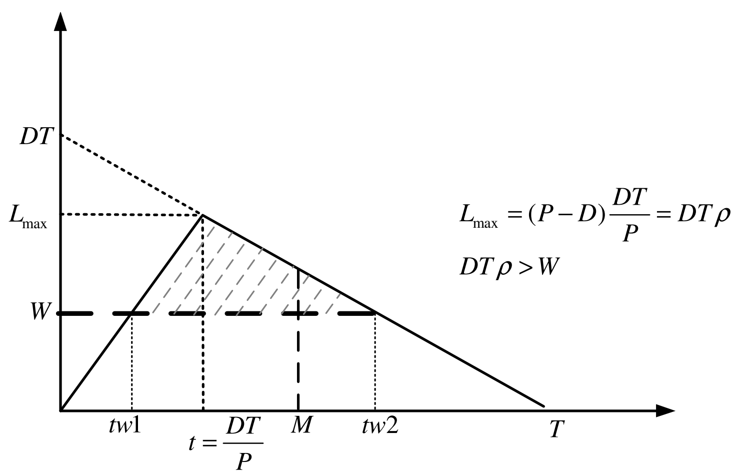

Figure 4. $DT\rho > W$ and $M \le T \le \frac{PM}{D}$.

iii) $T \le M$, shown in Figure 7.

Annual interest earned $= \dfrac{sI_e\left(\frac{DT^2}{2}+DT(M-T)\right)}{T}$

From the above arguments, the annual total relevant cost for the retailer can be expressed as $TVC(T) =$ ordering cost $+$ stock-holding cost $+$ interest payable $-$ interest earned. Three situations arise:

(1) $M \ge \frac{W}{D\rho}$,

(2) $\frac{W}{D\rho} > M$ and $\frac{PM}{D} \ge \frac{W}{D\rho}$ (That is equivalent to $\frac{PM}{D} \ge \frac{W}{D\rho} > M$),

(3) $\frac{W}{D\rho} > M$ and $\frac{W}{D\rho} \ge \frac{PM}{D}$ (That is equivalent to $\frac{W}{D\rho} \ge \frac{PM}{D} > M$)

Case I: $M \ge \frac{W}{D\rho}$

We show that the annual total relevant cost, $TVC(T)$, is given by

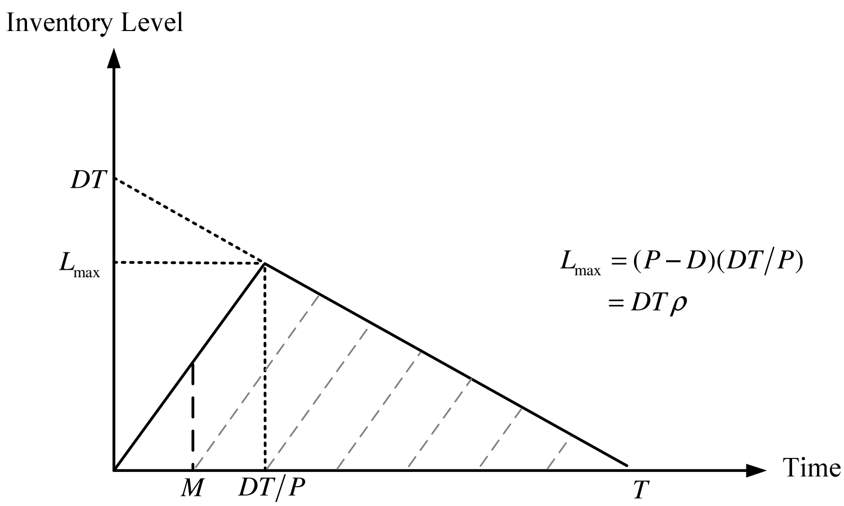

Figure 5. The total accumulation of interest payable when $PM/D \le T$.

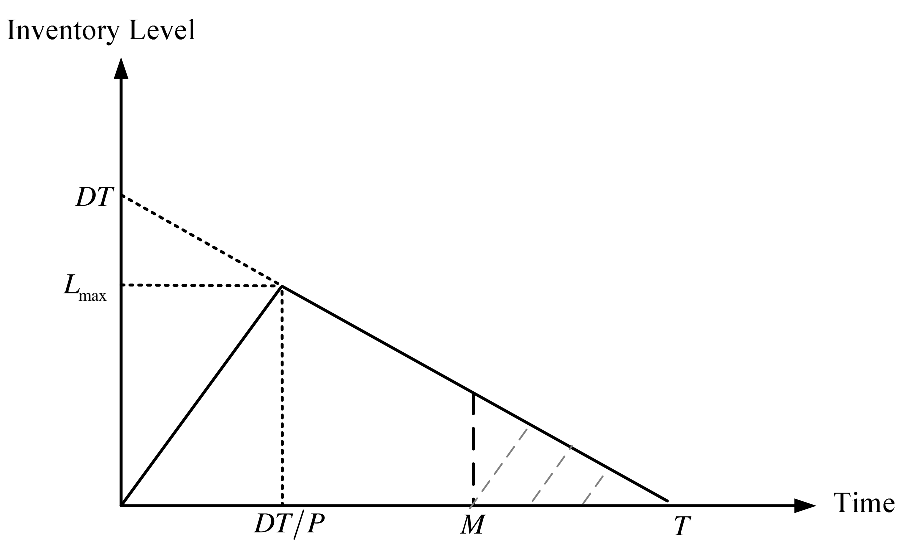

Figure 6. The total accumulation of interest payable when $M \le T \le PM/D$.

$$TVC(T)=\begin{cases} TVC_5(T) & \text{if} \quad 0<T\le \dfrac{W}{D\rho} & \text{(1a)} \\ TVC_6(T) & \text{if} \quad \dfrac{W}{D\rho}\le T\le M & \text{(1b)} \\ TVC_4(T) & \text{if} \quad M\le T\le \dfrac{PM}{D} & \text{(1c)} \\ TVC_2(T) & \text{if} \quad T\ge \dfrac{PM}{D} & \text{(1d)} \end{cases}$$

Where

$$TVC_5(T)=\frac{A}{T}+\frac{DTh\rho}{2}-\frac{sI_e\left(\dfrac{DT^2}{2}+DT(M-T)\right)}{T}, \tag{2}$$

$$TVC_6(T)=\frac{A}{T}+\frac{Pk(DT\rho-W)^2}{2DT(P-D)}+Wh-\frac{PW^2h}{2DT(P-D)}-\frac{sI_e\left(\dfrac{DT^2}{2}+DT(M-T)\right)}{T}, \tag{3}$$

$$TVC_4(T)=\frac{A}{T}+\frac{Pk(DT\rho-W)^2}{2DT(P-D)}+Wh-\frac{PW^2h}{2DT(P-D)}+\frac{cI_k\left(\dfrac{D(T-M)^2}{2}\right)}{T}-\frac{sI_e\left(\dfrac{DM^2}{2}\right)}{T}, \tag{4}$$

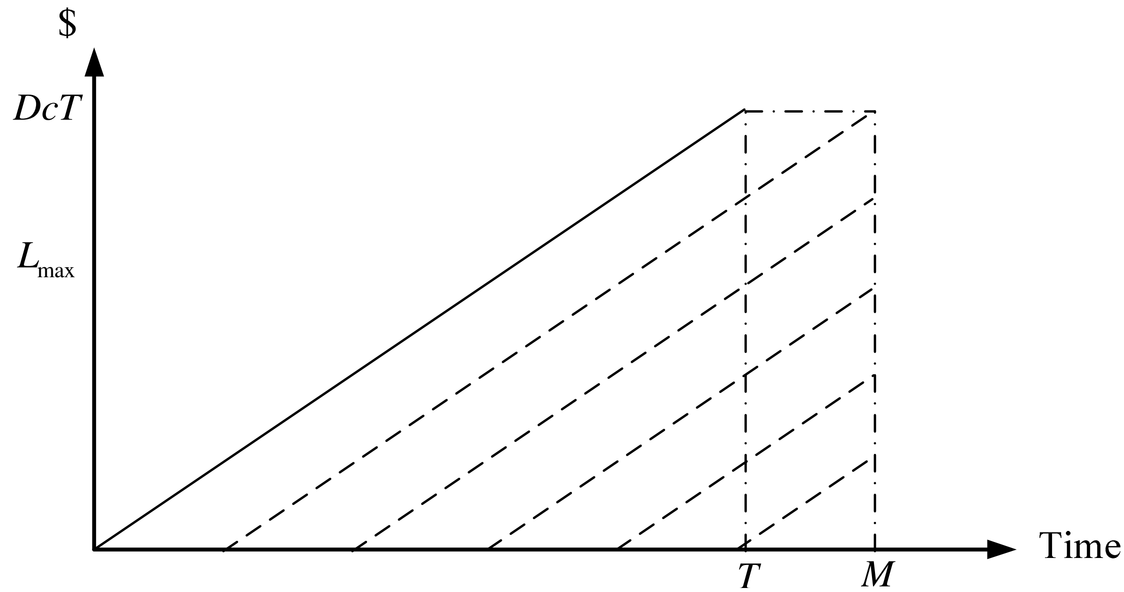

Figure 7. The total accumulation of interest payable when $T \le M$.

$$TVC_2(T) = \frac{A}{T} + \frac{Pk(DT\rho - W)^2}{2DT(P-D)} + Wh - \frac{PW^2h}{2DT(P-D)} + \frac{cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)}{T} - \frac{sI_e\left(\frac{DM^2}{2}\right)}{T}. \quad (5)$$

Case II: $\frac{PM}{D} \ge \frac{W}{D\rho} > M$

We show that the annual total relevant cost, $TVC(T)$, is given by

$$TVC(T) = \begin{cases} TVC_5(T) & \text{if} \quad 0 < T \le M & (6a) \\ TVC_3(T) & \text{if} \quad M \le T \le \frac{W}{D\rho} & (6b) \\ TVC_4(T) & \text{if} \quad \frac{W}{D\rho} \le T \le \frac{PM}{D} & (6c) \\ TVC_2(T) & \text{if} \quad T \ge \frac{PM}{D} & (6d) \end{cases}$$

Where

$$TVC_3(T) = \frac{A}{T} + \frac{DTh\rho}{2} + \frac{cI_k\left(\frac{D(T-M)^2}{2}\right)}{T} - \frac{sI_e\left(\frac{DM^2}{2}\right)}{T} \quad (7)$$

Case (III): $\frac{W}{D\rho} > \frac{PM}{D} > M$

We show that the annual total relevant cost, $TVC(T)$, is given by

$$TVC(T) = \begin{cases} TVC_5(T) & \text{if} \quad 0 < T \le M & (8a) \\ TVC_3(T) & \text{if} \quad M \le T \le \frac{PM}{D} & (8b) \\ TVC_1(T) & \text{if} \quad \frac{PM}{D} \le T \le \frac{W}{D\rho} & (8c) \\ TVC_2(T) & \text{if} \quad T \ge \frac{W}{D\rho} & (8d) \end{cases}$$

Where,

$$TVC_1(T) = \frac{A}{T} + \frac{DTh\rho}{2} + \frac{cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)}{T} - \frac{sI_e\left(\frac{DM^2}{2}\right)}{T} \quad (9)$$

Equations (2)-(5), (7) and (9) yield

$$TVC_1'(T) = -\frac{2A + DM^2(cI_k - sI_e) - PM^2cI_k}{2T^2} + D\rho\left(\frac{h + cI_k}{2}\right) \quad (10)$$

$$TVC_1''(T) = \frac{2A + DM^2(cI_k - sI_e) - PM^2cI_k}{T^3} \quad (11)$$

$$TVC_2'(T) = -\frac{\left(2A + DM^2(cI_k - sI_e) - PM^2cI_k\right) + \frac{W^2P(k-h)}{D(P-D)}}{2T^2} + D\rho\left(\frac{k + cI_k}{2}\right) \quad (12)$$

$$TVC_2''(T) = \frac{\left(2A + DM^2(cI_k - sI_e) - PM^2cI_k\right) + \frac{W^2P(k-h)}{D(P-D)}}{T^3} \quad (13)$$

$$TVC_3'(T) = -\frac{2A + DM^2(cI_k - sI_e)}{2T^2} + D\left(\frac{h\rho + cI_k}{2}\right) \quad (14)$$

$$TVC_3''(T) = \frac{2A + DM^2(cI_k - sI_e)}{T^3} \quad (15)$$

$$TVC_4'(T) = -\frac{2A + DM^2(cI_k - sI_e) + \frac{W^2P(k-h)}{D(P-D)}}{2T^2} + D\left(\frac{k\rho + cI_k}{2}\right) \quad (16)$$

$$TVC_4''(T) = \frac{2A + DM^2(cI_k - sI_e) + \frac{W^2P(k-h)}{D(P-D)}}{T^3} \quad (17)$$

$$TVC_5'(T) = -\frac{A}{T^2} + D\left(\frac{h\rho + sI_e}{2}\right) \quad (18)$$

$$TVC_5''(T) = \frac{2A}{T^3} > 0 \quad (19)$$

$$TVC_6'(T) = -\frac{2A + \frac{W^2P(k-h)}{D(P-D)}}{2T^2} + D\left(\frac{k\rho + sI_e}{2}\right) \quad (20)$$

$$TVC_6''(T) = \frac{2A + \frac{W^2P(k-h)}{D(P-D)}}{T^3} > 0 \quad (21)$$

Equations (19) and (21) reveal that $TVC_5(T)$ and $TVC_6(T)$ are convex on $T > 0$.

THE DETERMINATION OF THE OPTIMAL CYCLE TIME T^*

The determination of the optimal cycle time T^* can be divided into three cases:

Case I: $M \geq \frac{W}{D\rho}$,

Case II: $\frac{PM}{D} \geq \frac{W}{D\rho} > M$, and

Case III: $\frac{W}{D\rho} \geq \frac{PM}{D} > M$.

For convenience, the domains of all $TVC_i(T)$ $(i = 1,\ 2,\ 3,\ 4,\ 5 \text{ and } 6)$ can be treated as $T > 0$. Also, let

$$\alpha = 2A + DM^2(cI_k - sI_e) - PM^2cI_k + \frac{W^2P(k-h)}{D(P-D)} \quad (22)$$

$$\beta = 2A + DM^2(cI_k - sI_e) + \frac{W^2P(k-h)}{D(P-D)} \quad (23)$$

$$\gamma = 2A + DM^2(cI_k - sI_e) \quad (24)$$

and

$$\chi = 2A + DM^2(cI_k - sI_e) - PM^2cI_k \quad (25)$$

Then, we have

$$\beta > \gamma > \chi \quad (26)$$

$$\beta > \alpha \geq \chi \quad (27)$$

and

$$\gamma > \chi. \quad (28)$$

Furthermore, let T_i^* denote the solution of equation (29)

$$TVC_i(T) = 0 \quad (29)$$

for all $i = 1,\ 2,\ 3,\ 4,\ 5 \text{ and } 6$. Then

$$T_1^* = \sqrt{\frac{2A + DM^2(cI_k - sI_e) - PM^2cI_k}{D\rho(h + cI_k)}} \quad \text{if} \quad \chi > 0 \quad (30)$$

$$T_2^* = \sqrt{\frac{2A + DM^2(cI_k - sI_e) - PM^2cI_k + \frac{W^2P(k-h)}{D(P-D)}}{D\rho(k + cI_k)}} \quad \text{if} \quad \alpha > 0 \quad (31)$$

$$T_3^* = \sqrt{\frac{2A + DM^2(cI_k - sI_e)}{D(h\rho + cI_k)}} \quad \text{if} \quad \gamma > 0 \quad (32)$$

$$T_4^* = \sqrt{\frac{2A + DM^2(cI_k - sI_e) + \frac{W^2P(k-h)}{D(P-D)}}{D(k\rho + cI_k)}} \quad \text{if} \quad \beta > 0 \quad (33)$$

$$T_5^* = \sqrt{\frac{2A}{D(h\rho + sI_e)}} \quad (34)$$

and

$$T_6^* = \sqrt{\frac{2A + \frac{W^2P(k-h)}{D(P-D)}}{D(k\rho + sI_e)}} \quad (35)$$

If T_i^* exists, then $TVC_i(T)$ is convex on $T > 0$. We also have

$$TVC_i'(T) = \begin{cases} <0 & \text{if} \quad T < T_i^* & (36a) \\ =0 & \text{if} \quad T = T_i^* & (36b) \\ >0 & \text{if} \quad T > T_i^* & (36c) \end{cases}$$

Equations 36 (a, b, c) imply that $TVC_i(T)$ is decreasing on $\left(0, T_i^*\right]$ and increasing on $\left[T_i^*, \infty\right)$ for

all $i = 1,\ 2,\ 3,\ 4,\ 5 \text{ and } 6$.

Case I: $M \geq \frac{W}{D\rho}$

In this case, equations 1(a, b, c, d) imply $TVC_2\left(\frac{PM}{D}\right) = TVC_4\left(\frac{PM}{D}\right)$,

$TVC_4(M) = TVC_6(M)$ and

$TVC_6\left(\frac{W}{D\rho}\right) = TVC_5\left(\frac{W}{D\rho}\right)$. So, $TVC(T)$ is continuous and well-defined. Furthermore, we have

$$TVC_2'\left(\frac{PM}{D}\right) = TVC_4'\left(\frac{PM}{D}\right) = \frac{\Delta_1}{2\left(\frac{PM}{D}\right)^2} \quad (37)$$

$$TVC_4'(M) = TVC_6'(M) = \frac{\Delta_2}{2M^2}, \ and \quad (38)$$

$$TVC_6'\left(\frac{W}{D\rho}\right) = TVC_5'\left(\frac{W}{D\rho}\right) = \frac{\Delta_3}{2\left(\frac{W}{D\rho}\right)^2}, \quad (39)$$

where

$$\Delta_1 = -2A + \frac{M^2}{D}\left(P(P-D)k + cI_k(P^2 - D^2) + D^2 sI_e\right) - \frac{W^2P(k-h)}{D(P-D)} \quad (40)$$

$$\Delta_2 = -2A + DM^2\left(k\rho + sI_e\right) - \frac{W^2P(k-h)}{D(P-D)} \quad (41)$$

(35)

and

$$\Delta_3 = -2A + \frac{PW^2\left(h + \frac{sI_e}{\rho}\right)}{D(P-D)}. \quad (42)$$

Then, we have

$$\Delta_1 > \Delta_2 \geq \Delta_3 \ \left(\text{since } M \geq \frac{W}{D\rho}\right). \quad (43)$$

So, we obtain the following results.

Lemma 1. (A) If $\beta \leq 0$, then $TVC(T)$ is convex on $(0, M]$ and concave on $[M, \infty)$. Furthermore, we have $TVC_2'(T) > 0$ and $TVC_4'(T) > 0$. So $TVC_2(T)$ and $TVC_4(T)$ are increasing on $T > 0$.(B) If $\alpha \leq 0$ and $\beta > 0$, then $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore,

$TVC_2'(T) > 0$ and $TVC_2(T)$ is increasing on $T > 0$.

(C) If $\alpha > 0$, then $TVC(T)$ is convex on $(0, \infty)$.

Appendix A1 shows Proof.

Lemma 2. Suppose $M \geq \frac{W}{D\rho}$ and $\beta \leq 0$. Hence,

(A) If $\Delta_3 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_3 \leq 0$, then $T^* = T_6^*$.

Appendix A2 shows Proof.

Lemma 3. Suppose $M \geq \frac{W}{D\rho}$, $\beta > 0$ and $\alpha \leq 0$.

(A) If $\Delta_3 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_2 > 0$ and $\Delta_3 \leq 0$, then $T^* = T_6^*$.

(C) If $\Delta_2 \leq 0$, then $T^* = T_4^*$.

Appendix A3 shows Proof.

Lemma 4. Suppose $M \geq \frac{W}{D\rho}$, $\alpha > 0$. Hence,

(A) If $\Delta_3 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_2 > 0$ and $\Delta_3 \leq 0$, then $T^* = T_6^*$.

(C) If $\Delta_1 > 0$ and $\Delta_2 \leq 0$, then $T^* = T_4^*$.

(D) If $\Delta_1 \leq 0$, then $T^* = T_2^*$.

Appendix A4 shows Proof.

Combining all arguments of Lemmas 2-4 constitutes the complete proof of the following theorem.

Theorem 1: Suppose that $M \geq \frac{W}{D\rho}$. Hence,

(A) If $\Delta_3 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_2 > 0$ and $\Delta_3 \leq 0$, then $T^* = T_6^*$.

(C) If $\Delta_1 > 0$ and $\Delta_2 \leq 0$, then $T^* = T_4^*$.

(D) If $\Delta_1 \leq 0$, then $T^* = T_2^*$.

Case II: $\frac{PM}{D} \geq \frac{W}{D\rho} > M$

In this case, equations 6(a, b, c, d) imply $TVC_2\left(\frac{PM}{D}\right) = TVC_4\left(\frac{PM}{D}\right)$, $TVC_4\left(\frac{W}{D\rho}\right) = TVC_3\left(\frac{W}{D\rho}\right)$ and $TVC_3(M) = TVC_5(M)$. So, $TVC(T)$ is continuous and well-defined. Furthermore, we have

$$TVC_2'\left(\frac{PM}{D}\right) = TVC_4'\left(\frac{PM}{D}\right) = \frac{\Delta_1}{2\left(\frac{PM}{D}\right)^2} \quad (44)$$

$$TVC_4'\left(\frac{W}{D\rho}\right) = TVC_3'\left(\frac{W}{D\rho}\right) = \frac{\Delta_4}{2\left(\frac{W}{D\rho}\right)^2} \quad (45)$$

and

$$TVC_3'(M) = TVC_5'(M) = \frac{\Delta_5}{M^2}, \quad (46)$$

where

$$\Delta_4 = -2A - DM^2(cI_k - sI_e) + \frac{PM^2\left(h + \frac{cI_k}{\rho}\right)}{D(P-D)}, \text{ and} \quad (47)$$

$$\Delta_5 = -2A + DM^2(h\rho + sI_e). \quad (48)$$

Then, we have

$$\Delta_1 \geq \Delta_4 > \Delta_5 \quad \left(\text{since } \frac{PM}{D} \geq \frac{W}{D\rho}\right). \quad (49)$$

Equations (26) and (27) imply that $\beta > \gamma$ and $\beta > \alpha$. Then we have the following results.

Lemma 5.

(A) If $\beta \leq 0$, then $TVC(T)$ is convex on $(0, M]$ and concave on $[M, \infty)$. Furthermore, we have $TVC_2'(T) > 0$, $TVC_3'(T) > 0$ and $TVC_4'(T) > 0$. So $TVC_2(T)$, $TVC_3(T)$ and $TVC_4(T)$ are increasing on $T > 0$.

(B) If $\alpha < 0$, $\beta > 0$ and $\gamma \leq 0$, then $TVC(T)$ is convex on $(0, M]$ and concave on $\left[M, \frac{W}{D\rho}\right]$, convex on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, we have $TVC_2'(T) > 0$ and $TVC_3'(T) > 0$. So, $TVC_2(T)$ and $TVC_3(T)$ are increasing on $T > 0$.

(C) If $\alpha < 0$, $\beta > 0$ and $\gamma > 0$, then $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, we have $TVC_2'(T) > 0$. So, $TVC_2(T)$ is increasing on $T > 0$.

(D) If $\alpha \geq 0$ and $\gamma \leq 0$, then $TVC(T)$ is convex on $(0, M]$, concave on $\left[M, \frac{W}{D\rho}\right]$ and concave on $\left[\frac{W}{D\rho}, \infty\right)$. Furthermore, we have $TVC_3'(T) > 0$. So, $TVC_3(T)$ is increasing on $T > 0$.

(E) If $\alpha \geq 0$ and $\gamma > 0$, then $TVC(T)$ is convex on $(0, \infty)$.

Appendix A5 shows Proof.

Lemma 6. If $\frac{PM}{D} \geq \frac{W}{D\rho} > M$ and $\beta \leq 0$, then $T^* = T_5^*$.

Appendix A6 shows Proof.

Lemma 7. If $\frac{PM}{D} \geq \frac{W}{D\rho} > M$, $\alpha > 0$, $\beta > 0$ and $\gamma \leq 0$, then $T^* = T_5^*$.

Appendix A7 shows Proof .

Lemma 8. Suppose $\frac{PM}{D} \geq \frac{W}{D\rho} > M$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_4 > 0$, and $\Delta_5 \leq 0$, then $T^* = T_3^*$.

(C) If $\Delta_4 \le 0$, then $T^* = T_4^*$.
Appendix A8 shows Proof.

Lemma 9 If $\frac{PM}{D} \ge \frac{W}{D\rho} > M$, $\alpha \ge 0$ and $\gamma \le 0$, then $T^* = T_5^*$.
Appendix A9 shows Proof.

Lemma 10. Suppose $\frac{PM}{D} \ge \frac{W}{D\rho} > M$, $\alpha \ge 0$ and $\gamma > 0$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_4 > 0$, and $\Delta_5 \le 0$, then $T^* = T_3^*$.

(C) If $\Delta_1 > 0$, and $\Delta_4 \le 0$, then $T^* = T_4^*$.

(D) If $\Delta_1 \le 0$, then $T^* = T_2^*$.

Appendix A10 shows Proof.

Combining all arguments of Lemmas 6-10 constitutes the complete proof of the following theorem.

Theorem 2: Suppose $\frac{PM}{D} \ge \frac{W}{D\rho} > M$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_4 > 0$ and $\Delta_5 \le 0$, then $T^* = T_3^*$.

(C) If $\Delta_1 > 0$ and $\Delta_4 \le 0$, then $T^* = T_4^*$.

(D) If $\Delta_1 \le 0$, then $T^* = T_2^*$.

Case (III:) $\frac{W}{D\rho} \ge \frac{PM}{D} > M$

In this case, equations 8(a, b, c, d) imply $TVC_2\left(\frac{W}{D\rho}\right) = TVC_1\left(\frac{W}{D\rho}\right)$, $TVC_1\left(\frac{PM}{D}\right) = TVC_3\left(\frac{PM}{D}\right)$ and $TVC_3(M) = TVC_5(M)$. So, $TVC(T)$ is continuous and well-defined. Furthermore, we have

$$TVC_2'\left(\frac{W}{D\rho}\right) = TVC_1'\left(\frac{W}{D\rho}\right) = \frac{\Delta_6}{2\left(\frac{W}{D\rho}\right)^2} \tag{50}$$

$$TVC_1'\left(\frac{PM}{D}\right) = TVC_3'\left(\frac{PM}{D}\right) = \frac{\Delta_7}{2\left(\frac{PM}{D}\right)^2} \tag{51}$$

and

$$TVC_3'(M) = TVC_5'(M) = \frac{-2A + DM^2(h\rho + sI_e)}{2M^2} \tag{52}$$

Furthermore, we let

$$\Delta_6 = -2A - DM^2(cI_k - sI_e) + PM^2 cI_k + \frac{PW^2(h + cI_k)}{D(P-D)}, \ and \tag{53}$$

$$\Delta_7 = -2A + \frac{M^2}{D}\left(P(P-D)h + cI_k(P^2 - D^2) + D^2 sI_e\right) \tag{54}$$

Then, we have

$$\Delta_6 > \Delta_7 > \Delta_5 \ \left(\text{since } \frac{W}{D\rho} > \frac{PM}{D} > M\right). \tag{55}$$

Equations (27) and (28) imply that $\gamma > \chi$ and $\alpha \ge \chi$. Then we have the following results.

Lemma 11.

(A) If $\alpha \le 0$ and $\gamma \le 0$, then $TVC(T)$ is convex on $(0, M]$ and concave on $[M, \infty)$. Furthermore, we have $TVC_1'(T) > 0$, $TVC_2'(T) > 0$ and $TVC_3'(T) > 0$. So, $TVC_1(T)$, $TVC_2(T)$ and $TVC_3(T)$ are increasing on $T > 0$.

(B) If $\alpha \le 0$ and $\gamma \le 0$, then $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, we have $TVC_1'(T) > 0$ and $TVC_2'(T) > 0$. So, $TVC_1(T)$ and $TVC_2(T)$ are increasing on $T > 0$.

(C) If $\alpha > 0$ and $\gamma \le 0$, then $TVC(T)$ is convex on $(0, M]$ and concave on $\left[M, \frac{W}{D\rho}\right]$ and convex on $\left[\frac{W}{D\rho}, \infty\right)$. Furthermore, we have $TVC_1'(T) > 0$ and $TVC_3'(T) > 0$. So, $TVC_1(T)$ and $TVC_3(T)$ are increasing on $T > 0$.

(D) If $\alpha > 0$, $\chi \le 0$ and $\gamma > 0$, then $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$, concave on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$ and convex on $\left[\frac{W}{D\rho}, \infty\right)$. Furthermore, we have $TVC_1'(T) > 0$. So, $TVC_1(T)$ is increasing on $T > 0$.

(E) If $\chi > 0$, then $TVC(T)$ is convex on $(0, \infty)$.

Appendix A11 shows Proof.

Lemma 12. If $\frac{W}{D\rho} > \frac{PM}{D} > M$, $\alpha \le 0$ and $\gamma \le 0$, then $T^* = T_5^*$.

Appendix A12 shows Proof.

Lemma 13. Suppose $\frac{W}{D\rho} > \frac{PM}{D} > M$, $\alpha \le 0$ and $\gamma > 0$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_5 \le 0$, then $T^* = T_3^*$.

Appendix A13 shows Proof.

Lemma 14. If $\frac{W}{D\rho} > \frac{PM}{D} > M$, $\alpha > 0$ and $\gamma \le 0$, then $T^* = T_5^*$

Appendix A14 shows Proof.

Lemma 15. Suppose $\frac{W}{D\rho} > \frac{PM}{D} > M$, $\alpha > 0$, $\chi \le 0$ and $\gamma > 0$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_5 \le 0$, then $T^* = T_3^*$.

Appendix A15 shows Proof.

Lemma 16. Suppose $\frac{W}{D\rho} > \frac{PM}{D} > M$ and $\chi > 0$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_7 > 0$ and $\Delta_5 \le 0$, then $T^* = T_3^*$.

(C) If $\Delta_6 > 0$ and $\Delta_7 \le 0$, then $T^* = T_1^*$.

(D) If $\Delta_6 \le 0$, then $T^* = T_2^*$.

Appendix A16 shows Proof.

Combining all arguments of Lemmas 12-16 constitutes the complete proof of the following theorem.

Theorem 3. Suppose $\frac{W}{D\rho} > \frac{PM}{D} > M$. Hence,

(A) If $\Delta_5 > 0$, then $T^* = T_5^*$.

(B) If $\Delta_7 > 0$ and $\Delta_5 \le 0$, then $T^* = T_3^*$.

(C) If $\Delta_6 > 0$ and $\Delta_7 \le 0$, then $T^* = T_1^*$.

(D) If $\Delta_6 \le 0$, then $T^* = T_2^*$.

SPECIAL CASES

(1) When $h = k$ and $s = c$, then equations 1(a, b, c, d), 6(a, b, c, d) and 8(a, b, c, d) will also be reduced as follows:

$$\bar{T}VC(T) = \begin{cases} \bar{T}VC_3(T) & \text{if } 0 < T \le M & \text{(56a)} \\ \bar{T}VC_2(T) & \text{if } M \le T \le \frac{PM}{D} & \text{(56b)} \\ \bar{T}VC_1(T) & \text{if } T \ge \frac{PM}{D} & \text{(56c)} \end{cases}$$

Where

$$\bar{T}VC_1(T) = \frac{A}{T} + \frac{DTh\rho}{2} + \frac{cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)}{T} - \frac{cI_e\left(\frac{DM^2}{2}\right)}{T}$$

$$\bar{T}VC_2(T) = \frac{A}{T} + \frac{DTh\rho}{2} + \frac{cI_k\rho\left(\frac{D(T-M)^2}{2}\right)}{T} - \frac{cI_e\left(\frac{DM^2}{2}\right)}{T}$$

$$\bar{T}VC_3(T) = \frac{A}{T} + \frac{DTh\rho}{2} - \frac{cI_e\left(\frac{DT^2 + DT(M-T)}{2}\right)}{T}.$$

Equations 56(a, b, c) will be consistent with equations (7), (8) and (9) in Chung and Huang (2003), respectively. So, this paper generalizes Chung and Huang (2003).

(2) When $h = k$, $s = c$ and $P \to \infty$, equations 1(a, b, c, d), 6(a, b, c, d) and 8(a, b, c, d) will also be reduced as follows:

$$\tilde{T}VC(T) = \begin{cases} \tilde{T}VC_2(T) & \text{if } 0 \le T \le M & \text{(57a)} \\ \tilde{T}VC_1(T) & \text{if } M \le T & \text{(57b)} \end{cases}$$

where

$$\tilde{T}VC_1(T) = \frac{A}{T} + \frac{DTh}{2} + \frac{cI_k\left(\frac{D(T-M)^2}{2}\right)}{T} - \frac{cI_e\left(\frac{DM^2}{2}\right)}{T}$$

$$\tilde{T}VC_2(T) = \frac{A}{T} + \frac{DTh}{2} - \frac{cI_e\left(\frac{DT^2}{2} + DT(M-T)\right)}{T}.$$

Equations 57(a, b) will be consistent with Equations (1) and (4) in Goyal (1985). Hence, Goyal (1985) will be a special case of this paper. Similarly, Chung (1998) is a special case of this paper as well.

NUMERICAL EXAMPLES

Twenty-nine numerical examples are used to explain all results in this paper. The necessary parameters and the optimal solutions of the twenty-nine examples are presented in Tables 1 and 2, respectively. All dimensions of parameters involved in Table 1 are the same as those of Huang (2006). Examples 1-9 concern (Lemmas 2-4 and Theorem 1) related to Case (I): $M \ge \frac{W}{D\rho}$. Examples 10-19 concern all results (Lemmas 6-10 and Theorem 2)

Table 1. Given parameters values.

Example	D	P	c	s	h	k	I_e	I_k	A	M	W
1	7300	25000	10	210	5	7	0.13	0.15	2000	0.15	700
2	7000	25000	50	300	5	7	0.13	0.15	1500	0.15	500
3	7000	25000	50	265	5	7	0.13	0.15	2500	0.15	700
4	7000	25000	50	300	5	7	0.13	0.15	3000	0.15	700
5	7000	25000	50	300	5	7	0.13	0.15	3500	0.15	600
6	5000	14000	50	600	5	30	0.13	0.15	3800	0.14	440
7	5000	14000	50	600	5	30	0.13	0.15	4000	0.14	440
8	5000	14000	50	600	5	30	0.13	0.15	4100	0.14	440
9	5000	10000	50	600	5	10	0.13	0.15	6700	0.15	350
10	7000	25000	10	170	5	7	0.13	0.15	1500	0.15	700
11	7000	25000	10	255	5	7	0.13	0.15	2400	0.15	1000
12	7000	25000	50	265	5	7	0.13	0.15	2500	0.15	1000
13	7000	25000	50	300	5	7	0.13	0.15	3500	0.15	1000
14	7000	25000	50	300	5	7	0.13	0.15	4100	0.15	1000
15	7000	15000	10	160	5	25	0.13	0.15	1500	0.15	1000
16	8000	16000	50	300	5	9	0.13	0.15	3700	0.15	1000
17	7000	25000	50	300	5	7	0.13	0.15	4400	0.15	1150
18	7000	25000	50	300	5	7	0.13	0.15	4400	0.15	1000
19	6000	15000	50	300	5	7	0.13	0.15	7000	0.15	600
20	15000	25000	9	100	5	7	0.13	0.15	1910	0.15	1600
21	20000	25000	50	100	5	7	0.13	0.15	2800	0.15	800
22	20000	25000	180	200	5	7	0.13	0.15	6500	0.15	800
23	22000	25000	10	150	15	30	0.13	0.15	4450	0.15	2200
24	15000	25000	50	300	5	7	0.13	0.15	6700	0.15	2200
25	14000	25000	50	300	5	7	0.13	0.15	6500	0.15	2200
26	15450	16000	20	300	5	7	0.13	0.15	6800	0.15	2200
27	14000	19000	50	300	5	7	0.13	0.15	6600	0.15	2200
28	12000	18000	50	300	5	7	0.13	0.15	7100	0.15	2200
29	12000	18000	50	300	5	7	0.13	0.15	7400	0.15	1000

related to Case (II): $\frac{PM}{D} \geq \frac{W}{D\rho} > M$. Finally, Examples 20-29 concern all results (Lemmas 12-16 and Theorem 3) related Case (III): $\frac{W}{D\rho} > \frac{PM}{D} > M$. Table 2 reveals the following observations:

Case (I): $M \geq \frac{W}{D\rho}$

Observation (A): $T^* = T_5^*$ if $0 < T_5^* \leq \frac{W}{D\rho}$

Observation (B): $T^* = T_6^*$ if $\frac{W}{D\rho} \leq T_6^* \leq M$

Observation (C): $T^* = T_4^*$ if $M \leq T_4^* \leq \frac{PM}{D}$

Observation (D): $T^* = T_2^*$ if $\frac{PM}{D} \leq T_2^*$

Case (II): $\frac{PM}{D} \geq \frac{W}{D\rho} > M$

Observation (E): $T^* = T_5^*$ if $0 < T_5^* \leq M$

Observation (F): $T^* = T_3^*$ if $M \leq T_3^* \leq \frac{W}{D\rho}$

Observation (G): $T^* = T_4^*$ if $\frac{W}{D\rho} \leq T_4^* \leq \frac{PM}{D}$

Observation (H): $T^* = T_2^*$ if $\frac{PM}{D} \leq T_2^*$

Case (III): $\frac{W}{D\rho} > \frac{PM}{D} > M$

Observation (I): $T^* = T_5^*$ if $0 < T_5^* \leq M$

Observation (J): $T^* = T_3^*$ if $M \leq T_3^* \leq \frac{PM}{D}$

Table 2. The optimal solutions.

Example	Case	Lemma	Theorem	α	β	γ	χ	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	T_1^*	T_2^*	T_3^*	T_4^*	T_5^*	T_6^*	T^*
1	(I)	2(A)	1	<0	<0	-	-	>0	>0	>0	-	-	-	-	-	N	-	N	(*)	Y	0.13329
2	(I)	2(B)	1	<0	<0	-	-	>0	>0	<0	-	-	-	-	-	N	-	N	Y	(*)	0.10027
3	(I)	3(A)	1	<0	>0	-	-	>0	>0	>0	-	-	-	-	-	N	-	Y	(*)	Y	0.13701
4	(I)	3(B)	1	<0	>0	-	-	>0	>0	<0	-	-	-	-	-	N	-	Y	Y	(*)	0.14175
5	(I)	3(C)	1	<0	>0	-	-	>0	<0	<0	-	-	-	-	-	N	-	(*)	Y	Y	0.15765
6	(I)	4(A)	1	>0	>0	-	-	>0	>0	>0	-	-	-	-	-	Y	-	Y	(*)	Y	0.13681
7	(I)	4(B)	1	>0	>0	-	-	>0	>0	<0	-	-	-	-	-	Y	-	Y	Y	(*)	0.13979
8	(I)	4(C)	1	>0	>0	-	-	>0	<0	<0	-	-	-	-	-	Y	-	(*)	Y	Y	0.14451
9	(I)	4(D)	1	>0	>0	-	-	<0	<0	<0	-	-	-	-	-	(*)	-	Y	Y	Y	0.30336
10	(II)	6	2	<0	<0	<0	-	>0	-	-	>0	>0	-	-	-	N	N	N	(*)	-	0.12914
11	(II)	7	2	<0	>0	<0	-	>0	-	-	>0	>0	-	-	-	N	N	Y	(*)	-	0.13660
12	(II)	8(A)	2	<0	>0	>0	-	>0	-	-	>0	>0	-	-	-	N	Y	Y	(*)	-	0.13701
13	(II)	8(B)	2	<0	>0	>0	-	>0	-	-	>0	<0	-	-	-	N	(*)	Y	Y	-	0.16198
14	(II)	8(C)	2	<0	>0	>0	-	>0	-	-	<0	<0	-	-	-	N	Y	(*)	Y	-	0.20351
15	(II)	9	2	>0	>0	<0	-	>0	-	-	>0	>0	-	-	-	Y	N	Y	(*)	-	0.13514
16	(II)	10(A)	2	>0	>0	>0	-	>0	-	-	>0	>0	-	-	-	Y	Y	Y	(*)	-	0.14930
17	(II)	10(B)	2	>0	>0	>0	-	>0	-	-	>0	<0	-	-	-	Y	(*)	Y	Y	-	0.22227
18	(II)	10(C)	2	>0	>0	>0	-	>0	-	-	<0	<0	-	-	-	Y	Y	(*)	Y	-	0.21966
19	(II)	10(D)	2	>0	>0	>0	-	<0	-	-	<0	<0	-	-	-	(*)	Y	Y	Y	-	0.37639
20	(III)	12	3	<0	-	<0	<0	-	-	-	-	>0	>0	>0	N	N	N	-	(*)	-	0.13030
21	(III)	13(A)	3	<0	-	>0	<0	-	-	-	-	>0	>0	>0	N	N	Y	-	(*)	-	0.14142
22	(III)	13(B)	3	<0	-	>0	<0	-	-	-	-	<0	>0	>0	N	N	(*)	-	Y	-	0.15498
23	(III)	14	3	>0	-	<0	<0	-	-	-	-	>0	>0	>0	N	Y	N	-	(*)	-	0.13781
24	(III)	15(A)	3	>0	-	>0	<0	-	-	-	-	>0	>0	>0	N	Y	Y	-	(*)	-	0.14761
25	(III)	15(B)	3	>0	-	>0	<0	-	-	-	-	<0	>0	>0	N	Y	(*)	-	Y	-	0.15054
26	(III)	16(A)	3	>0	-	>0	>0	-	-	-	-	>0	>0	>0	Y	Y	Y	-	(*)	-	0.14991
27	(III)	16(B)	3	>0	-	>0	>0	-	-	-	-	<0	>0	>0	Y	Y	(*)	-	Y	-	0.16296
28	(III)	16(C)	3	>0	-	>0	>0	-	-	-	-	<0	>0	<0	(*)	Y	Y	-	Y	-	0.23054
29	(III)	16(D)	3	>0	-	>0	>0	-	-	-	-	<0	<0	<0	Y	(*)	Y	-	Y	-	0.25453

Case: Which Case is discussed?; Lemma: Which Lemma is applied?; Theorem: Which Theorem is applied? N: Does not exist; Y: Exists; (*): The optimal solution; - : Does not relate to this example.

Observation (K): $T^* = T_1^*$ if $\frac{PM}{D} \le T_1^* \le \frac{W}{D\rho}$

Observation (L): $T^* = T_2^*$ if $\frac{PM}{D} \le T_2^*$

Furthermore, Table 2 can also reveal situations of existences and interrelations of all $T_i^* (i = 1, 2, 3, 4, 5, 6)$. Basically, Table 2 is rather informative and meaningful.

CONCLUSIONS

This paper establishes a new economic production model with trade credit, finite replenishment rate and limited storage capacity to generalize some existing articles. There are three cases: (1) $M \ge \frac{W}{D\rho}$, (2) $\frac{PM}{D} \ge \frac{W}{D\rho} > M$ and (3) $\frac{W}{D\rho} > \frac{PM}{D} > M$ to be discussed throughout the whole paper. Three main theorems are used to characterize the optimal solutions and provide three easy-to-use criterions to find the optimal replenishment cycle times under various circumstances. Several numerical examples are given to verify the theoretical results.

Recently, Chang et al. (2008) present a review of the advances in inventory literature under conditions of permissible delay in payments since 1985. They classify all related articles into five categories based on: (a) without deterioration, (b) with deterioration, (c) with allowable shortage, (d) linked to order quantity, and (e) with inflation. Our model can be extended to more supply chain systems by incorporating one or more of the above (a)-(e). In fact, deteriorating items and the order quantity as a function of trade credit period will be considered in the proposed model in the future works.

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Appendix

Appendix A1: Proof of Lemma 1

Proof.

(A) Equation (19) and (21) reveal that $TVC_5''(T) > 0$ and $TVC_6''(T) > 0$ for all $T > 0$. Equations 1(a, b) imply that $TVC(T)$ is convex on $(0, M]$. If $\beta \le 0$, then $TVC_2''(T) < 0$ and $TVC_4''(T) \le 0$ for all $T > 0$. Equations 1(c, d) imply that $TVC(T)$ is concave on $[M, \infty)$. Furthermore, equations (12) and (16) imply that $TVC_2'(T) > 0$ and $TVC_4'(T) > 0$. So, $TVC_2(T)$ and $TVC_4(T)$ are increasing on $T > 0$.

(B) If $\beta > 0$, then equations (17), (19) and (21) reveal that $TVC_5''(T) > 0$, $TVC_6''(T) > 0$ and $TVC_4''(T) > 0$ for all $T > 0$. Equations 1(a, b, c) imply that $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$. Furthermore, if $\alpha \le 0$, then $TVC_2''(T) \le 0$ for all $T > 0$. Equation (1d) implies that $TVC(T)$ is concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, equation (12) implies $TVC_2'(T) > 0$. So, $TVC_2(T)$ is increasing on $T > 0$.

(C) If $\alpha > 0$, equations (13), (17), (19) and (21) reveal that $TVC_5''(T) > 0$, $TVC_6''(T) > 0$, $TVC_4''(T) > 0$ and $TVC_2''(T) > 0$. Equations 1(a, b, c, d) imply that $TVC(T)$ is convex on $(0, \infty)$.

Combining all arguments of (A)-(C), we have completed the proof of Lemma 1.

Appendix A2: Proof of Lemma 2

Proof. If $\beta \le 0$, then $\alpha < 0$. Equation (40) and (41) reveal that $\Delta_1 > \Delta_2 > 0$.

(A) If $\Delta_3 > 0$, with Lemma 1, we have

(i) $TVC_5(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, \frac{W}{D\rho}\right]$.

(ii) $TVC_6(T)$ is increasing on $\left[\frac{W}{D\rho}, M\right]$.

(iii) $TVC_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 1(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$. So, $T^* = T_5^*$.

(B) If $\Delta_3 \le 0$, with Lemma 1, we have

(v) $TVC_5(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$.

(vi) $TVC_6(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_6^*\right]$ and increasing on $\left[T_6^*, M\right]$.

(vii) $TVC_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

(viii) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 1(a, b, c, d) and (v)-(viii), we conclude that $TVC(T)$ is decreasing on $\left(0, T_6^*\right]$ and increasing on $\left[T_6^*, \infty\right)$. So, $T^* = T_6^*$.

Incorporating the above arguments, we have completed the proof of Lemma 2.

Appendix A3: Proof of Lemma 3

Proof. If $\alpha \le 0$, equation (40) implies $\Delta_1 > 0$.

(A) If $\Delta_3 > 0$, the proof of (A) is the same as that of Lemma 2(A).

(B) If $\Delta_2 > 0$ and $\Delta_3 \le 0$, the proof of (B) is the same as that of Lemma 2(B).

(C) If $\Delta_2 \le 0$, with Lemma 1, we have

(i) $TVC_5(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$.

(ii) $TVC_6(T)$ is decreasing on $\left[\frac{W}{D\rho}, M\right]$.

(iii) $TVC_4(T)$ is decreasing on $\left[M, T_4^*\right]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 1(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_4^*\right]$ and increasing on $\left[T_4^*, \infty\right)$. So, $T^* = T_4^*$.

Incorporating the above arguments, we have completed the proof of Lemma 3.

Appendix A4: Proof of Lemma 4

Proof.

(A) If $\Delta_3 > 0$, the proof of (A) is the same as that of Lemma 2(A).

(B) If $\Delta_2 > 0$ and $\Delta_3 \le 0$, the proof of (B) is the same as that of Lemma 2(B).

(C) If $\Delta_1 > 0$ and $\Delta_2 \le 0$, the proof of (C) is the same as that of Lemma 3(C).

(D) If $\Delta_1 \le 0$, then $0 \ge \Delta_1 > \Delta_2 > \Delta_3$. With Lemma 1, we have

(i) $TVC_5(T)$ is decreasing on $\left(0, \frac{W}{D\rho}\right]$.

(ii) $TVC_6(T)$ is decreasing on $\left[\frac{W}{D\rho}, M\right]$.

(iii) $TVC_4(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is decreasing on $\left[\frac{PM}{D}, T_2^*\right]$.

Combining equations 1(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$. So, $T^* = T_2^*$.

Incorporating the above arguments, we have completed the proof of Lemma 4.

Appendix A5: Proof of Lemma 5

Proof.

(A) If $\beta \le 0$, then $\gamma < 0$ and $\alpha < 0$. Equations (13), (15), (17) and (19) reveal that $TVC_2''(T) < 0$, $TVC_3''(T) < 0$, $TVC_4''(T) \le 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 6(a, b, c, d) imply that $TVC(T)$ is convex on $(0, M]$ and concave on $[M, \infty)$. Furthermore, equations (12), (14) and (16) imply that $TVC_2'(T) > 0$, $TVC_3'(T) > 0$ and $TVC_4'(T) > 0$. So, $TVC_2(T)$, $TVC_3(T)$ and $TVC_4(T)$ are increasing on $T > 0$.

(B) If $\alpha < 0$, $\beta > 0$ and $\gamma \le 0$, then equation (13), (15), (17) and (19) reveal that $TVC_2''(T) < 0$, $TVC_3''(T) \le 0$, $TVC_4''(T) > 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equation 6(a, b, c, d) imply that $TVC(T)$ is convex on $(0, M]$ and concave on $\left[M, \frac{W}{D\rho}\right]$, convex on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, we have $TVC_2'(T) > 0$ and $TVC_3'(T) > 0$. So, $TVC_2(T)$ and $TVC_3(T)$ are increasing on $T > 0$.

(C) If $\alpha < 0$, $\beta > 0$ and $\gamma > 0$, then equations (13), (15), (17) and (19) reveal that $TVC_2''(T) < 0$, $TVC_3''(T) > 0$, $TVC_4''(T) > 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 6(a, b, c, d) imply that $TVC(T)$ is convex on $\left(0, \frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D}, \infty\right)$. Furthermore, we have $TVC_2'(T) > 0$. So, $TVC_2(T)$ is increasing on $T > 0$.

(D) If $\alpha \ge 0$, then $\beta > 0$. So, we have $\alpha \ge 0$, $\beta > 0$ and $\gamma \le 0$. Equations (13), (15), (17) and (19) reveal that $TVC_2''(T) \ge 0$, $TVC_3''(T) \le 0$, $TVC_4''(T) > 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 6(a, b, c, d) imply that $TVC(T)$ is convex on $(0, M]$ and concave on $\left[M, \frac{W}{D\rho}\right]$ and concave on $\left[\frac{W}{D\rho}, \infty\right)$. Furthermore, we have $TVC_3'(T) > 0$. So, $TVC_3(T)$ is increasing on $T > 0$.

(E) If $\alpha \ge 0$, then $\beta > 0$. So, we have $\alpha \ge 0$, $\beta > 0$ and $\gamma > 0$. Equations (13), (15), (17) and (19) reveal that $TVC_2''(T) \ge 0$, $TVC_3''(T) > 0$, $TVC_4''(T) > 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 6(a, b, c, d) imply that $TVC(T)$ is convex on $(0, \infty)$.

Combining all arguments of (A)-(E), we have completed the proof of Lemma 5.

Appendix A6: Proof of Lemma 6

Proof. Lemma 5(A), equations (40), (47) and (48) reveal that $\Delta_1 \ge \Delta_4 > \Delta_5 > 0$. Then we have

(i) $TVC_5(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, M\right]$.

(ii) $TVC_3(T)$ is increasing on $\left[M, \frac{W}{D\rho}\right]$.

(iii) $TVC_4(T)$ is increasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 6(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$. So, $T^* = T_5^*$.

This completes the proof of Lemma 6.

Appendix A7: Proof of Lemma 7

Proof. Lemma 5(B), equations (40), (47) and (48) reveal that $\Delta_1 \ge \Delta_4 > \Delta_5 > 0$. Then, the proof of Lemma 7 is the same as that of Lemma 6.

Appendix A8: Proof of Lemma 8

Proof. Lemma 5(C) and equation (40) reveal $\Delta_1 > 0$. Then, we have

(A) If $\Delta_5 > 0$, then $\Delta_1 \geq \Delta_4 > \Delta_5 > 0$. The proof of (A) is the same as that of Lemma 6.

(B) If $\Delta_4 > 0$ and $\Delta_5 \leq 0$, with Lemma 5(C), we have

(i) $TVC_5(T)$ is decreasing on $(0, M]$.

(ii) $TVC_3(T)$ is decreasing on $\left[M, T_3^*\right]$ and increasing on $\left[T_3^*, \frac{W}{D\rho}\right]$.

(iii) $TVC_4(T)$ is increasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 6(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_3^*\right]$ and increasing on $\left[T_3^*, \infty\right)$. So, $T^* = T_3^*$.

(C) If $\Delta_4 \leq 0$, with Lemma 5(C), we have

(v) $TVC_5(T)$ is decreasing on $(0, M]$.

(vi) $TVC_3(T)$ is decreasing on $\left[M, \frac{W}{D\rho}\right]$.

(vii) $TVC_4(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_4^*\right]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$.

(viii) $TVC_2(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Combining equations 6(a, b, c, d) and (v)-(viii), we conclude that $TVC(T)$ is decreasing on $\left(0, T_4^*\right]$ and increasing on $\left[T_4^*, \infty\right)$. So, $T^* = T_4^*$.

Incorporating the above arguments, we have completed the proof of Lemma 8.

Appendix A9: Proof of Lemma 9

Proof. Lemma 5(D) and equations (47) and (48) reveal that $\Delta_1 \geq \Delta_4 > \Delta_5 > 0$. So, the proof of Lemma 9 is the same as Lemma 6.

Appendix A10: Proof of Lemma 10

Proof.

(A) If $\Delta_5 > 0$, then $\Delta_1 \geq \Delta_4 > \Delta_5 > 0$. With Lemma 5(E),

the proof of (A) is the same as that of lemma 6.

(B) If $\Delta_4 > 0$ and $\Delta_5 \leq 0$, with Lemma 5(E), the proof of (B) is the same as that of Lemma 8(B).

(C) If $\Delta_1 > 0$ and $\Delta_4 \leq 0$, with lemma 5(E), the proof of (C) is the same as that of Lemma 8(C).

(D) If $\Delta_1 \leq 0$, with Lemma 5(E), we have

(i) $TVC_5(T)$ is decreasing on $(0, M]$.

(ii) $TVC_3(T)$ is decreasing on $\left[M, \frac{W}{D\rho}\right]$.

(iii) $TVC_4(T)$ is decreasing on $\left[\frac{W}{D\rho}, \frac{PM}{D}\right]$.

(iv) $TVC_2(T)$ is decreasing on $\left[\frac{PM}{D}, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$.

Combining equations 6(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$. So, $T^* = T_2^*$.

Incorporating the above arguments, we have completed the proof of Lemma 10.

Appendix A11: Proof of Lemma 11

Proof.

(A) If $\alpha \leq 0$ and $\gamma \leq 0$, then $\chi \leq 0$. Equations (11), (13), (15) and (19) reveal that $TVC_1''(T) \leq 0$, $TVC_2''(T) \leq 0$, $TVC_3''(T) \leq 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 8(a, b, c, d) imply that $TVC(T)$ is convex on $(0, M]$ and concave on $[M, \infty)$. Furthermore, equations (10), (12) and (14) imply that $TVC_1'(T) > 0$, $TVC_2'(T) > 0$ and $TVC_3'(T) > 0$. So, $TVC_1(T)$, $TVC_2(T)$ and $TVC_3(T)$ are increasing on $T > 0$.

(B) If $\alpha \leq 0$ and $\gamma > 0$, then $\chi \leq 0$. Equations (11), (13), (15) and (19) reveal that $TVC_1''(T) \leq 0$, $TVC_2''(T) \leq 0$, $TVC_3''(T) > 0$ and $TVC_5''(T) > 0$ for all $T > 0$. Equations 8(a, b, c, d) imply that $TVC(T)$ is convex on $\left(0,\frac{PM}{D}\right]$ and concave on $\left[\frac{PM}{D},\infty\right)$. Furthermore, equations (10) and (12) imply that $TVC_1'(T)>0$ and $TVC_2'(T)>0$. So, $TVC_1(T)$ and $TVC_2(T)$ are increasing on $T>0$.

(C) If $\gamma\le 0$, then $\chi\le 0$. Equations (11), (13), (15) and (19) reveal that $TVC_1''(T)\le 0$, $TVC_2''(T)>0$, $TVC_3''(T)\le 0$ and $TVC_5''(T)>0$ for all $T>0$. Equations 8(a, b, c, d) imply that $TVC(T)$ is convex on $(0,M]$, concave on $\left[M,\frac{W}{D\rho}\right]$ and convex on $\left[\frac{W}{D\rho},\infty\right)$. Furthermore, equations (10) and (14) imply that $TVC_1'(T)>0$ and $TVC_3'(T)>0$. So, $TVC_1(T)$ and $TVC_3(T)$ are increasing on $T>0$.

(D) If $\alpha>0$, $\chi\le 0$ and $\gamma>0$. Equations (11), (13), (15) and (19) reveal that $TVC_1''(T)\le 0$, $TVC_2''(T)>0$, $TVC_3''(T)>0$ and $TVC_5''(T)>0$ for all $T>0$. Equations 8(a, b, c, d) imply that $TVC(T)$ is convex on $\left(0,\frac{PM}{D}\right]$, concave on $\left[\frac{PM}{D},\frac{W}{D\rho}\right]$ and convex on $\left[\frac{W}{D\rho},\infty\right)$. Furthermore, equations (10) implies that $TVC_1'(T)>0$. So, $TVC_1(T)$ is increasing on $T>0$.

(E) If $\chi>0$, $\alpha>0$ and $\gamma>0$. Equations (11), (13), (15) and (19) reveal that $TVC_1''(T)>0$, $TVC_2''(T)>0$, $TVC_3''(T)>0$ and $TVC_5''(T)>0$ for all $T>0$. Equations 8(a, b, c, d) imply that $TVC(T)$ is convex on $(0,\infty)$.

Combining all arguments of (A)-(E), we have completed the proof of Lemma 11.

Appendix A12: Proof of Lemma 12

Proof. Lemma 11(A), equations (53), (54) and (48) reveal that $\Delta_6>\Delta_7>\Delta_5>0$. We have

(i) $TVC_5(T)$ is decreasing on $\left(0,T_5^*\right]$ and increasing on $\left[T_5^*,M\right]$.

(ii) $TVC_3(T)$ is increasing on $\left[M,\frac{PM}{D}\right]$.

(iii) $TVC_1(T)$ is increasing on $\left[\frac{PM}{D},\frac{W}{D\rho}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{W}{D\rho},\infty\right)$.

Combining equations 8(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0,T_5^*\right]$ and increasing on $\left[T_5^*,\infty\right)$. Consequently, $T^*=T_5^*$.

This completes the proof of Lemma 12.

Appendix A13: Proof of Lemma 13

Proof. Lemma 11(B), equations (53) and (54) reveal that $\Delta_6>\Delta_7>0$.

(A) If $\Delta_5>0$, we have $\Delta_6>\Delta_7>\Delta_5>0$. The proof of (A) is the same as that of Lemma 12.

(B) If $\Delta_5\le 0$, with Lemma 11(B), we have

(i) $TVC_5(T)$ is decreasing on $(0,M]$.

(ii) $TVC_3(T)$ is decreasing on $\left[M,T_3^*\right]$ and increasing on $\left[T_3^*,\frac{PM}{D}\right]$.

(iii) $TVC_1(T)$ is increasing on $\left[\frac{PM}{D},\frac{W}{D\rho}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{W}{D\rho},\infty\right)$.

Combining equations 8(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0,T_3^*\right]$ and increasing on $\left[T_3^*,\infty\right)$. Consequently, $T^*=T_3^*$.

Incorporating the above arguments, we have completed the proof of Lemma 13.

Appendix A14: Proof of Lemma 14

Proof. Lemma 11(C), equations (53), (54) and (48) reveal that $\Delta_6>\Delta_7>\Delta_5>0$. So, the proof of Lemma 14 is the same as that of Lemma 12.

Appendix A15: Proof of Lemma 15

Proof. Lemma 11(D), equations (53) and (54) reveal that $\Delta_6>\Delta_7>0$.

(A) If $\Delta_5>0$, the proof of (A) is the same as the proof of Lemma 12.

(B) If $\Delta_5\le 0$, the proof of (B) is the same as the proof of Lemma 13(B).

Appendix A16: Proof of Lemma 16

Proof.

(A) If $\Delta_5>0$, then $\Delta_6>\Delta_7>\Delta_5>0$. With Lemma 11(E), the proof of (A) is the same as that of Lemma 12.

(B) If $\Delta_7 > 0$ and $\Delta_5 \le 0$, with Lemma 11(E), the proof of (B) is the same as that of Lemma 13(B).

(C) If $\Delta_6 > 0$ and $\Delta_7 \le 0$, with Lemma 11(E), we have

(i) $TVC_5(T)$ is decreasing on $\left(0, M\right]$.

(ii) $TVC_3(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$.

(iii) $TVC_1(T)$ is decreasing on $\left[\frac{PM}{D}, T_1^*\right]$ and increasing on $\left[T_1^*, \frac{W}{D\rho}\right]$.

(iv) $TVC_2(T)$ is increasing on $\left[\frac{W}{D\rho}, \infty\right)$.

Combining equations 8(a, b, c, d) and (i)-(iv), we conclude that $TVC(T)$ is decreasing on $\left(0, T_1^*\right]$ and increasing on $\left[T_1^*, \infty\right)$. Consequently, $T^* = T_1^*$.

(D) If $\Delta_6 \le 0$, with Lemma 11(E), we have

(v) $TVC_5(T)$ is decreasing on $\left(0, M\right]$.

(vi) $TVC_3(T)$ is decreasing on $\left[M, \frac{PM}{D}\right]$.

(vii) $TVC_1(T)$ is decreasing on $\left[\frac{PM}{D}, \frac{W}{D\rho}\right]$.

(viii) $TVC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$.

Combining equations 8(a, b, c, d) and (v)-(viii), we conclude that $TVC(T)$ is decreasing on $\left(0, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$. Consequently, $T^* = T_2^*$.

Incorporating the above arguments, we have completed the proof of Lemma 16.