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Full Length Research Paper

Continuous-time evolutionary stock and bond markets with time-dependent strategies

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This paper develops a general continuous-time evolutionary finance model with time-dependent strategies based on evolutionary game theory. We show that the continuous model, which is a limit of a general discrete model, is well-defined and if there exists one completely diversified strategy in the market, then there is no sudden bankruptcy. We study in detail, a deterministic evolutionary bond market and certify that a bond market is evolutionary stable if and only if the total returns across all assets are the same. By this way, we derive an explicit expression for the bond valuation and provide an approach to recover the benchmark interest rate from an effective bond market.

Key words: Evolutionary finance, evolutionary bond market, continuous-time model, time-dependent strategies.

INTRODUCTION

The Darwinian principle "Survival of the Fittest" is well known to evolutionary biologists but rather less known for its applicability in financial market theory. However, this major principle, as applied to financial markets, means nothing else that those traders with the most successful trading strategies will dominate the market at last, after an evolutionary process has taken place. This evolutionary process can be understood as a process of adaptation and imitation, rather than a process of inheritance in evolutionary biology. From an evolutionary point, the market is completely determined by the corresponding *evolutionary stable* trading strategy. Evolutionary financial market models have been considered in Blume and Easley (1992), Evstigineev et al. (2006), and Farmer and Lo (1999). The main point in setting up an evolutionary financial market model is the specification of an evolutionary dynamic which determines the market shares of the relevant trading strategies in time. In general, such a dynamic will depend on the stochastic payoffs, dividends or prices of the underlying assets, as well as the trading strategies, which are assumed to be adapted to the underlying information.

The models developed by Evstigineev et al. (2006) and Hens and Schenk-Hoppé (2005) assume that stochastic dividends or payoffs of the underlying assets are exogenously given, but that in contrast to other models, the asset prices are determined by the trading strategies and a market clearing condition.

This paper employs a similar approach as in Evstigineev et al. (2006) and Hens and Schenk-Hoppé (2005) but sets up a model in continuous time rather than in discrete time. The choice of continuous time brings with it the usual technical problems which lie in the analytical formulation of the model, in particular in a probabilistic framework, but has the major benefit. In particular, the methods from classical analysis such as PDE and stochastic calculus become applicable and provide powerful tools for the solution of problems. It is therefore necessary to extend discrete time evolutionary Evstigineev et al.'s (2006) finance models to continuous time framework.

As a first step into this direction, Yang and Ewald (2008) consider a model in which the trading strategies are assumed to be fix-mix, that is, the relative budget fractions are constant in time. In this model, Yang and Ewald (2008) set up a continuous time stochastic dynamic which describes the evolution of market shares in a population of finitely many trading strategies. Yang and Ewald (2008) identify *evolutionary stable* investment strategies, that is, those strategies that prevent entrants to the financial market from gaining wealth in the long

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run. Under the assumption that the relative dividends are first-order stationary and ergodic, Yang and Ewald (2008) derive an *evolutionary stable* investment rule. Fix strategies are simple and also suitable in some economic environments but are of course too restrictive.

Along this research line, Buchmann and Weber (2007) derive a continuous time approximation of the evolutionary market selection model of Blume and Easley (1992). Conditions on the payoff structure of the assets are identified that guarantee convergence. It is shown that the continuous time approximation equals the solution of an integral equation in a random environment. For constant asset returns, the long-run asymptotic behavior is discussed in detail.

However, the paper by Buchmann and Weber (2007) is substantially simpler than what studied in this paper. First Buchmann and Weber (2007) consider a market with only short-lived assets and thus capital gains are omitted, which is much less interesting. Secondly, to guarantee convergence, Buchmann and Weber (2007) impose an unnatural condition on the payoff structure of the assets and so, strictly speaking, its continuous-time model is not a limit of the model of Blume and Easley (1992). Lastly, the assumption of constant asset returns, which is much simpler than the one supposed by this paper, is extremely restrictive and thus its strategies are fix or time-invariant as well.

Taking a step further into the continuous evolutionary finance model, this paper considers a model in which strategies are time-dependent[1] rather than time-invariant. A continuous time approximation of the evolutionary market selection of a general discrete-time model is first derived, which is a generalization of Evstigineev et al. (2006). It is shown that the continuous model is well-defined and only if one of the strategies is completely diversified - this condition almost imposes no restrictiveness, there is no sudden bankruptcy in the market. Secondly, a bond market is studied in which the dividend process paid by each asset is deterministic and prices and wealth vary due to market interaction. It is certified that a bond market is *evolutionary stable* if and only if all the *total return*s of bonds defined in this paper are equal to each other although the same *total return* may change along the time. Any other market can be invaded just by a portfolio that invests all wealth in an asset, which pays off the largest *total return*. When introduced on the market with arbitrarily small initial wealth, this portfolio increases its market share at the incumbent's expense. By this way, the necessary and sufficient conditions are derived for the *evolutionary stable* portfolio rule. It is shown that a bond market is *evolutionary stable* if and only if each bond is evaluated by an improper integral in which the integrand is a discounted value of the dividend payoff with the discount rate being market consumption parameter. A formula is provided to evaluate the discount rate or the market consumption rate based on the trading prices in a relative effective bond market. By this way, an approach to compute benchmark interest rate is presented.

The structure of the article is as follows. This study sets up a general discrete-time market selection dynamic where the length of the trading time interval is arbitrarily positive. Then a continuous-time evolutionary stock market model is derived and discussed. Furthermore, it investigates an evolutionary bond market in which the dividend process is deterministic, after which two formulas to price bonds were provided and the benchmark interest rate was recovered respectively. Finally, the main conclusions of this paper are summarized.

CONTINUOUS-TIME MARKET SELECTION PROCESS WITH TIME-DEPENDENT STRATEGIES

Following Lucas (1978), this paper introduces an infinite horizon asset market model. First, a discrete market model is established with an arbitrary trading interval and then a general continuous-time model is derived by taking a limit of that the trading interval converges to zero.

Let (Ω, F, P) be a probability space endowed with the filtration $(F_t)_{0\le t\le\infty}$, where F_t is an information filtration for market participants up to time t. All random variables and stochastic processes in this paper will be defined on this base.

Consider an asset market with $K \ge 1$ long-lived assets and a single perishable consumption good. The assets are indexed by $k \in \Lambda \equiv \{1, \cdots, K\}$. Time is discrete and denoted by $t \in \Pi \equiv \{l\Delta t,\ l = 0, 1, 2, \cdots\}$, where Δt is the length of the time interval. In this market there are finitely many players (investors) and each player plays one strategy. All players or strategies indexed by $i = 1, \cdots, I, I \ge 2$ compete with each other for the market capital, and the amount of the wealth managed by strategy i is F_t-adapted and denoted by $w_t^i, i = 1, \cdots, I$. Each asset pays off a dividend in cash or just a single perishable consumption good as implied by Lucas (1978). As seen further, whether the dividend is cash or good, does not matter only if it is supposed to be totally consumed. The amount of the dividend paid by asset k at period t is F_t-adapted and denoted by D_t^k. In this paper, the following assumption is imposed:

Assumption 1: All strategies have a common consumption rate at each period t, denoted by $c_t(\Delta t)$,

[1]In a time-dependent strategy, the fraction of the remaining wealth after consumption, which the strategy assigns to the purchase of an asset in one period, may change with time. On the contrary, in a fix, time-invariant or time-independent strategy, the fraction must keep unchanged. Of course, the time-dependent strategy is much more reasonable than the latter.

which is F_t – adapted satisfying $0 < c_t(\Delta t) < 1, t \in \Pi$.

The assumption on common consumption rate is obviously necessary in order to compare performances among strategies, since one of the aims of the paper is to study what strategy will dominate the market at last. If, on the contrary, the consumption rate is different, then the strategy dominates the market probably also because it consumes less. Moreover, the assumption that $0 < c_t(\Delta t) < 1$ says every strategy or player must consume at a strictly positive consumption rate at each period but is not permitted to consume all his wealth.

Denote by $\lambda_{t,k}^i$ the fraction of the remaining wealth after consumption, $w_t^i(1 - c_t(\Delta t))$, which strategy i assigns to the purchase of the asset k in period t. Formally, a strategy is a stochastic process, which is F_t – adapted. Further, the next assumption is made:

Assumption 2: (i) For every strategy, the fraction invested in each asset is F_t – adapted and non-negative, that is, $\lambda_{t,k}^i \geq 0$ for almost every sample path and all t, k, i. (ii) there is at least one completely diversified strategy, that is, given that $t \in \Pi$ or $t \geq 0$ for the continuous time model, there is a $j \in \{1, \cdots, I\}$ with $w_t^j > 0$ such that $0 < \lambda_{t,k}^j < 1$ for almost every sample path and all k.

Assumption 2 means that short selling is prohibited (though which is quite possibly unnecessary to get the main conclusions of this paper). Meanwhile, Assumption 2 also makes sure that the prices of all assets are strictly positive, thanks to equation (1), which is actually just the reason why it is assumed.

In this framework, all the assets are just K long-lived assets, which pay off dividends in cash or produce the same perishable consumption good, say milk. The cash or perishable consumption good are consumed completely and especially can not be used to reinvest. For this reason the total amount of assets traded in the market keeps constant. On account of a stock split, the following assumption is imposed without loss of generality:

Assumption 3: The supply of each asset is normalized to 1.

According to this assumption, the market-clearing price, denoted by ρ_t^k, is given by:

$$\rho_t^k = \sum_{i=1}^{I} \lambda_{t,k}^i w_t^i \left(1 - c_t(\Delta t)\right). \tag{1}$$

Denote the aggregate market wealth by W_t, that is, $W_t = \sum_{i=1}^{I} w_t^i$, then the market-clearing price of consumption good or cash, denoted by $\rho_t^0, \forall t \in \Pi$, is determined by:

$$W_t c_t(\Delta t) = \rho_t^0 D_t, \tag{2}$$

where D_t is the aggregate dividend, that is, $D_t = \sum_{k=1}^{K} D_t^k$. In contrast to (2), Evstigineev et al. (2006) take a normalized price for cash, that is, $\rho_t^0 \equiv 1$. Following this, (2) is written as $W_t c_t(\Delta t) = D_t$, which is however a bit more artificial, although the difference is not important.

Let $\Lambda \equiv \{1, \cdots, K\}$. For a given strategy profile and the wealth $w_t^i, i = 1, \cdots, I$, the percentage of all shares issued of asset k that strategy i invests during period t is:

$$\pi_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^j w_t^j}, t \in \Pi, k \in \Lambda. \tag{3}$$

It is evident that:

$$\sum_{k=1}^{K} \lambda_{t,k}^i = 1, \sum_{i=1}^{I} \pi_{t,k}^i = 1. \tag{4}$$

Since the change of a wealth results only from dividends and capital gains, the wealth of strategy i at the beginning of period $t + \Delta t$, that is, w_{t+1}^i is determined by:

$$w_{t+\Delta t}^i - \left(1 - c_t(\Delta t)\right) w_t^i = \sum_{k=1}^{K} \left(\rho_{t+\Delta t}^0 D_{t+\Delta t}^k + (\rho_{t+\Delta t}^k - \rho_t^k)\right) \pi_{t,k}^i, \tag{5}$$

or according to (1) and (3)□ (4), determined by:

$$w_{t+\Delta t}^i = \sum_{k=1}^{K} \left(\rho_{t+\Delta t}^0 D_{t+\Delta t}^k + \rho_{t+\Delta t}^k\right) \pi_{t,k}^i. \tag{6}$$

While the total amount of assets traded in the market keeps constant, the aggregate market wealth W_t does change with time because of trading randomly. However, social welfare has evidently no relation to the level of the market wealth W_t and a big W_t only means higher prices of the assets and consumption good or cash. Actually, there exist infinitely many solutions to the wealth dynamics (6) since it is by (2) homogeneous for all wealth variables, that is, $w_t^i, t \in \Pi, i = 1, \cdots, I$. In particular, any solution multiplied by an arbitrary non-zero number is still a solution. For this reason and an economic consideration, the aggregate market wealth W_t is taken as a numéraire in the following.

Based on this numéraire, the normalized prices for asset k and consumption good are $p_t^k \equiv \rho_t^k W_t$ and $p_0^k \equiv \rho_0^k W_t$ respectively for each period t. Further, the normalized wealth or the market share of strategy i is $r_t^i \equiv w_t^i W_t$. Accordingly, it follows directly from (1) and (3) that:

$$p_t^k = \sum_{i=1}^{I} \lambda_{t,k}^i r_t^i \left(1 - c_t(\Delta t)\right), \pi_{t,k}^i = \frac{\left(1 - c_t(\Delta t)\right) \lambda_{t,k}^i r_t^i}{p_t^k} \text{ and } \sum_{i=1}^{I} r_t^i = 1. \quad (7)$$

Obviously, the market share process r_t^i is the most important index that tells how strategy i performs. For example, the market share process determines whether the strategy is dominated or dominating. Consequently, the dynamics of market share process is studied further in detail.

Similar to Yang and Ewald (2008), by an elementary manipulation, for all i, k, t it follows from (1)□ (2) and (6) that:

$$r_{t+\Delta t}^i = \sum_{k=1}^{K} \left(c_t\left(\Delta t\right) d_{t+\Delta t}^k + \left(1 - c_t\left(\Delta t\right)\right) \sum_{j=1}^{I} \lambda_{t+\Delta t,k}^j r_{t+\Delta t}^j \right) \pi_{t,k}^i, \quad (8)$$

where d_t^k represents the relative dividend payment of asset k , that is, $d_t^k \equiv D_t^k D_t$. Equation (8) is called *market selection equation*.

Obviously, $r_t^i \equiv 0$ thanks to (8) if $r_0^i = 0$. In order to avoid this uninteresting case, it is supposed throughout the paper that $r_0^i > 0$ for each $i \in \{1, \cdots, I\}$. Further, it is not difficult to prove that (8) is well-defined. Formally, the following theorem holds.

Theorem 1

Given a strategy profile and all the relative dividend payment processes, $\{d_t^k\}_{t\in\Pi}, k \in \Lambda$, and provided that Assumption 1 and 2 hold, there exists a path-wise unique market share process $\{r_t^i\}_{t\in\Pi}$ satisfying (8) for each strategy $i, i = 1, \cdots, I$.

Proof

A sketch of a proof is given here (Evstigineev et al., 2006). In fact, this theorem can be proved by solving (8) step by step forward. At each step, say period $t + \Delta t$, (8) is a system of I linear equations in I variables, that is $r_{t+\Delta t}^i, i \in \{1, \cdots, I\}$ and the coefficient matrix can be proved to be invertible by verifying that it has a column dominant diagonal thanks to Assumption 1 and 2 and (4). Thus the theorem is proved.

A continuous-time evolutionary finance model with time-dependent strategies is established in the following by letting $\Delta t \to 0$ in (8). To achieve this goal, more assumptions are needed and shown thus:

Assumption 4: For all $t \in \Pi, k \in \Lambda, i \in \{1, \cdots, I\}$ and every sample path, the following limits exist and the equalities hold:

$$\lim_{\Delta t \to 0} d_{t+\Delta t}^k = d_t^k, \lim_{\Delta t \to 0} \lambda_{t+\Delta t,k}^i = \lambda_{t,k}^i \text{ and further, } \lim_{\Delta t \to 0} \frac{\lambda_{t+\Delta t,k}^i - \lambda_{t,k}^i}{\Delta t} \equiv \dot{\lambda}_{t,k}^i \quad (9)$$

Throughout this paper, "dot" notation is used for derivatives. Assumption 4 seems too restrictive but actually not. The first equality of the assumption says the relative dividend process is continuous as well as stochastic, which describes the economic phenomenon that the relative dividend changes gradually instead of suddenly. In addition, it is well-known that a continuous-time portfolio in mathematical finance is mostly left continuous with right hand limits. Consequently, the portfolio is generally not continuous, let alone being differentiable. On the contrary, asset prices in this setup are endogenous instead of exogenous as in mathematical finance and they do change along with the portfolio. Accordingly, once a portfolio is performed, the wealth managed by any players or strategies will be changed in the same time through a complicated rebalancing process for market-clearing. This rebalancing process will make the amount of wealth managed by any players changed after trading, which is totally different from the self-financing trading assumed by mathematical finance. As a result, every alteration of a portfolio will incur a strong resistance from the market and the speed of alteration improbably gets too quick - that is to say, the portfolio is altered slowly and thus the differential assumptions in (9) are also acceptable.

Following the seemly strongly restrictive Assumption 4, the next assumption is intuitively reasonable:

Assumption 5: The common consumption rate for each player or strategy is a function of the time interval Δt and the following limits exist for almost every sample path and all $t \in \Pi$:

$$\lim_{\Delta t\to 0} c_t(\Delta t) = 0 \text{ and further, } \lim_{\Delta t\to 0} \frac{c_t(\Delta t)}{\Delta t} \equiv \overline{c}_t. \tag{10}$$

This assumption says that the longer the time interval, the more the consumption rate and there exists an instantaneous speed of the change of the consumption rate, which is $\overline{c}_t$ for all $t \in \Pi$.

Now, it is ready to derive the continuous-time evolutionary finance model with time-dependent strategies. In order to get a deep insight, the following text turns to the case $I = 2$, which is particularly important from evolutionary game theory. Under this case, only two strategies compete with each other for the market capital and the market shares satisfy that $r_t^1 + r_t^2 = 1$ for all $t \in \Pi$. For this reason, the dynamics of only one market share, say $r_t^2, t \in \Pi$ will be studied in detail subsequently

The following theorem is the main conclusion of "continuous-time market selection process with time-dependent strategies" in which (12) is called *continuous-time market selection equation* with time-dependent strategies.

Theorem 2

Suppose a strategy profile $\{\lambda_{t,k}^i\}_{t\in\Pi}, i = 1,2, k \in \Lambda$, is given and Assumption 1□5 hold:

(i) For all $t \in \Pi$ and $i = 1,2$, $0 < r_{t+\Delta t}^i < 1$ if and only if $0 < r_t^i < 1$;

(ii) For almost every sample path:

$$\lim_{\Delta t\to 0} r_{t+\Delta t}^1 = r_t^1, \lim_{\Delta t\to 0} r_{t+\Delta t}^2 = r_t^2; \tag{11}$$

(iii) Further, for almost every sample path, the market share $r_t^i, i = 1,2$ is differentiable and r_t^2 satisfies the following random differential equation:

$$\dot{r}_t^2 = \frac{-\overline{c}_t r_t^2 + \sum_{k=1}^K \frac{\lambda_{t,k}^2 r_t^2\left(\dot{\lambda}_{t,k}^1\left(1-r_t^2\right)+\dot{\lambda}_{t,k}^2 r_t^2+\overline{c}_t d_t^k\right)}{\lambda_{t,k}^1\left(1-r_t^2\right)+\lambda_{t,k}^2 r_t^2}}{\sum_{k=1}^K \frac{\lambda_{t,k}^1\lambda_{t,k}^2}{\lambda_{t,k}^1\left(1-r_t^2\right)+\lambda_{t,k}^2 r_t^2}}, \tag{12}$$

with initial value r_0^2, while $\dot{r}_t^1 = -\dot{r}_t^2$.

***Proof*[2]**

It suffices to verify the conclusions about the market share of strategy 2.First, It follows from (8) and the case $I = 2$ that:

$$r_{t+\Delta t}^2 = \frac{c_t(\Delta t)\sum_{k=1}^K d_{t+\Delta t}^k \pi_{t,k}^2 + \left(1-c_t(\Delta t)\right)\sum_{k=1}^K \lambda_{t+\Delta t,k}^1 \pi_{t,k}^2}{c_t(\Delta t)+\left(1-c_t(\Delta t)\right)\sum_{k=1}^K\left(\lambda_{t+\Delta t,k}^1\pi_{t,k}^2+\lambda_{t+\Delta t,k}^2\pi_{t,k}^1\right)}. \tag{13}$$

As a result, part (i) of this theorem is immediately proved thanks to the assumptions of the theorem and:

$$r_t^1 + r_t^2 = 1, t \in \Pi. \tag{14}$$

From (4) and (7), it is concluded that:

$$r_t^1\sum_{k=1}^K \lambda_{t,k}^1\pi_{t,k}^2 + r_t^2\sum_{k=1}^K \lambda_{t,k}^2\pi_{t,k}^2 = \sum_{k=1}^K\left(r_t^1\lambda_{t,k}^1 + r_t^2\lambda_{t,k}^2\right)\pi_{t,k}^2 = r_t^2. \tag{15}$$

Consequently, one gets that:

$$r_{t+\Delta t}^2 - r_t^2 = \frac{c_t(\Delta t)\sum_{k=1}^K (d_{t+\Delta t}^k - r_t^2)\pi_{t,k}^2 + \left(1-c_t(\Delta t)\right)\sum_{k=1}^K \pi_{t,k}^2 \sum_{i=1}^2 r_t^i(\lambda_{t+\Delta t,k}^i - \lambda_{t,k}^i)}{c_t(\Delta t)+\left(1-c_t(\Delta t)\right)\sum_{k=1}^K\left(\lambda_{t+\Delta t,k}^1\pi_{t,k}^2+\lambda_{t+\Delta t,k}^2\pi_{t,k}^1\right)}. \tag{16}$$

Next, let the time interval shrink to zero, that is, $\Delta t \to 0$, then by (9), (10) and (14) the numerator converges to zero and the denominator of the right-hand side of (16) converges to:

$$\sum_{k=1}^K \lambda_{t,k}^1\pi_{t,k}^2 + \sum_{k=1}^K \lambda_{t,k}^2\pi_{t,k}^1 = \sum_{k=1}^K \frac{\lambda_{t,k}^1\lambda_{t,k}^2}{\lambda_{t,k}^1(1-r_t^2)+\lambda_{t,k}^2 r_t^2} > 0. \tag{17}$$

Accordingly, the second equality of (11) is derived. Furthermore, one infers from (9), (10) and (16) that:

$$\dot{r}_t^2 = \lim_{\Delta t\to 0}\frac{r_{t+\Delta t}^2 - r_t^2}{\Delta t} = \frac{\overline{c}_t\sum_{k=1}^K (d_t^k - r_t^2)\pi_{t,k}^2 + \sum_{k=1}^K \pi_{t,k}^2\sum_{i=1}^2 r_t^i\dot{\lambda}_{t,k}^i}{\sum_{k=1}^K \frac{\lambda_{t,k}^1\lambda_{t,k}^2}{\lambda_{t,k}^1(1-r_t^2)+\lambda_{t,k}^2 r_t^2}}.$$

Therefore, part (iii) of the theorem is shown after a simple substitution from (7).

Market selection equation (12) differs from Itô Stochastic differential equation since its integral can be

[2]We only give a short proof from intuition here. Recently, Palczewski and Schenk-Hoppé (2010) provide a strict proof for almost the same conclusion.

defined path by path as a Riemann integral. (12) demonstrates explicitly that the evolution of each market share is determined by all strategies as well as the relative dividend and it is well defined in the sense that for an arbitrary initial value $0 \le r_0^2 \le 1$, its solution exists and is path-wise unique. With a view to proving this conclusion, a general existence and uniqueness are shown in the following. Let:

$$h(t,\omega,x) \equiv \frac{-\overline{c}_t + \sum_{k=1}^{K} \frac{\lambda_{t,k}^2\left(\dot{\lambda}_{t,k}^1(1-x)+\dot{\lambda}_{t,k}^2 x+\overline{c}_t d_t^k\right)}{\lambda_{t,k}^1(1-x)+\lambda_{t,k}^2 x}}{\sum_{k=1}^{K} \frac{\lambda_{t,k}^1\lambda_{t,k}^2}{\lambda_{t,k}^1(1-x)+\lambda_{t,k}^2 x}},$$

and $g(t,\omega,x) \equiv xh(t,\omega,x)$ then (12) can be equivalently written as:

$$\dot{x} = g(t,\omega,x) \text{ or } \dot{x} = xh(t,\omega,x), \qquad (18)$$

in which initial value $t_0 = 0, x(t_0) = r_0^2$ and $r_t^2 \equiv x(t)$. Generally speaking, there is a path-wise unique solution to (18) for an arbitrary initial value $(t_0, x(t_0)) \in [0,\infty)\times(-\infty,\infty)$ instead of only $t_0 = 0, x(t_0) = r_0^2 \in [0,1]$. In order to get this result, the next assumption instead of Assumption 2 is sufficient:

Assumption 6: The consumption rate and the derivatives of the fraction invested in each asset are almost surely path-wise continuous, finite and F_t-adapted.

That is to say, $0 < \overline{c}_t < \infty, |\dot{\lambda}_{t,k}^i| < \infty$ and $\overline{c}_t, \dot{\lambda}_{t,k}^i$ are continuous functions of t for all $t \ge 0, k \in \Lambda, i \in \{1,2\}$ and almost every sample path.

This assumption nearly puts no extra restriction from an economic viewpoint on account of the comments following Assumption 4. Now, the existence and uniqueness of (18) are formally presented thus.

Theorem 3

If Assumption 6 holds for a given strategy profile, then for an arbitrary initial value $(t_0, x(t_0)) \in [0,\infty)\times(-\infty,\infty)$ with $\lambda_{t_0,k}^1(1-x(t_0)) + \lambda_{t_0,k}^2 x(t_0) \ne 0$ for all $k \in \Lambda$, there exists a path-wise unique local solution of random differential equation (18).

Proof

According to the assumption of the theorem, there is a null set $N \in F$, such that for every $\omega \in N^c$, one can find an open subset U satisfying $(t_0, x(t_0)) \in U \subset [0,\infty)\times(-\infty,\infty)$, on which function $g(t,\omega,x)$ is Lipschitz continuous in the third argument, uniformly with respect to the first (for more details refer to the proof of Theorem 4 below). Hence, the theorem is proved thanks to the well-known Picard-Lindelöf theorem.

Based on Theorem 3 and Assumption 2, the existence and uniqueness of *market selection equation* (12) are discussed next and a more insightful result is summed up in the following.

Theorem 4

For a given strategy profile, Assumption 2 and 6 hold:

(i) For an arbitrary initial value $0 < r_0^i < 1, i = 1,2$, there exists a path-wise unique solution of market selection equation (12), which satisfies $0 < r_t^i < 1, i = 1,2$ for all $t > 0$;
(ii) For almost every sample path, if $r_0^i = 0$, then $r_t^i = 0$ for $i = 1,2$ and $t \ge 0$; if $r_0^i = 1$, then $r_t^i = 1$ for $i = 1,2$ and $t \ge 0$.

Proof

According to Theorem 3 part (ii) of the theorem is obvious and thus, without loss of generality, only part (i) for $i = 2$ is shown in the following. (12) can be written as:

$$\dot{r}_t^2 = g(t,\omega,r_t^2) \text{ or } \dot{r}_t^2 = r_t^2 h(t,\omega,r_t^2), \qquad (19)$$

with initial value $0 < r_0^2 < 1$. Thanks to Theorem 3 and part (ii) of the theorem, the phase space of (19) is included in the interval $[0,1]$ for an arbitrary initial value $0 < r_0^2 < 1$. First, there exists a positive $t_1 > 0$ such that there is a path-wise unique solution of the initial value problem (19) for a given sample path up to t_1 owing to Theorem 3 and in particular, $0 < r_t^2 < 1$ can be guaranteed for $0 \le t \le t_1$. Secondly, the interval $[0, t_1)$ is extended as wide as possible in the following way.

For a given strategy profile and an arbitrary positive integer $n \ge 2$, define stopping time:

$$T_n(r_0^2,\omega) \equiv \inf\left\{ t \ge 0 \,\middle|\, \exists k \in \Lambda, \text{s.t. } 0 < \lambda_{t,k}^1(1-r_t^2) + \lambda_{t,k}^2 r_t^2 < \frac{1}{n} \right.$$
$$\left. \text{or } \max\{\overline{c}_t, |\dot{\lambda}_{t,k}^1|, |\dot{\lambda}_{t,k}^2|\} > n \right\},$$

where the infimum of the empty set is understood to be ∞. Note that for all $x \in [0,1]$ and $0 \le t < T_n(r_0^2, \omega)$,

$$\frac{1}{n} \le \min\{\lambda_{t,k}^1, \lambda_{t,k}^2\} \le \lambda_{t,k}^1(1-x) + \lambda_{t,k}^2 x \le \max\{\lambda_{t,k}^1, \lambda_{t,k}^2\} \le 1$$

and

$$\frac{1}{n} \le \min\{\lambda_{t,k}^1, \lambda_{t,k}^2\} \le \frac{\lambda_{t,k}^1 \lambda_{t,k}^2}{\lambda_{t,k}^1(1-x) + \lambda_{t,k}^2 x} \le \max\{\lambda_{t,k}^1, \lambda_{t,k}^2\} \le 1,$$

and thus, it follows from a lengthy but elementary calculation that for all $x \in [0,1]$, $0 \le t < T_n(r_0^2, \omega)$ and $\omega \in \Omega$,

$$| f_x'(t, \omega, x) | < 6n^4, \text{ and } h(t, \omega, x) > -2n^3.$$

For this reason, there exists a path-wise unique solution of (19) up to $T_n(r_0^2, \omega)$ by means of extensibility of solutions. Moreover, according to the second equality of (19), the solution satisfies:

$$r_t^2 = r_0^2 \exp\left(\int_0^t h(u, \omega, r_u^2) du\right) > r_0^2 \exp\left(\int_0^t -2n^3 du\right) > 0 \quad (20)$$

for $0 \le t \le T_n(r_0^2, \omega)$. On the other hand, it is evident that $r_t^2 < 1$ since $r_t^1 + r_t^2 = 1$ and one can also show $r_t^1 > 0$ in the same way.

Lastly, it can be verified that $\lim_{n\to\infty} T_n(r_0^2, \omega) = \infty, a.s.$. In fact, if there is a measurable set $A \in F$ with $P(A) > 0$, such that $\lim_{n\to\infty} T_n(r_0^2, \omega) = \tau(\omega) < \infty$ for all $\omega \in A$, then there exists $k \in \Lambda$ and $i \in \{1,2\}$, such that $\lambda_{\tau(\omega),k}^1 (1 - r_{\tau(\omega)}^2) + \lambda_{\tau(\omega),k}^2 r_{\tau(\omega)}^2 = 0$, or $\overline{c}_{\tau(\omega)} = \infty$, or $| \dot{\lambda}_{\tau(\omega),k}^i | = \infty$ for all $\omega \in A$. The last two equalities contradict Assumption 6 and the first equality leads to $r_{\tau(\omega)}^2 = 0$, that is, $w_{\tau(\omega)}^2 = 0$ and $\lambda_{\tau(\omega),k}^1 = 0$ or $r_{\tau(\omega)}^1 = 0$, i.e. $w_{\tau(\omega)}^1 = 0$ and $\lambda_{\tau(\omega),k}^2 = 0$, which contradicts Assumption 2. Hence the proof is finished. □

It follows directly from Theorem 4 that if one of the strategies is completely diversified then there is no sudden death or bankruptcy in the economy even some strategies are not completely diversified.

For example, suppose in a market there are only two assets: One pays 1 and another pays nothing all the time, that is, $d_t^1 = 1$ and $d_t^2 = 0$ respectively. The consumption parameter is constant, i.e. $\overline{c}_t \equiv c > 0$. Strategy 1 is completely diversified with $\lambda_{t,1}^1 = \mu$ and $\lambda_{t,2}^1 = 1 - \mu$, $0 < \mu < 1$ all the time but strategy 2 is not with $\lambda_{t,1}^2 = 0$ and $\lambda_{t,2}^2 = 1$. Strategy 2 is clearly very bad since it invests all the wealth in asset 2, which pays nothing all the time. However, the bad strategy will not go bankrupt forever only if its initial wealth is strictly positive, that is, $0 < r_0^2 < 1$. In fact, a simple calculation leads to:

$$r_t^2 = \frac{(1-\mu) r_0^2 \exp(-ct)}{1 - \mu + \mu r_0^2 \left(1 - \exp(-ct)\right)} > 0, \text{ for all } t \ge 0,$$

although $\lim_{t\to\infty} r_t^2 = 0$ and thus $\lim_{t\to\infty} r_t^1 = 1$, that is, strategy 1 dominates the market at last.

EVOLUTIONARY STABLE BOND MARKET

Market selection equation (12) is a rather general continuous-time evolutionary finance model, in which it is difficult to answer what is the "optimal" strategy since the strategies are time-dependent in a stochastic environment. In order to avoid a too complex problem suddenly while studying this general model, Yang and Ewald (2008) deal with the case of constant proportions strategies though in a stochastic world, which is substantially simpler than the general model established in subsequently. For the same reason, nevertheless from another point of view, the following text focuses on a deterministic bond market but keeps the strategies time-dependent.

By a deterministic bond market we mean nothing other than the market discussed above in which the relative dividend process is a deterministic function of time t. While fundamentals are fixed, prices and wealth vary due to market interaction. This market appears too restrictive but in fact includes a variety of marketable treasury securities issued by the United States Department of the Treasury through the Bureau of the Public Debt: Treasury bills, Treasury notes, Treasury bonds, and Treasury Inflation Protected Securities (TIPS). This market also includes Account Treasury bonds issued by the China Department of Finance, which were first issued in 1997 but the amount increases with an extremely quick speed.

In a deterministic world, the dividend is decided in advance and thus for a given strategy profile, (12) is an ordinary differential equation. Let:

$$f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) \equiv \frac{-\overline{c}_t r_t^2 + \sum_{k=1}^K \frac{\lambda_{t,k}^2 r_t^2 \left(\dot{\lambda}_{t,k}^1 (1 - r_t^2) + \dot{\lambda}_{t,k}^2 r_t^2 + \overline{c}_t d_t^k\right)}{\lambda_{t,k}^1 (1 - r_t^2) + \lambda_{t,k}^2 r_t^2}}{\sum_{k=1}^K \frac{\lambda_{t,k}^1 \lambda_{t,k}^2}{\lambda_{t,k}^1 (1 - r_t^2) + \lambda_{t,k}^2 r_t^2}}.$$

The *market selection equation* (12) in a deterministic world is rewritten as:

$$\dot{r}_t^2 = f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) \text{ and } \dot{r}_t^1 = -\dot{r}_t^2. \quad (21)$$

There are obviously only two fixed points in (20), that is, $(1,0)$ and $(0,1)$ respectively according to part (ii) of Theorem 4. Without loss of generality, the stability of fixed point (1,0) is discussed below. Namely we investigate a bond market where there are two players: One is an incumbent or the dominant market strategy, strategy 1 with an initial market share $r_0^1 \approx 1$ and another is a mutant, strategy 2 with an initial market share $r_0^2 \approx 0$. It is our aim to answer whether the incumbent is *evolutionary stable*.

At first sight, the assumption that there are only such two strategies in a market seems unrealistic but in fact not. For example, one can interpret strategy 1 as the market portfolio and strategy 2 as an arbitrary strategy, say one played by an individual or even a financial institution.

For this reason, whether the incumbent is *evolutionary stable* equals to whether the market is effective. And if the incumbent is *evolutionary stable* then a "fair price" of each asset can be derived by the following (24). Accordingly, a study on a market even with only such two strategies can still lead to a lot of interesting conclusions.

It is clear that the evolution of (20) is determined by the sign of function $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$. In particular, if the sign is positive the market share of strategy 2 will increase but decrease on the contrary. Therefore, the following definition is presented.

Definition 1: (i) Strategy 1 is called *evolutionary stable*, if there exists a positive $\varepsilon > 0$ such that, for every strategy, say strategy 2, with an initial market share $0 < r_0^2 < \varepsilon$, it is impossible to find a time interval, say $(a,b) \subset (0,\infty)$ such that $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) > 0$ for $t \in (a,b)$ while $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) \geq 0$ for $t \in (a,b)^c$ and $t \geq 0$; (ii) On the contrary, strategy 1 is called *evolutionary unstable*, if there exists a time interval, say $(a,b) \subset (0,\infty)$ and a strategy denoted by strategy 2 with an arbitrary initial market share $0 < r_0^2 < 1$, such that $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) > 0$ for $t \in (a,b)$ while $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) \geq 0$ for $t \in (a,b)^c$ and $t \geq 0$.

It follows from this definition that an incumbent is evolutionary stable if and only if there is no arbitrage opportunity in the market. Note that at the beginning of the entry of a mutant, the market shares satisfy $r_t^1 \approx 1, r_t^2 \approx 0$ and thus the sign of $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$ is determined by its linear approximation:

$$f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) \approx \left(-\overline{c}_t + \sum_{k=1}^{K} \frac{\lambda_{t,k}^2}{\lambda_{t,k}^1} \left(\overline{c}_t d_t^k + \dot{\lambda}_{t,k}^1 \right) \right) r_t^2. \quad (22)$$

Definition 2: For a given incumbent $\{\lambda_{t,k}^1 \mid k \in \Lambda; t \geq 0\}$, the *total return* of asset k is defined by:

$$\Gamma_t(k) \equiv \frac{\overline{c}_t d_t^k + \dot{\lambda}_{t,k}^1}{\lambda_{t,k}^1}, k \in \Lambda. \quad (23)$$

$\Gamma_t(k)$ is called the *total return* because it completely determine the investment value of the asset as seen in the following Theorem 5. In fact, $\Gamma_t(k)$ is a sum of the capital gain and the dividend return. To make this clear, recall (7) and (10) and note that $r_t^1 \approx 1, r_t^2 \approx 0$. Then one finds:

$$p_t^k = \lambda_{t,k}^1 r_t^1 + \lambda_{t,k}^2 r_t^2 \approx \lambda_{t,k}^1, t \geq 0, k \in \Lambda, \quad (24)$$

which means the relative price of an asset approximately equals to the fraction invested in this asset by the incumbent. Especially, if the market portfolio is taken as the unique strategy in a market, that is, $r_t^1 = 1$ all the time, then the relative price is just the fraction invested by the market portfolio according to (24).

It follows from (24) that $\dot{\lambda}_{t,k}^1 / \lambda_{t,k}^1 \approx \dot{p}_t^k / p_t^k$ and $\overline{c}_t d_t^k / \lambda_{t,k}^1 \approx \overline{c}_t d_t^k p_t^k$. Clearly $\dot{p}_t^k p_t^k$ is a capital gain while $\overline{c}_t d_t^k p_t^k$ represents a dividend return. (23) says that the bigger the changing speed $\overline{c}_t$ of consumption rate, the more dependent the *total return* on the relative dividend but the less dependent on the capital gain. Hence, it explains that in a society with excess consumption, the dividend paid by an asset should be much more appreciated.

Theorem 5

For a given incumbent, provided that the *total return* of a market with the incumbent is not identical for all assets at some time t, this incumbent is evolutionary unstable.

Proof. In fact, if a mutant invests all wealth in an asset with the largest *total return* at first and then gradually revise the portfolio to equal to the incumbent, then the mutant will gain the market. Namely $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$ is strictly positive in a small neighborhood of $(1,0)$ and keep non-negative all the time. Hence this theorem is shown from Definition 1.

This theorem means that a market is arbitrage-free only if the *total return*s of an incumbent defined by (23) are the same across all assets. To take one step further, next we prove that if the *total return*s of an incumbent defined by (23) are the same across all assets, then the incumbent is evolutionary stable, i.e. the market is arbitrage-free:

Assumption 7: For a given incumbent, strategy 1, the *total return* defined by (23) is identical for all assets all the time.

This assumption also appears to be too restrictive but as proved in Theorem 5, if the assumption does not hold then there exists an arbitrage opportunity in the market, which is not interesting to study also in mathematical finance.

By virtue of Assumption 7, the value $\Gamma_t(k)$ is independent of k and it is therefore denoted by v_t in the following. Accordingly, one gets:

$$\overline{c}_t d_t^k + \dot{\lambda}_{t,k}^1 = v_t \lambda_{t,k}^1, k \in \Lambda. \tag{25}$$

Note that:

$$\sum_{k=1}^{K} d_t^k = 1, \sum_{k=1}^{K} \lambda_{t,k}^1 = 1, \sum_{k=1}^{K} \dot{\lambda}_{t,k}^1 = 0. \tag{26}$$

Aggregating (25) over assets, one finds $v_t = \overline{c}_t$, namely:

$$\overline{c}_t d_t^k + \dot{\lambda}_{t,k}^1 = \overline{c}_t \lambda_{t,k}^1. \tag{27}$$

Consequently, under Assumption 7, the linear approximation in (22) is always zero, which, however, does not directly mean strategy 1 is *evolutionary stable*. Hence a more detailed analysis of the second and even higher order approximation is necessary. For this purpose, a second-order approximation of function $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$ is shown thus:

$$\dot{r}_t^2 = f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2) = \left(r_t^2\right)^2 \sum_{k=1}^{K} -\frac{\overline{c}_t}{\lambda_{t,k}^1}\left(\lambda_{t,k}^2\right)^2 + \frac{\dot{\lambda}_{t,k}^2 + \overline{c}_t d_t^k}{\lambda_{t,k}^1}\lambda_{t,k}^2 + O((r_t^2)^3), \tag{28}$$

where a function $h(x)$ is said to be $O(g(x))$ if $\lim_{x\to 0} |h(x)/g(x)| < \infty$. Equivalently, for an arbitrary terminal time T, one has:

$$\frac{1}{r_T^2} = \frac{1}{r_0^2} + \int_0^T \left(\sum_{k=1}^{K} \frac{\overline{c}_t}{\lambda_{t,k}^1}\left(\lambda_{t,k}^2\right)^2 - \frac{\dot{\lambda}_{t,k}^2 + \overline{c}_t d_t^k}{\lambda_{t,k}^1}\lambda_{t,k}^2 \right) dt + O(r_t^2). \tag{29}$$

Therefore, if the target of a mutant is to maximize its market share at a given terminal time T, that is, r_T^2 then the optimal mutant $\{\lambda_{t,k}^2\}$ is a solution of the following variation problem:

$$\min_{\lambda_{t,k}^2; 0\le t\le T, k\in\Lambda} \int_0^T \sum_{k=1}^{K} \lambda_{t,k}^2 \left(\frac{\overline{c}_t \lambda_{t,k}^2}{\lambda_{t,k}^1} - \frac{\dot{\lambda}_{t,k}^2 + \overline{c}_t d_t^k}{\lambda_{t,k}^1} \right) dt$$
$$s.t. \begin{cases} \sum_{k=1}^{K} \lambda_{t,k}^2 = 1, \\ 0 \le \lambda_{t,k}^2 \le 1, 0 \le t \le T, k \in \Lambda, \end{cases} \tag{30}$$

where strategy 2 is constrained to be differential according to Assumption 4. Regarding this problem, there exists an $\varepsilon-$optimal strategy shown further.

Theorem 6

Let λ_{T,k^*}^1 be a minimum value of the set $\{\lambda_{T,k}^1 | k \in \Lambda\}$. Then for any given small $\varepsilon > 0$, an ε-optimal solution to (30) is given by:

$$\begin{cases} \lambda_{t,k}^2 = \lambda_{t,k}^1, 0 \le t \le T - \delta(\varepsilon), \\ \lambda_{T,k}^2 = 0, k \ne k^*; \lambda_{T,k^*}^2 = 1. \end{cases} \tag{31}$$

In addition, $\lambda_{t,k}^2, k \in \Lambda - \{k^*\}$, which passes $\left(T-\delta(\varepsilon), \lambda_{T-\delta(\varepsilon),k}^1\right)$ and $(T,0)$, and λ_{t,k^*}^2, which passes $\left(T-\delta(\varepsilon), \lambda_{T-\delta(\varepsilon),k^*}^1\right)$ and $(T,1)$, are differentiable at $T-\delta(\varepsilon)$ and T, where $\delta(\varepsilon)$ is a sufficiently small positive number. The optimal objective function equals to $1/2 - 1/(2\lambda_{t,k^*}^1) + \varepsilon$, which can be strictly negative since ε is a sufficiently small positive number.

Proof

It follows from (27) that:

$$\left(\frac{1}{\lambda_{t,k}^1}\right)_t' = \frac{\overline{c}_t d_t^k - \overline{c}_t \lambda_{t,k}^1}{\left(\lambda_{t,k}^1\right)^2},$$

Consequently, applying the rule for integration by parts, (30) is equivalent to:

$$\min_{\lambda_{t,k}^2;0\le t\le T,k\in\Lambda} \sum_{k=1}^{K}\left(\frac{\left(\lambda_{0,k}^2\right)^2}{2\lambda_{0,k}^1}-\frac{\left(\lambda_{T,k}^2\right)^2}{2\lambda_{T,k}^1}\right)+$$

$$\int_0^T \sum_{k=1}^{K}\lambda_{t,k}^2\left(\left(\frac{\overline{c}_t}{\lambda_{t,k}^1}+\frac{\overline{c}_t d_t^k-\overline{c}_t\lambda_{t,k}^1}{2\left(\lambda_{t,k}^1\right)^2}\right)\lambda_{t,k}^2-\frac{\overline{c}_t d_t^k}{\lambda_{t,k}^1}\right)\mathrm{d}t$$

$$s.t.\begin{cases}\sum_{k=1}^{K}\lambda_{t,k}^2=1\\ 0\le\lambda_{t,k}^2\le 1,\ 0\le t\le T,\ k\in\Lambda.\end{cases}\tag{32}$$

Obviously, the optimal strategy is myopic. In addition, the objective function is quadric and the constrained conditions are convex. Thus there exists a unique globally optimal solution to (32). More concretely, three optimization problems are solved below: The first one is:

$$\min_{\lambda_{0,k}^2;k\in\Lambda}\sum_{k=1}^{K}\frac{\left(\lambda_{0,k}^2\right)^2}{2\lambda_{0,k}^1}$$

$$s.t.\begin{cases}\sum_{k=1}^{K}\lambda_{0,k}^2=1,\\ 0\le\lambda_{0,k}^2\le 1,k\in\Lambda,\end{cases}$$

in which, there is a unique global solution - that is $\lambda_{0,k}^2=\lambda_{0,k}^1,k\in\Lambda$, and the optimal value of the objective function is $1/2$. The second one is:

$$\max_{\lambda_{T,k}^2;k\in\Lambda}\sum_{k=1}^{K}\frac{\left(\lambda_{T,k}^2\right)^2}{2\lambda_{T,k}^1}$$

$$s.t.\begin{cases}\sum_{k=1}^{K}\lambda_{T,k}^2=1,\\ 0\le\lambda_{T,k}^2\le 1,k\in\Lambda,\end{cases}$$

in which, there is a unique global solution - that is $\lambda_{T,k^*}^2=1,\text{ while }\lambda_{T,k}^2=0,k\in\Lambda-\{k^*\}$, and the optimal value of the objective function is $(2\lambda_{t,k^*}^1)^{-1}$. The last one is:

$$\min_{\lambda_{t,k}^2;0\le t\le T,k\in\Lambda}\int_0^T \sum_{k=1}^{K}\lambda_{t,k}^2\left(\left(\frac{\overline{c}_t}{\lambda_{t,k}^1}+\frac{\overline{c}_t d_t^k-\overline{c}_t\lambda_{t,k}^1}{2\left(\lambda_{t,k}^1\right)^2}\right)\lambda_{t,k}^2-\frac{\overline{c}_t d_t^k}{\lambda_{t,k}^1}\right)\mathrm{d}t$$

$$s.t.\begin{cases}\sum_{k=1}^{K}\lambda_{t,k}^2=1\\ 0\le\lambda_{t,k}^2\le 1,0\le t\le T,k\in\Lambda,\end{cases}\tag{33}$$

in which there is a unique global solution, i.e. $\lambda_{t,k}^2=\lambda_{t,k}^1,0\le t\le T,k\in\Lambda$ and the optimal value of the objective function is zero. Accordingly, on account of that the strategies must be differential, the theorem is proved.

This theorem shows that the optimal mutant is almost the same as the incumbent, except that, while approaching to the terminal time, the mutant gradually but quickly invests all the wealth in only one asset, of which the fraction of the wealth the incumbent assigns to the purchase is minimum at the terminal time. It is also shown that the optimal objective function is strictly negative. This implies from (29) that the market share of the mutant is strictly increasing at the terminal time although it keeps unchanged prior to the end. However, that does not represent the incumbent is evolutionary unstable.

To make this point clear, one notes that, if the mutant plays the ε-optimal strategy given by (31), the *total return* on asset k^* must be the smallest among all assets at the terminal time in the new market portfolio including the investment of the mutant after the entry. As a result, Assumption 7 does not hold anymore at time T and the only asset the mutant purchases has the least value for investment since no rational investor will buy it at the purchase price of the mutant according to Theorem 5. Meanwhile, if the portfolio of the mutant keeps invariant, the right-hand side of (22) is strictly negative at least in a short period after the terminal time and so the mutant's market share will strictly decrease.

For example, suppose an institutional investor buys an asset on a large scale during a short period. Undoubtedly the asset price will shot up and his market share will increase. However, this phenomenon will not last long and the price will definitely fall. Afterwards, he will probably lose what he gained.

Accordingly, another condition is imposed on strategy 2 played by the mutant. The aim is to make sure that the new extended market portfolio after the entry satisfies Assumption 7 still at the terminal time T - that is to say, the constrain $\lambda_{T,k}^2=\lambda_{T,k}^1,k\in\Lambda$ is added in the following. Therefore, the variation problem (30) is changed into:

$$\min_{\lambda_{t,k}^2;0\le t\le T,k\in\Lambda} \int_0^T \sum_{k=1}^K \lambda_{t,k}^2 \left(\frac{\bar{c}_t \lambda_{t,k}^2}{\lambda_{t,k}^1} - \frac{\dot{\lambda}_{t,k}^2 + \bar{c}_t d_t^k}{\lambda_{t,k}^1} \right) dt$$

$$s.t. \begin{cases} \sum_{k=1}^K \lambda_{t,k}^2 = 1, \\ 0 \le \lambda_{t,k}^2 \le 1, 0 \le t \le T, k \in \Lambda, \\ \lambda_{T,k}^2 = \lambda_{T,k}^1, k \in \Lambda. \end{cases} \quad (34)$$

Based on Theorem 6, the following theorem is evident and hence the proof is omitted.

Theorem 7

If Assumption 7 holds, then there is a unique solution of (34), which is given by:

$$\lambda_{t,k}^2 = \lambda_{t,k}^1, 0 \le t \le T, k \in \Lambda. \quad (35)$$

The optimal objective function and $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$ equal to zero for all $t \in [0,T]$. For an arbitrary strategy, which is different from the strategy given by (35) the objective function and $f(r_t^2; \lambda_{t,k}^1, \lambda_{t,k}^2)$ at some time $t \in [0,T]$, are strictly negative in a small neighborhood of $(1,0)$. Hence, the incumbent satisfying Assumption 7 is *evolutionary stable*.

Notice that the terminal time T is arbitrarily selected and in particular if let $T \to \infty$, then one concludes (35) holds anytime. Therefore the following corollary is direct from Theorems 5 and 7:

Corollary 1: An incumbent is *evolutionary stable* if and only if the portfolio of the incumbent satisfies Assumption 7.

EXPLICIT EXPRESSION FOR BOND VALUATION AND BENCHMARK INTEREST RATE

This discuss provides an expression for bond valuation if the market is arbitrage-free. Owing to Definition 1, a market is arbitrage-free if and only if the incumbent, that is, the market portfolio is evolutionary stable or equivalently, if and only if Assumption 7 holds according to Corollary 1. Thus, here it is supposed that Assumption 7 holds.

Clearly, Assumption 7 leads to (27), which yields by integration:

$$\lambda_{T,k}^1 = \exp\left(\int_t^T \bar{c}_u du\right)\left[\lambda_{t,k}^1 - \int_t^T \bar{c}_u d_u^k \exp\left(-\int_t^u \bar{c}_s ds\right) du\right], t \ge 0.$$

or equivalently:

$$\lambda_{t,k}^1 = \lambda_{T,k}^1 \exp\left(-\int_t^T \bar{c}_u du\right) + \int_t^T \bar{c}_u d_u^k \exp\left(-\int_t^u \bar{c}_s ds\right) du, t \ge 0. \quad (36)$$

Following that, the main result here is provided subsequently.

Theorem 8

(i) A bond market is *evolutionary stable* if and only if each bond (relative) price is given by:

$$p_t^k = \int_t^\infty \bar{c}_u d_u^k \exp\left(-\int_t^u \bar{c}_s ds\right) du, t \ge 0, k \in \Lambda; \quad (37)$$

(ii) On the contrary, if the bond prices are collected already in an *evolutionary stable* market, then the discount rate (consumption parameter) is recovered by:

$$\bar{c}_t = \frac{\dot{p}_t^k}{p_t^k - d_t^k}, k \in \Lambda, t \ge 0. \quad (38)$$

Proof

In this bond market, the market portfolio can be considered as the unique strategy, say strategy 1 with $r_t^1 \equiv 1$. Therefore, we conclude from (24) that $p_t^k = \lambda_{t,k}^1$ for $k \in \Lambda$. Let $T \to \infty$ in (36), then (37) is derived since:

$$\lim_{T\to\infty} \lambda_{T,k}^1 \exp\left(-\int_t^T \bar{c}_u du\right) = 0.$$

Lastly, (38) follows directly from (27) and the equality $p_t^k = \lambda_{t,k}^1$ for all $k \in \Lambda$.

Since $p_t^k = \lambda_{t,k}^1$ as shown in the afore proof, the first part of the theorem accords with economic intuition. It says that in an evolutionary stable bond market, one should divide wealth across assets according to the relative asset prices, that is, the arbitrage-free prices, which equal to the discounted values of the relative dividends with the discount rate just being the consumption parameter $\bar{c}_t$.

However, the conclusion of the second part appears more interesting. For example, the benchmark interest rate is extremely important in finance industry, however, how to fix it is open to my best knowledge. In order to approach this problem, the third part of Theorem 8 suggests that in a relatively effective (that is, evolutionary stable) bond market, the relative bond prices are collected first and then the consumption parameter $\bar{c}_t$ are recovered by (38). This parameter should be a good candidate for the benchmark interest rate.

CONCLUSIONS

This paper develops a continuous-time evolutionary finance model with time-dependent strategies, from which an *evolutionary stable* bond market is studied. First, a general discrete-time evolutionary finance is established with an arbitrary trading time interval. A continuous-time model is derived by letting the time interval converge to zero. It is shown that the continuous-time model is well-defined and there is no sudden bankruptcy in the general market only if all the asset prices keep positive.

Second, as a special case of the general model, a bond market with deterministic dividend payoffs is discussed in detail. This bond market is not so artificial as one may consider at first sight since it includes a variety of marketable treasury securities, say Treasury bills, Treasury notes, Treasury bonds, and Treasury Inflation Protected Securities (TIPS) issued by the United States Department of the Treasury. This market also includes Account Treasury bonds issued by the China Department of Finance. It is certified that a bond market is *evolutionary stable*, which implies the market is arbitrage-free, if and only if the *total return* of each asset in the market is identical all the time.

Further, the arbitrage-free price of each bond is derived, which equals an improper integral with the integrand being a discounted value of the dividend payoff. The discount rate is identical for all bonds and equals the market consumption parameter. To my best knowledge, there is not an accepted approach to evaluate the discount rate or the benchmark interest rate although this rate is vital for asset pricing. Equation (38) at least provides a new attempt to attack this problem. For example, if one gets bond prices in a relatively effective or *evolutionary stable* bond market then the discount rate or the benchmark interest rate can be recovered from (38). In particular, along this research line, it must be very interesting to have this model tested with market data. Since this is not a simple work and will lead to another paper, so we leave it in future research.

Finally, although a general continuous-time evolutionary stock market with time-dependent strategies is established in this paper, we afterwards focus on an evolutionary bond market, in which the fundamentals are deterministic and both prices and wealth vary due to market interaction. Clearly a profound study on continuous-time evolutionary stock market in a stochastic world is more interesting and of course more challenging. This work is also left in future research.

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