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# The pricing and timing of the option to invest for cash flows with partial information

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**This paper extends real options theory to consider the situation where the mean appreciation rate of cash flows generated by an irreversible investment project is not observable and governed by an Ornstein-Uhlenbeck process. The main purpose of this study is to analyze the impact of the uncertainnty of the mean appreciation rate on the pricing and investment timing of the option to invest under incomplete markets with partial information. We assume that an investor aims to maximize expected discounted utility of lifetime consumption. Based on consumption utility indifference pricing method, stochastic control and filtering theory, under constant absolute risk aversion (CARA) utility, we derive the implied value of cash flows after investment, and then obtain the implied value and the optimal investment threshold of the option to invest, which are determined by a semi-closed-form solution of a free-boundary partial differential equations (PDE) problem. We show that the solutions are independent of the utility time-discount rate. We provide numerical results by finite difference methods and compare the results with those under a fully observable case. Numerical calculations show that partial information leads to a significant loss of the implied value of the option to invest. This loss increases with the uncertainty of the mean appreciation rate. In contrast to standard real options theory, a high volatility of cash flows decreases the implied value of the option to invest as well as the implied information value.**

**Key words:** Partial information, cash flows, consumption utility-based indifference pricing, real options, implied information value.

## INTRODUCTION

This paper extends the real options theory to consider the situation where the mean appreciation rate of cash flows generated by an irreversible investment project is an unobservable random variable. The objective of an investor is to maximize expected utility of consumption over an infinite time horizon with the option to invest in the project, which can be exercised by him at some endogenously chosen time $\tau$. We assume that cash flows evolve according to an arithmetic Brownian motion. Similar to the work of Miao and Wang (2007), we assume that the investor has access to only one risk-free asset. Based on consumption utility-based indifference pricing methodology and real options theory, we derive semi-closed-form solutions for the implied value and investment threshold of the option to invest in the project. We analyze the impact of the uncertainty of the mean appreciation rate on the implied value and threshold. Comparing the results with that under a fully observable case, we quantify the loss of the implied value resulting from partial information and obtain the implied information value.

The real options approach to investment under uncertainty originates from the work of Myers (1977) and presently becomes more and more popular. Major milestones in this development are McDonald and Siegal (1986), Myers and Majd (1990) and Dixit and Pindyck (1994) among others. Recently, Henderson and Hobson (2002), Miao and Wang (2007), Henderson (2007) and Ewald and Yang (2008) study the real options problem

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**Abbreviations: CARA,** Constant absolute risk aversion; **PDE,** partial differential equations; **NPV,** net present value; **IVD,** implied value difference; **IIV,** implied information values.

under incomplete markets by utility-based indifference pricing methodology. However, all the papers assume that the investor has access to full information. In other words, the mean appreciation rate of the project value or cash flows and the driving Brownian motion are assumed to be observable, which is of course unrealistic.

Following Yang and Yang (2010b), the special feature of this paper is that in contrast to the above papers, we shall not assume that the investor can observe the mean appreciation rate and the Brownian motion appearing in the stochastic differential equation on the project cash flows. This situation is called the case of partial information in the literature.

The "partial information" assumption in our model is quite realistic since the mean appreciation rate and the paths of Brownian motions are fictitious mathematical tools, which are of course not observable. On the contrary, the volatility/dispersion parameter for the project cash flows will be observable since one can prove that the volatility is adapted to the filtration generated by the project cash flows.

In fact, financial econometricians agree that it is feasible to obtain good estimates of volatility parameters but much harder to estimate expected returns or the mean appreciation rate (Merton, 1980). Take stock prices for example. According to Brennan (1998), the mean return or mean appreciation rate on common stocks since 1926 is often cited as the best estimate of the mean return, there are good reasons to doubt that this parameter has remained constant for almost three quarters of a century which has witnessed the most dramatic economic, technological and social change of any comparable period in history. Therefore, as a practical matter, it is of interest to consider how the optimal investment allocation is affected by uncertainty over this important, but hard to estimate, parameter.

Our model is closely related with the research line of optimal investment with partial information, say Gennotte (1986), Lakner (1998), Brennan (1998), Yang and Ma (2001), Xiong and Zhou (2007), Monoyios (2007), Monoyios (2008) and Wang(2009), to name just a few. But all of the papers did not consider investing in an irreversible investment project.

The differences between Décamps et al. (2005), Klein (2007, 2009), Yang and Yang (2010a) and this paper are also evident, although these papers do discuss the real options problems with partial information also.

First, we suppose that the mean appreciation rate follows a normal distribution other than a two-point distribution as assumed by these papers. Second and the most importantly, we solve the real options problem based on consumption utility-based indifference pricing methodology, while these papers assume that the investor is risk-neutral. To the best of our knowledge, this paper is most related with Yang and Yang (2010b). However, we assume in this paper that the investor obtains stochastic cash flows rather than a lump-sum payoff upon

investment as assumed by Yang and Yang (2010b). This difference is nuisance in a risk neutral world but significant in our model since we suppose that the investor is risk-averse and thus, one cannot get an equivalent lump-sum payment simply by discounting future cash flows. Last but not least, these papers also measure the implied information value with respect to investment under uncertainty and study its relation to other economical factors. While this study is important, it is also especially challenging. The additional main difficulty is that, we must first derive the filtering estimation of the parameter and the filtering process is valued in the infinite-dimensional space of probability measures: it satisfies the Zakai stochastic partial differential equation (Pham, 2005) and references therein for additional details. For this reason, we provide an elementary setting at the start and a more complicated model will be studied in the future. In the following text, we first develop a more general investment model than indicated above and then focus on the above-mentioned case in order to get a simpler semi-closed-form solution.

## MODEL SETUP

In this section, we develop an investment model under uncertainty with partial information. As studied by Wang (2009) and Yang and Yang (2010a), we consider the case where the investment payoff is given in cash flows form. Time is continuous and the horizon is infinite.

Suppose that we have a given complete probability space $(\Omega, F, P)$, and on it (i) two standard Brownian motion $(Z_t^1, Z_t^2)_{t \geq 0}$, where process $Z^1$ and $Z^2$ are, without loss of generality, assumed to be independent, as well as (ii) a random variable $\mu_0 : \Omega \to \Re$, independent of the process $Z^1$ and $Z^2$, which is normal with prior mean $m_0$ and variance $v_0$.

Similar to the work of Miao and Wang (2007), we consider the situation where an investor has access to only one risk-free asset. Specifically, the investor may borrow or lend at a constant risk-free $r > 0$. In addition, the investor can choose to invest or not to invest in an irreversible investment project, which can be undertaken at a time $\tau$ by him. Investment cost $I > 0$ paid at the exercising time $\tau$. After investment, the investor obtains a perpetual stream of payoffs $X_t : t \geq \tau$. Assume that the cash flow payoff process $(X_t)_{t \geq 0}$ is observable and governed by the following arithmetic Brownian motion equation:

$$dX_t = \mu_t dt + \sigma dZ_t^1, X_0 \text{ given} \tag{1}$$

where volatility $(\sigma)$ is a known positive constant and

$(\mu_t)_{t\geq0}$ is the mean appreciation rate process. This process implies that payoffs can take negative values and it refers to negative values as losses. In contrast to most of the literatures on real options theory but similar to Gennotte (1986) and Lakner (1998), we assume that the mean appreciation rate process is not observable and governed by

$$d\mu_t = (a_0 + a_1\mu_t)dt + b_1 dZ_t^1 + b_2 dZ_t^2,$$    (2)

where $a_0, a_1, b_1$ and $b_2$ are known constants. Equation 1 and 2 say that the project value is subject to two different sources of uncertainty. The first one is, $Z^1$, affects current project value, $dX_t$, and also future expected appreciation rate, $d\mu_t$.

The second one, $Z^2$, affects only future project value through its effect on future expected appreciation. Clearly, if $a_1$ is negative, then process $(\mu_t)_{t\geq0}$ will be an Ornstein-Uhlenbeck process with mean-reverting drift.

We denote by $F \equiv \{F_t : t \geq 0\}$ the $P$-augmentation of the filtration $\sigma(\mu_0, Z_s^1, Z_s^2; 0 \leq s \leq t)$ generated by process $Z^1, Z^2$ and the random variable $\mu_0$, and by $G \equiv \{G_t : t \geq 0\}$ the $P$-augmentation of the filtration $\sigma(X_s; 0 \leq s \leq t)$ generated by the cash flow payoff process $X$. Since the investor can only observe the project value process $(X_t)_{t\geq0}$, and the mean appreciation rate process $(\mu_t)_{t\geq0}$ together with process $(Z_t^1, Z_t^2)_{t\geq0}$ is not observable, the information available to the investor at time $t$ is partial and modeled by the $\sigma$-field $G_t$.

In other words, only $G$-adapted process is observable. Clearly, the assumptions are more reasonable than that assumed by [20] among others, although as seen below, it leads to a much more complicated decision problem.

Let $\tau$ be the stopping time of investment and $T$ be the set of $\{G_t : t \geq 0\}$-stopping times. Denote the wealth process of the investor by $W = (W_t)_{t\geq0}$, which is evidently $G$-adapted thanks to Equation 3 and 4 given below. Let $C$ be the space of $\{G_t : t \geq 0\}$-progressively measurable process $C$, taking value on $[0, \infty)$, such that $\int_0^t |C_s| ds < \infty$ for any $t \geq 0$. In this paper, $C_s$ represents the consumption rate selected by the investor at time $s$ and the consumption is taken from the bank account. We call a consumption process $C$ is admissible, if $C \in C$.

An investor is characterized by his initial wealth $W_0$, a time-discount rate $\beta$ and his preference $U(\cdot)$. He seeks to choose a stopping time $\tau \in T$, here $\tau$ represents the

time to invest in the project, and a consumption process $C \in C$ so as to maximize his expected lifetime time-additive utility of consumption conditional on all available information:

$$J(\tau, C) = E\left[\int_0^\infty \exp(-\beta s)U(C_s)ds \Big| G_0\right], (\tau, C) \in T \times C.$$    (3)

Since the saving is the only financial investment that the investor may use to smooth his consumption over time, this optimization problem is therefore, subject to the following budget constraint:

$$\begin{cases} dW_s = (rW_s - C_s)ds, 0 \leq s < \tau, \\ W_\tau = W_{\tau-} - I, \\ dW_s = (rW_s + X_s - C_s)ds, s > \tau, \\ W_0 \text{ given}, W_s > 0 \text{ for } s \geq 0, \end{cases}$$    (4)

where process $X$ is given by Equation 1 and 2, $U(\cdot)$ is an increasing, concave, twice differentiable von Neuman-Morgenstern utility function.

Clearly, this market is incomplete because the risk-free asset is the unique tradable asset while cash flows are uncertain.

As usual, we do not consider borrowing constraints and transaction costs so as to highlight the effects of market incompleteness and partial information.

In the following, we consider the optimization problem Equation 3 under CARA utility, that is, exponential utility with

$$U(c) = -\exp(-\gamma c)/\gamma, c \in \Re,$$    (5)

where $\gamma > 0$ is the absolution risk aversion parameter. As assumed by Henderson (2002), Ewald and Yang (2008), Miao and Wang (2007) and Henderson (2007) among others, we choose the exponential utility primarily for separating wealth out of the problem, which allows for semi-closed-form solutions for the value of the option to invest and the investment threshold in our partial information model.

Clearly, this is an optimization problem with a partially observable system since the state process $\mu$ is not observable.

According to a separation theorem established by Gennotte (1986) for instance, we can solve this problem by two steps.

We first derive the filtering estimation for the mean appreciation rate process and then an optimal strategy is chosen conditional on these estimates.

## Filtering estimation for the mean appreciation rate

In this here, we apply the filtering technique to estimate

the mean appreciation rate $\{\mu_t\}_{t\geq 0}$. Denote $m_t = E(\mu_t | G_t)$ and $v_t = E[(\mu_t - m_t)^2 | G_t]$. The following lemma follows from theorem 11.1 in Liptser and Shiryayev (1977).

**Lemma 1:** If the conditional distribution $\Psi_{G_0}(x) = P(\mu_0 \leq x | G_0)$ is normal with mean $m_0$ and variance $v_0$, a.s., then the conditional distribution $\Psi_{G_t}(x) = P(\mu_t \leq x | G_t)$ is normal with mean $m_t$ and variance $v_t$, a.s.. From this lemma we see that $m_t$ is the optimal estimate of $\mu_t$ under the observed information $G_t$.

**Lemma 2:** Let $\{\mu_t, X_t\}_{t\geq 0}$ be stochastic processes with differentials given by Equation 1 and 2. Suppose that $P(\mu_0 \leq x | G_0)$ is Gaussian with mean $m_0$ and variance $v_0$. Then $m_t$ and $v_t$ satisfy the following equations:

$$\begin{cases} dm_t &= (a_0 + a_1 m_t)dt + \dfrac{b_1\sigma + v_t}{\sigma^2}(dX_t - m_t dt), \\ \dot{v}_t &= 2a_1 v_t + b_1^2 + b_2^2 - \left(\dfrac{b_1\sigma + v_t}{\sigma}\right)^2, \end{cases} \quad t \geq 0. \tag{6}$$

**Proof:** The conclusion of this lemma follows directly from Theorem 12.1 in Liptser and Shiryayev (1977). □

By separation of variables, $v_t (t \geq 0)$ is solved explicitly as follows:

$$v_t = \begin{cases} v_- + 2\sqrt{\eta^2 + b_2^2\sigma^2}\left(1 - \dfrac{v_0 - v_+}{v_0 - v_-}\exp\left(-\dfrac{2\sqrt{\eta^2 + b_2^2\sigma^2}}{\sigma^2}t\right)\right)^{-1} & \text{if } \eta^2 + b_2^2 \neq 0, \\ \left(\dfrac{1}{v_0} + \dfrac{t}{\sigma^2}\right)^{-1} & \text{if } \eta = b_2 = 0, \end{cases}$$

$$\tag{7}$$

where, $\eta = a_1\sigma^2 - b_1\sigma$, $v_\pm = \eta \pm \sqrt{\eta^2 + b_2^2\sigma^2}$.

**Remark 1:** (i) The magnitude of $v_t$ is a characterization of the accuracy of the estimate $m_t$ of $\mu_t$. If $v_t$ converges to zero as $t \to \infty$, then we say that $m_t$ is a consistent estimate of $\mu_t$. (ii) It is clear that $v_t \to v_+$ as $t \to \infty$. Note that $v_+ = 0$ if and only if $b_2 = 0$ and $\eta \leq 0$. Namely, $m_t$ is not consistent in general.

Now we define the innovation process by its differential

$$d\bar{Z}_t = \frac{1}{\sigma}(dX_t - m_t dt). \tag{8}$$

It is well-known from the theory of filtering that $\bar{Z} = \{\bar{Z}_t\}_{t\geq 0}$ is a standard Brownian motion with respect to the stochastic basis $(\Omega, G_T, P, \{G_t\}_{t\geq 0})$.

According to Equation 8 and 6, we obtain immediately that

$$\begin{cases} dm_t = (a_0 + a_1 m_t)dt + \dfrac{b_1\sigma + v_t}{\sigma}d\bar{Z}_t, \\ dX_t = m_t dt + \sigma d\bar{Z}_t. \end{cases} \tag{9}$$

## Model solutions

In here, we discuss the problem on the pricing and investment timing of the option to invest in a project, of which the project value process is defined by Equation 1 and 2. We provide a PDE to determine the implied value and the optimal investment threshold of the option to invest by the consumption utility indifference pricing method.

Based on the filtering estimation and the separation theorem, equivalently, we restate the optimization problem formulated in previously as follows:

$$\sup_{(\tau, C)\in T\times C} J(\tau, C) \equiv E\left[\int_0^\infty \exp(-\beta s)U(C_s)ds \Big| G_0\right], \tag{10}$$

subject to

$$\begin{cases} dW_s = (rW_s - C_s)ds, 0 \leq s < \tau, \\ W_\tau = W_{\tau-} - I, \\ dW_s = (rW_s - C_s + X_s)ds, s > \tau, \\ dX_s = m_s ds + \sigma d\bar{Z}_s, \\ dm_s = (a_0 + a_1 m_s)dt + \dfrac{b_1\sigma + v_s}{\sigma}d\bar{Z}_s. \\ W_0 > 0, X_0 > 0, m_0 \text{ given and } W_s > 0 \text{ for } s > 0, \end{cases} \tag{11}$$

where $v_s$ is given by Equation 7.

We notice that if investment has taken place, the problem described by Equation 10 subject to Equation 11, which is an optimal consumption-saving problem with stochastic cash flows, which continuously changes the wealth levels and thus the consumption-saving strategy as well. Accordingly, in contrast to Yang and Yang (2010b), the corresponding value function must depend on the current filtering estimation based on the

incomplete information no matter whether investment has taken place or not, which of course makes the problem discussed here more challenging than that studied by Yang and Yang (2010b). In this way, we actually find that all relevant information on the current and future physical state of the economy is summarized in the current wealth level $W$, the current cash flow rate $X$, the filtering estimation $m$ and the time $t$. For this reason, if the current time is $t$, throughout the following text, we denote the value function of Equation 10 subject to 11 after and before investment has taken place, respectively by

$$V^0(W_t, X_t, m_t, t) \text{ and } V(W_t, X_t, m_t, t).$$

If we are able to find a stopping time $\tau^* \in T$ and an admissible consumption process $C^*$, such that

$$V(W_0, X_0, m_0, 0) \equiv J(\tau^*, C^*) = \sup_{(\tau,C) \in T \times C} J(\tau, C), \quad (12)$$

then the stopping time $\tau^*$ is obviously the optimal exercising time of the option to invest. In order to derive the consumption utility indifference price of the option to invest and the value of cash flows, we define:

$$J^0((C_s)_{s \geq t}) \equiv E\left[\int_t^\infty \exp(-\beta(s-t))U(C_s)ds \Big| G_t\right], C \in C, \quad (13)$$

and consider the following optimization problem:

$$\sup_{C \in C} J^0((C_s)_{s \geq t}), \quad (14)$$

Subject to:

$$dW_s = (rW_s - C_s)ds, s \geq t, W_t > 0 \text{ given}, W_s > 0 \text{ for } s > t. \quad (15)$$

Clearly, this optimization problem corresponds to the situation where the investment project never exists.

Noting that the problem formulated by Equation 14 and 15 is a deterministic control problem, similar to the work of Merton (1971), we obtain the following explicit solution by dynamic programming:

$$\begin{aligned}
G(W_t) &\equiv J^0((C_s^*)_{s \geq t}), \\
&= \sup_{C \in C} J^0((C_s)_{s \geq t}; W_t) \quad (16) \\
&= -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r W_t),
\end{aligned}$$

where $C_s^*$ is the optimal consumption rate selected at time $s$, and it is given by;

$$C_s^* = \frac{\beta - r}{\gamma r} + rW_s. \quad (17)$$

According to consumption utility indifference pricing principle, we therefore define the consumption utility indifference price or implied value of the option to invest at time $0$ by number $y$, which satisfies

$$V(W_0, X_0, m_0, 0) = G(W_0 + y). \quad (18)$$

Actually, Equation 18 says that the consumption utility indifference price for adding the option to invest in this project is defined as the compensating variation $y$ of his present wealth that leaves his (indirect) utility unchanged. Similarly, we define the consumption utility indifference price or implied value of cash flows after investment at time $t$ by number $z$, which satisfies

$$V^0(W_t, X_t, m_t, t) = G(W_t + z). \quad (19)$$

**Remark 2:** If the time-discount rate $\beta$ is greater than the bank interest rate $r$, and the investor still selects consumption rate according to Equation 17, then the investor's wealth will get negative after a finite time. Hence the consumption process given by Equation 17 is not admissible. For this reason, in order to get an optimal solution that is taken on the inside (other than boundary) of the admissible domain under the constraint conditions including Equation 11, we assume in this paper that $\beta \leq r$.

In general, if the investment project is not exercised as of date $t$, then the implied value $y$ of the option to invest at time $t$ is given by

$$V(W_t, X_t, m_t, t) = G(W_t + y), \quad (20)$$

where $V(W_t, X_t, m_t, t)$ is the value function defined by

$$V(W_t, X_t, m_t, t) \equiv \sup_{(\tau,C) \in T \times C} E\left[\int_t^\infty \exp(-\beta(s-t))U(C_s)ds \Big| G_t\right], \quad (21)$$

Subject to Equation 11 thanks to the assumption of CARA utility, we will show that the implied value of the option and cash flows at time $t$ are functions only of project value $X_t$, filtering estimation $m_t$ and time $t$. We denote these functions by $z = f(X_t, m_t, t)$ and $y = g(X_t, m_t, t)$ respectively. In the following, we provide two second-order semi-linear homogeneous PDEs with three independent variables, of which functions $f(x, m, t)$ and $g(x, m, t)$ are respective solutions.

To achieve this goal, we solve the investor's decision problem by dynamic programming. By the standard argument, $V^0(W_t, X_t, m_t, t)$ satisfies the following Hamilton-Jacobi-Bellman equation:

$$\sup_{c \geq 0} \left\{ (rw + x - c)V_w^0 + U(c) \right\} + V_t^0 + mV_x^0 + (a_0 + a_1 m)V_m^0$$
$$+ \frac{\sigma^2 V_{xx}^0}{2} + (\sigma b_1 + v_t)V_{mx}^0 + \frac{(\sigma b_1 + v_t)^2 V_{mm}^0}{2\sigma^2} - \beta V^0 = 0, \quad (22)$$

where the subscript of $V^0$ denotes the differentiation with respect to that variable. The usual transversality condition:

$$\lim_{t \to \infty} E[\exp(-\beta t)V^0(W_t, X_t, m_t, t)] = 0,$$

must be satisfied. This is a condition for convergence of integral. The first-order condition for the optimal consumption policy after the option is exercised should be:

$$U'(C) = V_W^0.$$

And then, let's turn to the case before the option is exercised. By Bellman principle, problem Equation 21 can be equivalently written as

$$V(W_t, X_t, m_t, t) = \sup_{(\tau, C) \in T \times C} E \left[ \int_t^\tau \exp(-\beta(s-t))U(C_s)ds \right. \quad (23)$$
$$\left. + \exp(-\beta(\tau - t))V^0(W_{\tau-} - I)\big|G_t \right],$$

subject to Equation 11.

This is a combined stochastic control and optimal stopping problem. According to IV(3.21) in Fleming and Soner (2006), the Hamilton-Jacobi-Bellman equation now has the form

$$\sup_{c \geq 0} \left\{ (rw - c)V_w + U(c) \right\} + V_t + mV_x + (a_0 + a_1 m)V_m$$
$$+ \frac{\sigma^2 V_{xx}}{2} + (\sigma b_1 + v_t)V_{mx} + \frac{(\sigma b_1 + v_t)^2 V_{mm}}{2\sigma^2} - \beta V = 0. \quad (24)$$

Similarly, the first-order condition for the optimal consumption policy before the option is exercised should be:

$$U'(C) = V_W.$$

In order to find a solution $V(w, x, m, t)$ of Equation 24, which is just the value function defined in (21), we must specify the following conditions. First, the no-bubble

condition ensures $\lim_{x \to -\infty} V(W_t, X_t, m_t, t) = G(W_t)$, which states that when cash flows goes to negative infinity before exercising the option, the investor will never exercise the investment option and his value function will be equal to that without the investment project. Second, at a free boundary point $(\bar{w}, \bar{x}, \bar{m}, \tau)$, the following value matching condition and smooth-pasting conditions are imposed, see Equation 14 and 3 for details:

$$\begin{cases} V(\bar{w}, \bar{x}, \bar{m}, \tau) = V^0(\bar{w} - I, \bar{x}, \bar{m}, \tau), \\ V_w(\bar{w}, \bar{x}, \bar{m}, \tau) = V_w^0(\bar{w} - I, \bar{x}, \bar{m}, \tau), \\ V_x(\bar{w}, \bar{x}, \bar{m}, \tau) = V_x^0(\bar{w} - I, \bar{x}, \bar{m}, \tau), \\ V_m(\bar{w}, \bar{x}, \bar{m}, \tau)) = V_m^0(\bar{w} - I, \bar{x}, \bar{m}, \tau), \\ V_t(\bar{w}, \bar{x}, \bar{m}, \tau) = V_t^0(\bar{w} - I, \bar{x}, \bar{m}, \tau). \end{cases} \quad (25)$$

On account of Equation 16, we guess that the value functions before and after the option is exercised take the following forms respectively

$$V^0(w, x, m, t) = -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + f(x, m, t))), \quad (26)$$

$$V(w, x, m, t) = -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + g(x, m, t))), \quad (27)$$

where functions $f$ and $g$ are to be determined. And the corresponding optimal consumption rules should be $c = \frac{\beta - r}{\gamma r} + r[w + f(x, m, t)]$ and $c = \frac{\beta - r}{\gamma r} + r[w + g(x, m, t)]$ respectively. Then substituting them into the Hamilton-Jacobi-Bellman equations, we get the function $f$ and $g$ are the respective solutions of the following second-order semi-linear homogeneous PDEs:

$$rf + x - f_t - mf_x - (a_0 + a_1 m)f_m + \frac{1}{2}\sigma^2[-f_{xx} + \gamma r(f_x)^2]$$
$$+ (\sigma b_1 + v_t)[\gamma rf_x f_m - f_{mx}] + \frac{(\sigma b_1 + v_t)^2}{2\sigma^2}[\gamma r(f_m)^2 - f_{mm}] = 0, \quad (28)$$

$$rg - g_t - mg_x - (a_0 + a_1 m)g_m + \frac{1}{2}\sigma^2[-g_{xx} + \gamma r(g_x)^2]$$
$$+ (\sigma b_1 + v_t)[\gamma rg_x g_m - g_{mx}] + \frac{(\sigma b_1 + v_t)^2}{2\sigma^2}[\gamma r(g_m)^2 - g_{mm}] = 0, \quad (29)$$

subject to the no-bubble condition $\lim_{x \to -\infty} g(x, m, t) = 0$ and the free boundary conditions:

$$\begin{cases} g(\overline{x},\overline{m},\tau) = f(\overline{x},\overline{m},\tau) - I, \\ g_x(\overline{x},\overline{m},\tau) = f_x(\overline{x},\overline{m},\tau), \\ g_m(\overline{x},\overline{m},\tau) = f_m(\overline{x},\overline{m},\tau), \\ g_t(\overline{x},\overline{m},\tau) = f_t(\overline{x},\overline{m},\tau), \end{cases} \qquad (30)$$

where the subscript of functions $f$ and $g$ denote the differentiation with respect to that variable.

Naturally, the optimal exercising time is the first time the implied value of the option to invest becomes less than the implied net return $f(x_t,m_t,t) - I$ getting from the investment. In summary, we obtain the following theorem: Theorem 1, Suppose that $\beta \le r$, $f(x,m,t)$ and $g(x,m,t)$ are solutions of the PDEs formulated by Equation 28, 29 and 30. Define stopping time $\tau^*$ by

$$\tau^* = \inf\left\{t \ge 0 : g(X_t,m_t,t) \le f(X_t,m_t,t) - I\right\}, \quad (31)$$

then $\tau^*$ is the optimal exercising time of the option to invest. The optimal consumption rate is given by

$$\begin{cases} c_t^* = \dfrac{\beta - r}{\gamma r} + r[W_t + g(X_t,m_t,t)], 0 \le t < \tau^*; \\ c_t^* = \dfrac{\beta - r}{\gamma r} + r[W_t + f(X_t,m_t,t)], t \ge \tau^*. \end{cases} \qquad (32)$$

The implied value $F(X_t,m_t,t)$ of the option to invest is given by

$$F(X_t,m_t,t) = \begin{cases} g(X_t,m_t,t), & t < \tau^*; \\ f(X_t,m_t,t) - I, & t \ge \tau^*. \end{cases} \qquad (33)$$

From Theorem 1, while the time-discount rate $\beta$ has impact on the optimal consumption rate and the total consumption utility, we get an interesting conclusion, which is shown this study.

**Corollary 1:** The implied value and the optimal exercising time of the option to invest are independent of the time-discount rate $\beta$ of the consumption utility, Remark 3. This conclusion differs from Dixit and Pindyck (1994), Henderson (2007), Ewald and Yang (2008) and Décamps et al. (2005) and many others, in which the pricing and exercising are based on the ordinary utility indifference instead of the consumption utility indifference we argue in this paper. Naturally, Corollary 1 is a good result we expect, since we do not hope that there are two different prices and exercising times for two investors with the same utility but only different time-discount rates.

According to this point only, we can also think that consumption utility-based indifference pricing method is superior to the ordinary utility-based method.

## A deterministic case

The purpose of this segment is two-fold: First, it aims to give an example for the application of the above conclusions; and second, in so doing, to provide a very simple model that allows us to develop intuition for how to price and exercise the option to invest based on consumption utility indifference pricing method.

As argued by Dixit and Pindyck (1994), although we will be mostly concerned with the ways in which the investment decision is affected by uncertainty, it is useful to first examine the deterministic case, that is $a_0 = a_1 = b_1 = b_2 = v_0 = \sigma = 0$ in Theorem 1. As we see below, there can still be a implied value of the option to invest.

Under this deterministic situation, an investor has a deterministic irreversible investment project. Once he invests in this project, he gets permanent cash flows. The project value is governed by a deterministic ordinary differential equation $dX_t = \mu_0 dt, X_0 > 0$ with known constant $\mu_0 > 0$. Clearly, $m_t = \mu_t = \mu_0$ and $v_t = 0$ for all $t \ge 0$ under this case. Hence, Equation 28 and 29 is simplified to

$$rf - x - f_t - \mu_0 f_x = 0. \qquad (34)$$

$$rg - g_t - \mu_0 g_x = 0. \qquad (35)$$

Combining Equation 34, 35 and 30, we get the solution

$$f(x,m,t) = \frac{x}{r} + \frac{\mu_0}{r^2}, \qquad (36)$$

$$g(x,m,t) = \frac{\mu_0}{r^2} \exp[\frac{r}{\mu_0}(x - rI)]. \qquad (37)$$

**Remark 4:** For the same reason with the argument in remark 2, we assume here that $r > \mu_0$. Making use of the boundary condition $g(x,m,t) = f(x,m,t) - I$ and $g_x(x,m,t) = f_x(x,m,t)$ again, we obtain the optimal investment threshold

$$x^* = rI. \qquad (38)$$

According to Equation 31, we get the optimal exercising time;

$$\tau^* = \inf\{t \geq 0 : X_t \geq x^*\} \tag{39}$$
$$= \max\left\{\frac{rI - x_0}{\mu_0}, 0\right\}.$$

Consequently, the optimal investment threshold $x^*$ divides the state space $[0, \infty)$ of the project value $X$ into two regions: the investment region, $[x^*, \infty)$, and the waiting region, $[0, x^*)$. Substituting Equation 36, 37 into 32, we obtain the optimal consumption rate at time $t$

$$\begin{cases} c_t^* = \dfrac{\beta - r}{\gamma r} + \dfrac{\mu_0}{r} \exp[\dfrac{r}{\mu_0}(x - rI)] + rW_t, t < \tau^*; \\ c_t^* = \dfrac{\beta - r}{\gamma r} + x + \dfrac{\mu_0}{r} + rW_t, t \geq \tau^*. \end{cases} \tag{40}$$

where $\tau^*$ is derived in Equation 39. In addition, thanks to Theorem 1, the implied value is given by

$$F(X_t, m_t, t) = \begin{cases} \dfrac{\mu_0}{r^2} \exp[\dfrac{r}{\mu_0}(x - rI)], & t < \tau^*; \\ \dfrac{x}{r} + \dfrac{\mu_0}{r^2} - I, & t \geq \tau^*. \end{cases} \tag{41}$$

As expected, both the implied value and the optimal investment threshold are independent of the risk aversion parameter $\gamma$ and the time-discount rate $\beta$.

**A stochastic case**

We will now return to the stochastic case, in which $v_0 > 0, \sigma > 0$ but $a_0 = a_1 = b_1 = b_2 = 0$. Thus, under this case, the mean appreciation rate is a Gaussian random variable with mean $m_0$ and variance $v_0$, which is independent of time.

**Partial information**

This is a special case discussed previously; we can naturally make use of above-mentioned conclusions to derive all the desired results. For example, according to Lemma 3, we have

$$\begin{cases} dm_t = \dfrac{v_t}{\sigma^2}(dx_t - m_t dt) = \dfrac{v_t}{\sigma} d\bar{Z}_t, \\ \dot{v}_t = -\left(\dfrac{v_t}{\sigma}\right)^2, \end{cases} \quad t \geq 0, \tag{42}$$

to which we get explicit solutions

$$\begin{cases} m_t = \left(\dfrac{m_0}{v_0} + \dfrac{x_t}{\sigma^2} - \dfrac{x_0}{\sigma^2}\right) v_t, \\ v_t = \dfrac{v_0 \sigma^2}{v_0 t + \sigma^2}, \end{cases} \tag{43}$$

The proof of Equation 43 is simple, so we omit here. Thanks to Equation 43, we find that the estimation $m_t$ is a function of $t$ and $X_t$. For this reason, the corresponding value function $V^0$ and $V$ depend only on $W, X$ and time $t$, and thus can be written as $V^0(W_t, X_t, t)$ and $V(W_t, X_t, t)$. By this way, we derive more simple Hamilton-Jacobi-Bellman equations than Equations 22 and 24 as follows:

$$\sup_{c \geq 0}\{(rw + x - c)V_w^0 + U(c)\} + V_t^0 + mV_x^0 + \frac{\sigma^2 V_{xx}^0}{2} - \beta V^0 = 0, \tag{44}$$

$$\sup_{c \geq 0}\{(rw - c)V_w + U(c)\} + V_t + mV_x + \frac{\sigma^2 V_{xx}}{2} - \beta V = 0, \tag{45}$$

with transversality condition

$$\lim_{t \to \infty} E[\exp(-\beta t)V^0(W_t, X_t, t)] = 0,$$

the no-bubble condition $\lim_{x \to -\infty} V(W_t, X_t, t) = G(W_t)$ and the following value matching condition and smooth-pasting conditions:

$$\begin{cases} V(\bar{w}, \bar{x}, \tau) = V^0(\bar{w} - I), \\ V_w(\bar{w}, \bar{x}, \tau) = V_w^0(\bar{w} - I), \\ V_x(\bar{w}, \bar{x}, \tau) = V_x^0(\bar{w} - I), \\ V_t(\bar{w}, \bar{x}, \tau) = V_t^0(\bar{w} - I). \end{cases} \tag{46}$$

As we did previously, we guess that the value function has the form

$$V^0(w, x, t) = -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + f(x, t))), \tag{47}$$

$$V(w, x, t) = -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + g(x, t))), \tag{48}$$

Further, similar but different to the analysis of Wang (2009), we have;

$$f(x,t) = \frac{x}{r} + \frac{m_t}{r^2} - \frac{\gamma\sigma^2}{2r^2} + h(t), \tag{49}$$

where $h(t)$ measures the impact of the incomplete information on the certainty equivalent wealth after the option is exercised. And then, we get function $h$ and $g$ are solutions of the following equations:

$$rh = h_t - \frac{\gamma\sigma^2 v_0}{r^2(v_0 t + \sigma^2)} - \frac{\gamma\sigma^2}{2r^3}(\frac{v_0}{v_0 t + \sigma^2})^2, \tag{50}$$

$$rg = \left(\frac{m_0}{v_0} + \frac{x}{\sigma^2} - \frac{x_0}{\sigma^2}\right)\frac{v_0\sigma^2}{v_0 t + \sigma^2} g_x + \frac{\sigma^2}{2}\left(g_{xx} - \gamma r g_x^2\right) + g_t, \tag{51}$$

subject to the no-bubble condition $\lim_{x\to-\infty} g(x,t) = 0$ and the boundary conditions

$$\begin{cases} g(\bar{x},\tau) = f(\bar{x},\tau) - I, \\ g_x(\bar{x},\tau) = f_x(\bar{x},\tau), \\ g_t(\bar{x},\tau) = f_t(\bar{x},\tau). \end{cases} \tag{52}$$

Additionally, in that there is no further belief updating once time $t$ goes to positive infinity, the termination condition for $h$ is

$$\lim_{t\to+\infty} h(t) = 0. \tag{53}$$

To sum up, following Theorem 1, we present a similar result as follows.

**Theorem 2:** Suppose that $\beta \le r$ and $f(x,t)$ and $g(x,t)$ are solutions of the PDE formulated by Equations 50 and 51. Define stopping time $\tau^*$ by

$$\tau^* = \inf\left\{t \ge 0 : g(X_t,t) \le f(X_t,t) - I\right\}, \tag{54}$$

then $\tau^*$ is the optimal exercising time of the option to invest.
   The optimal consumption rate is given by

$$\begin{cases} c_t^* = \frac{\beta - r}{\gamma r} + r[W_t + g(X_t,t)], & 0 \le t < \tau^*; \\ c_t^* = \frac{\beta - r}{\gamma r} + r[W_t + f(X_t,t)], & t \ge \tau^*. \end{cases} \tag{55}$$

The impled value $F(X_t,t)$ of the option to invest is given by;

$$F(X_t,t) = \begin{cases} g(X_t,t), & t < \tau^*; \\ f(X_t,t) - I, & t \ge \tau^*. \end{cases} \tag{56}$$

## Net present value (NPV) of cash flows with partial information

In this here, we derive the corresponding net present value (NPV) of cash flows with partial information and compare it with the implied value of cash flows derived in this paper.
   By this way, we can make clear the effect of the risk aversion on the value of cash flows in a new viewpoint.
   Following Friedman (1957) and Hall (1978), the net present value of cash flows is defined as the expectation of the sum of discounted cash flows at the risk-free rate. That is,

$$f^0(x,t) \equiv E(\int_t^\infty e^{-r(s-t)} X_s ds \mid G_t). \tag{57}$$

Since we take $a_0 = a_1 = b_1 = b_2 = 0$ here, it follows from Equation 9 and 42 that

$$\begin{cases} dm_t = \frac{v_t}{\sigma} d\bar{Z}_t, \\ dX_t = m_t dt + \sigma d\bar{Z}_t. \end{cases} \tag{58}$$

Thus, we may solve $f^0(x,t)$ in the following way:

$$\begin{aligned} f^0(x,t) &= \int_t^\infty e^{-r(s-t)} x_t ds + \int_t^\infty e^{-r(s-t)} \int_t^s E(m_u \mid G_t) du ds \\ &= \frac{x_t}{r} + \int_t^\infty e^{-r(s-t)} \int_t^s E((m_t + \int_t^u \frac{v_s}{\sigma} dZ_s) \mid G_t) du ds \\ &= \frac{x_t}{r} + \int_t^\infty e^{-r(s-t)} \int_t^s m_t du ds \\ &= \frac{x_t}{r} + \frac{m_t}{r^2}. \end{aligned} \tag{59}$$

Now, we can rewrite the implied value of cash flows given in (49) as follows:

$$f(x,t) = f^0(x,t) - \frac{\gamma\sigma^2}{2r^2} + h(t). \tag{60}$$

Since the payoffs are given in the form of cash flows, the investor continues to face undiversifiable idiosyncratic cash flow risk after investment. From Equation 59, we can see that this net present value does not incorporate the effect of idiosyncratic risk, and the certainty-equivalent (risk-adjusted) wealth $f(x,t)$ decreases in the

the risk aversion coefficient $\gamma$ and also in income volatility $\sigma$. let $Lf(x,t) \equiv f^0(x,t) - f(x,t) = \frac{\gamma\sigma^2}{2r^2} - h(t)$, which represents the difference between NPV of cash flows and the implied value of cash flows after the option is exercised. Notice that if $\gamma = 0$, then $h(t) = 0$ and thus $Lf(x,t) = 0$. We get that the estimation risk has no effect on the value of cash flows after the option is exercised when the investor is risk neutral.

**Implied information value**

Intuitively, an investor who only has access to partial information instead of full information usually makes incorrect decisions on investment timing and thus, partial information leads to a loss of the implied value of the option to invest. We call the loss the implied information value since it represents the cost to pay for getting full information that leaves his (indirect) utility unchanged. Naturally, we want to know how much the implied information value is. This segment will answer this question. To this end, we first present a corollary of theorem 2 under full information assumption, and then provide a method to compute the implied information value. As said before, the initial value $\mu_0$ is a random variable but is known to the investor at the outset who has full information. Thus, $\mu_0$ can be considered as a known constant, which corresponds to the segment "partial information" with $m_0 = \mu_0$ and $v_0 = 0$. Under this situation, the controlled system and object function are independent of time and so, the pricing and timing of the option to invest are determined by the present project value and do not depend on time. We therefore obtain the following corollary of Theorem 2:

**Corollary 2:** Suppose that $\beta \leq r$, the implied value of cash flows is given by

$$f(x) = \frac{x}{r} + \frac{\mu_0}{r^2} - \frac{\gamma\sigma^2}{2r^2}. \tag{61}$$

and $g(x; \mu_0)$ is a solution to the free boundary problem:

$$rg = \mu_0 g_x(x) + \frac{1}{2}\sigma^2\left(g_{xx} - \gamma r(g_x)^2\right) \tag{62}$$

subject to the no-bubble condition $\lim_{x \to -\infty} g(x) = 0$ and the boundary conditions

$$\begin{cases} g(\bar{x}) = f(\bar{x}) - I, \\ g_x(\bar{x}) = f_x(\bar{x}) = \frac{1}{r}. \end{cases} \tag{63}$$

Define stopping time $\tau^*$ by

$$\tau^* = \inf\left\{t \geq 0 : g(X_t; \mu_0) \leq f(X_t; \mu_0) - I\right\}, \tag{64}$$

then $\tau^*$ is the optimal exercising time of the option to invest with full information. The optimal consumption rate is given by

$$\begin{cases} c_t^* = \frac{\beta - r}{\gamma r} + r[W_t + g(X_t; \mu_0)], 0 \leq t < \tau^*; \\ c_t^* = \frac{\beta - r}{\gamma r} + r[W_t + f(X_t; \mu_0)], t \geq \tau^*. \end{cases} \tag{65}$$

The impled value of the option to invest with full information depends on the random variable $\mu_0$ and is given by;

$$F^{Full}(X_t; \mu_0) = \begin{cases} g(X_t; \mu_0), & t < \tau^*; \\ f(X_t; \mu_0) - I, & t \geq \tau^*. \end{cases} \tag{66}$$

This corollary tells us that, for an investor with full information, the implied value $F^{Full}(X_t; \mu_0)$ is a function of the random variable $\mu_0$ as well as the current value $X_t$ of cash flows. However, under partial information assumption, the implied values $F(X_t, t)$ depends on $m_0$ and $v_0$ but not on $\mu_0$ as seen in Theorem 2. More often than not, $F^{Full}(X_t, \mu_0)$ will be greater than $F(X_t, t)$ since full information is helpful to make a right choice. But one cannot expect that it happens all the time since if the investor with full information is 'unlucky', he might be given a 'bad' $\mu_0$, that is one $\mu_0$ small enough, which leads to $F^{Full}(X_t; \mu_0) < F(X_t, t)$. For this reason, we consider as the implied information value for the mathematical expectation, instead of a single sample point, of the implied value difference (IVD) defined by $F^{Full}(X_t; \mu_0) - F(X_t, t)$. Combining with the consumption utility-based indifference pricing method, we define the implied information value by

$$IIV \equiv \int_{-\infty}^{\infty} F^{Full}(X_t, u)\phi(u)du - F(X_t, t), \tag{67}$$

where $\phi(\cdot)$ is the normal probability density function with mean $m_0$ and variance $v_0$.

Since there is no closed-form solution of Equation 51 with Equation 52 and 62 with Equation 63, we provide numerical calculations in the next segment.

**Table 1.** Impact of changes in prior variance ($v_0$) on implied values, investment thresholds, IIV and standard deviations of the IVD. Baseline parameter values are set as $r = 0.05, X_0 = 0.5, I = 10, m_0 = 0.02, \sigma = 0.3$ and $\gamma = 1$.

| $v_0$ | $0.005^2$ | $0.01^2$ | $0.015^2$ | $0.02^2$ | $0.03^2$ | $0.04^2$ |
|---|---|---|---|---|---|---|
| Implied value | 4.7926 | 4.7531 | 4.6859 | 4.6115 | 4.4965 | 4.4792 |
| Investment threshold | 1.9500 | 2.0000 | 2.0000 | 2.0500 | 2.1500 | 2.2500 |
| IIV | 2.2922 | 2.3286 | 2.3727 | 2.4991 | 2.5873 | 2.6160 |
| Standard deviation of IVD | 7.4635 | 7.4636 | 7.4492 | 7.4650 | 7.4753 | 7.5581 |

## Comparative statics and numerical simulations

Here, we perform a numerical simulation of the results from the segment "a stochastic case". The baseline parameter values are set as follows

$$r = 0.05, X_0 = 0.5, I = 10, m_0 = 0.02, v_0 = 0.02^2, \sigma = 0.3$$

and $\gamma = 1$.

First, we compute the implied value $f(x,t)$ of cash flows given in Equation 49, then solve the free boundary problem Equation 51 with Equation 52 and 62 with Equation 63 respectively by a finite difference method. After that, we get the implied values and investment thresholds of the option to invest with partial and full information respectively. In order to obtain the implied information value under a given prior normal distribution, $N(m_0, v_0)$, we compute the implied value $F(X_0, \mu_0)$ for every $\mu_0 \in A \equiv \{m_0 - 4\sqrt{v_0} + id_m : i = 0,1,\cdots,5000\}$, where $d_m \equiv 0.0002$. We then utilize a random number generator to generate 100,000 samples of the distribution $N(m_0, v_0)$ and hence get 100,000 samples of the implied value $F(X_0, \mu_0)$. Naturally, if the sample of $\mu_0$ does not belong to set $A$, we get an approximate implied value by linear interpolation. Following that, we obtain the sample distribution from the 100,000 samples of $F(X_0, \mu_0)$. In the end, we get the implied information values and the standard deviation of the IVD, that is $F^{Full}(X_t, \mu_0) - F(X_t, t)$, by (67). Most of the results are presented in Tables 1 and 5, which explain the impact of changes in several different parameters respectively, on implied values and investment thresholds of the option to invest with partial information, implied information values (IIV) and standard deviations of the IVD. An implementation of all algorithms used in this segment in Matlab is available upon request.

The numerical results show that the IIV is usually significant and in particular, IIV as well as the standard deviations of IVD $[(F^{Full}(X_t, \mu_0) - F(X_t, t)]$ gets bigger with the increase of $v_0$, that is the uncertainty of the mean appreciation rate of cash flows. Also, the implied values (investment thresholds) of the option to invest decrease (increase) slowly under a high-uncertainty mean appreciation rate of cash flows. These conclusions are quite in agreement with economic intuition but indicate that the loss for an investor with partial information is probably bigger than that we normally think.

For example, according to Table 1, IIV is up to 47.8% of the implied value of the option to invest even for a very small value $v_0 = 0.005^2$. By the way, although the results are computed at time t=0, it actually holds anytime. In particular, since the estimation is not consistent, thanks to Remark 1. The effects of the estimation error of the mean appreciate rate cannot be ignored even if the realized cash flows are observed continuously over an infinite time period Gennotte (1986).

It is well-known that in a risk-neutral world, a higher volatility of project values necessarily leads to a bigger implied value of the option to invest. However, Table 2 indicates that, both IIV and the implied values of the option to invest decrease with a growth of the volatility of cash flows. The intuition is as follows. The investor in our model is risk-averse and thus a high volatility, which means a high risk, might conversely decrease the implied value of the option to invest and IIV as well at last. In addition, as we expect, an investor always delay investment with regard to a higher volatility according to Table 2.

Tables 3 and 4 present the impacts of changes in risk aversion index $\gamma$ and the prior mean $m_0$ of the mean appreciation rate respectively. The results say that for a more risk averse investor or given a smaller prior mean, the implied value of the option to invest decrease quickly but increase the investment threshold substantially. However, IIV are not globally monotonic with respect to the variation of risk aversion index $\gamma$, which implies that the effect of the risk aversion level on the implied information value are ambiguous. But it is shown that the more the prior mean, the smaller the IIV. Table 5 means the implied values of the option to invest increase quickly but the investment threshold raise very slowly as the

**Table 2.** Impact of changes in volatility ($\sigma$) of project valus on implied values, investment thresholds, IIV and standard deviations of the IVD. Baseline parameter values are set as $r = 0.05, X_0 = 0.5, I = 10, m_0 = 0.02, v_0 = 0.02^2$ and $\gamma = 1$.

| $\sigma$ | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.4 |
|---|---|---|---|---|---|---|
| Implied value | 5.9803 | 5.4419 | 5.4075 | 5.1499 | 4.6115 | 3.6114 |
| Investment threshold | 0.7000 | 1.0000 | 1.3500 | 1.7000 | 2.0500 | 2.9000 |
| IIV | 5.3667 | 4.9800 | 3.9187 | 3.0305 | 2.4991 | 1.5951 |
| Standard deviation of IVD | 12.4732 | 11.4505 | 10.1666 | 8.7945 | 7.4798 | 5.2051 |

**Table 3.** Impact of changes in risk aversion ($\gamma$) on implied values, investment thresholds, IIV and standard deviations of the IVD. Baseline parameter values are set as $r = 0.05, X_0 = 0.5, I = 10, m_0 = 0.02, v_0 = 0.02^2$ and $\sigma = 0.3$.

| $\gamma$ | 0.3 | 0.5 | 1 | 1.5 | 2 | 2.3 |
|---|---|---|---|---|---|---|
| Implied value | 9.8804 | 7.9981 | 4.6115 | 2.6583 | 1.5621 | 1.1399 |
| Investment threshold | 1.5500 | 1.7000 | 2.0500 | 2.4500 | 2.8500 | 3.1000 |
| IIV | 1.9437 | 2.2393 | 2.4991 | 2.1837 | 1.7155 | 1.4494 |
| Standard deviation of IVD | 10.9334 | 9.8809 | 7.4798 | 5.4623 | 3.9200 | 3.1958 |

**Table 4.** Impact of changes in prior mean ($m_0$) on implied values, investment thresholds, implied information values (IIV) and standard deviations of the implied value differences (IVD). Baseline parameter values are set as $r = 0.05, X_0 = 0.5, I = 10, v_0 = 0.02^2, \sigma = 0.3$ and $\gamma = 1$.

| $m_0$ | 0 | 0.01 | 0.015 | 0.02 | 0.025 | 0.03 |
|---|---|---|---|---|---|---|
| Implied value | 2.3959 | 3.3869 | 3.9683 | 4.6115 | 5.3170 | 6.0942 |
| Investment threshold | 2.3000 | 2.1500 | 2.1000 | 2.0500 | 2.0000 | 1.9500 |
| IIV | 4.6989 | 3.6998 | 3.1088 | 2.0499 | 1.7751 | 1.0026 |
| Standard deviation of IVD | 7.4660 | 7.4617 | 7.4477 | 7.4633 | 7.4591 | 7.4665 |

**Table 5.** Impact of changes in project initial value ($X_0$) on implied values, investment thresholds, IIV and standard deviations of the IVD. Baseline parameter values are set as $r = 0.05, I = 10, m_0 = 0.02, v_0 = 0.02^2, \sigma = 0.3$ and $\gamma = 1$.

| $X_0$ | 0.1 | 0.5 | 1 | 1.5 | 2 | 2.5 |
|---|---|---|---|---|---|---|
| Implied value | 3.3099 | 4.6115 | 7.1267 | 11.2613 | 18.8459 | 28.0726 |
| Investment threshold | 2.0500 | 2.0500 | 2.0500 | 2.1000 | 2.1500 | 2.1500 |
| IIV | 2.1226 | 2.4991 | 3.0009 | 3.3720 | 1.9749 | 0.6542 |
| Standard deviation of IVD | 6.0204 | 7.4798 | 9.8587 | 12.8551 | 15.8128 | 17.9882 |

current cash flow value increases. Moreover, the impaction of the current cash flow value on IIV is not monotonic also.

Following that, we provide a figure to describe the probability densities of $F(X_0, 0)$ and $F^{Full}(X_0, \mu_0)$, that is the implied values of the option to invest with partial and full information respectively. The parameter values are set as $r = 0.05, X_0 = 15, I = 10, m_0 = 0.02, v_0 = 0.0004, \sigma = 0.4$

and $\gamma = 1$. According to the segment "a stochastic case", the implied value $F(X_0, 0)$ is deterministic for the given parameters but $F^{Full}(X_0, \mu_0)$ is a random variable. Therefore, as seen in Figure 1a, the probability density of $F(X_0, 0)$ is represented by a vertical line while $F^{Full}(X_0, \mu_0)$ is described by a curve. It is seen in the figures that $F(X_0, 0) > F^{Full}(X_0, \mu_0)$ sometimes but this does not
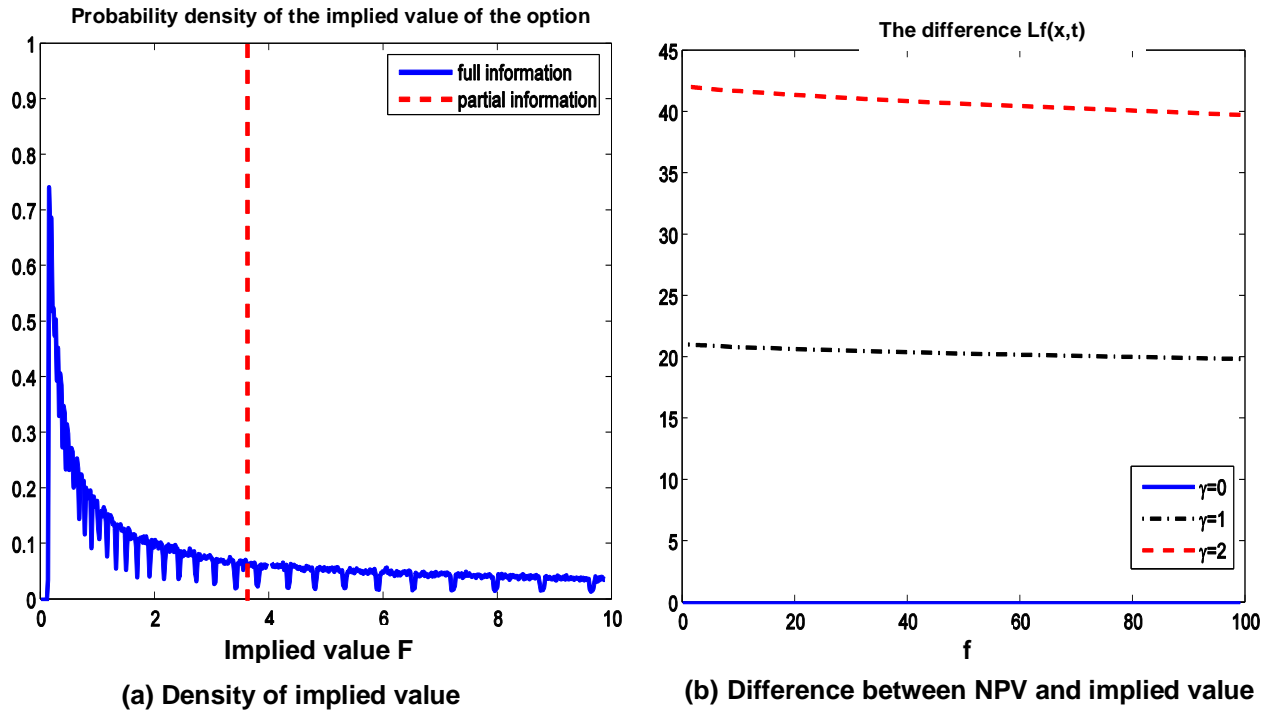
**(a) Density of implied value**

**(b) Difference between NPV and implied value**

**Figure 1.** a, the probability density of the implied value of the option to invest with full and partial information respectively. Baseline parameter values are set as $r = 0.05, X_0 = 0.5, I = 10, m_0 = 0.02, \sigma = 0.4, v_0 = 0.02^2$ and $\gamma = 1$; b, the difference $Lf(x,t)$ between NPV of cash flows and the implied value of cash flows after the option is exercised under different risk aversion level $\gamma = 0, \gamma = 1$, and $\gamma = 2$.

indicate that information is worthless since $F(x_0, 0)$ represents a mathematical expectation while $F^{Full}(X_0, \mu_0)$ is a sample point.

Finally, Figure 1(b) plots the difference $Lf(x,t)$ between NPV of cash flows and the implied value of cash flows after the option is exercised under risk aversion level $\gamma = 0, \gamma = 1$ and $\gamma = 2$. As we expect, the higher the risk aversion, the bigger the difference. In particular, for a risk neutral investor that is $\gamma = 0$, the difference is zero.

## Conclusions

To the best of our knowledge, the current real options approach to investment under uncertainty is based on one of the three assumptions: Complete markets, risk neutrality and full information. However, in practice, it is common for a risk-averse investor to invest under an incomplete market with partial information. For this reason, we discuss in this paper the real options problem under an incomplete market with partial information for a risk-averse investor. We extend the real options theory to consider the situation where the mean appreciation rate of cash flows generated by an irreversible investment is

not observable and is governed by an Ornstein-Uhlenbeck process. We analyze the impact of the uncertainty of the mean appreciation rate on the pricing and investment timing of the option to invest under an incomplete market with partial information. To this end, we assume that an investor aims to maximize expected discounted utility of lifetime consumption. Based on consumption utility indifference pricing method, we obtain under CARA utility the implied value and the optimal investment threshold of the option to invest, which are determined by a semi-closed-form solution of a free-boundary PDE problem. The solution indicates that optimal investment strategy and the implied (information) value are independent of the time-discount rate of the utility.

Since there is no closed-form solution of the free-boundary PDE problem, we provide numerical results by finite difference methods and compare the results with those in the fully observable case. Numerical calculations show that partial information leads to a significant loss of the implied value of the option to invest. This loss is naturally called by us the implied information value. Numerical simulations explain that the implied information value is usually significant and increase quickly with the uncertainty of the mean appreciation rate of cash flows. For instance, numerical analysis shows the implied

information value is more than a half of the implied value of the option to invest, even under middle-uncertainty of the mean appreciation rate in cash flows, let alone high-uncertainty.

In contrast to standard real options theory, our numerical results say that a growth of volatility of cash flows decrease the implied value of the option to invest as well as the implied information value. This happens because a high volatility leads to a high risk at the same time, which decreases the implied value at last for a risk-averse investor.

Several opportunities exist for future research. Firstly, in many practical projects, the initial investment cost is not predictable, and uncertainty about it can be incorporated into our model. Secondly, in order to separate wealth out of the problem, this paper chooses the exponential utility. However, as highlighted by Rouge and El Karoui (2000), this is not always desirable since it is unrealistic to assume that investors with different endowments have the same attitude toward risk. Consequently, it is worth considering the same problem under CRRA utility that is power utility. Finally, there is no tradeable risk asset in our model to hedge the cash flow risk. This is a shortcoming to be overcome. It is our hope that our work will encourage future research in these interesting and important directions.

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