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Continuous frame in Hilbert space and its applications

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In this paper, we study continuous frames in Hilbert spaces using a family of linearly independent vectors called coherent state (CS) and applying it in any physical space. To accomplish this goal, the standard theory of frames in Hilbert spaces, using discrete bases, is generalized to one where the basis vectors may be labeled using discrete, continuous or a mixture of the two types of indices. A comprehensive analysis of such frames is presented and illustrated by the examples drawn from a toy example Sea Star and the affine group.

Key words: Frame, continuous frame, unitary representation, coherent state (CS), sea star, affine group.

INTRODUCTION

The Hilbert space is the natural framework for the mathematical description of many areas of Physics and Mathematics. The most economical solution, is of course to use an orthonormal basis, $\{\phi_n, n \in \mathrm{N}\}$which gives in addition the uniqueness of decomposition:

$$\phi = \sum_{n \in N} \langle \phi_n | \phi \rangle \phi_n \,. \tag{1}$$

The uniqueness of the decomposition and the orthogonality of the basis vectors, while maintaining its others useful properties: fast convergence, numerical stability of the reconstruction $\{\langle \phi_n | \phi \rangle\} \to \phi$, etc. The resulting object is called a discrete frame, a concept introduced by Duffin and Schaeffer (1952) in the context of non-harmonic Fourier analysis. Latter the concept of generalization of frames was proposed by Daubechies et al. (1986) and then independently by Ali et al., (1993). Let H be an abstract, separable Hilbert space (over the complexes C) and $GL(H)$the group of all bounded linear operators on H which have bounded inverses, throughout this paper.

The definition is very simple: a family of vectors $\{\phi_n, n \in N\} \subset H$ (the Hilbert space) is a frame if there are two constants $A, B > 0$ such that, for all $\phi \in \mathrm{H}$, one has:

$$A \| \phi \|^2 \le \sum_{n \in N} |\langle \phi_n | \phi \rangle|^2 \le B \| \phi \|^2 \tag{2}$$

where the constant A and B are called frame bounds and if $A = B$, then the frame is called tight. Writing

$$S\phi = \sum_{n \in N} \langle \phi_n | \phi \rangle \phi_n, \tag{3}$$

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Here, S is a positive bounded operator with bounded inverse S^{-1}. Additionally, the set $\{\phi_n^{'}, n \in N\}$, where $\phi_n^{'} = S^{-1}\phi_n$, is called dual or reciprocal frame, with frame bounds B^{-1}, A^{-1}. Combining the two allows for the recovery of any element ϕ from its frame components:

$$\phi = \sum_{n \in N} \langle \phi_n^{'} | \phi \rangle \phi_n = \sum_{n \in N} \langle \phi_n | \phi \rangle \phi_n^{'} \tag{4}$$

In mathematical physics, the coherent states, $\eta_g = U(g)\eta$ actually yield the following continuous resolution of the identity on the group, G with the (left) Haar measure dg :

$$\int_G |\eta_g\rangle \langle \eta_g | dg = I, \tag{5}$$

where U being a strongly continuous unitary representation of G on the H, and η a fixed, suitable vector in H. Rewriting Equation 5, we obtain:

$$\int_G \left|\langle \eta_g | \phi \rangle\right|^2 = \|\phi\|^2, \ \forall \phi \in H \tag{6}$$

Analogy with a tight frame is clear and it seems natural to call the set of vectors $\{\eta_g, g \in G\}$ a continuous tight frame.

One should mention furthermore that the theory of frames has been elaborated by Ali et al. (1993, 2000) and Daubechies et al. (1986) and Daubechies (1990). Moreover, the interested reader can refer to Antoine and Grossmann (1976), Duffin and Schaeffer (1952), Gazeau (2016), and Christensen (2003). In this paper, continuous frames in Hilbert space were studied and applied to any physical space.

MATERIALS AND METHODS

Here, mathematical formulation of frames in H was discussed by taking the famous articles by Ali et al., (1993) and Rahimi et al., (2017).

Mathematical formulation of frames

A frame H is the union of a choice of linearly independent sets of vectors, satisfying a specific completeness-or rather over completeness-condition. Each set of vectors could be labeled by a set of discrete, continuous, or mixed indices. Let X be a locally compact space (which could be partly, or completely discrete) and let ν be a regular Borel measure on X with support equal to X. A set of vectors $\eta_x^i \in H, \quad i = 1,2,\ldots\ldots\ldots,n < \infty, \ x \in X$ is said to constitute a rank n frame F if the following conditions hold:

(i) For all $x \in X$, $\{\eta_x^i, i = 1,2,\ldots\ldots n\}$ is a linearly independent set; and

(ii) There exists a positive operator $A \in GL(H)$ such that,

$$\sum_{i=1}^{n} \int_X | \eta_x^i \rangle \langle \eta_x^i | d\nu(x) = A, \tag{7}$$

the integral converging weakly.

To be more explicit, we shall denote the frame so defined by $F\{\eta_x^i, A, n\}$. Note that if $X = J$ is a discrete set and ν is a counting measure, then the condition of Equation 7 yields the following equation.

$$\sum_{i=1}^{n} \sum_{j \in J} |\eta_j^i\rangle \langle \eta_j^i | = A \tag{8}$$

Or simply,

$$\sum_{k \in J} |\eta_k\rangle \langle \eta_k | = A \tag{9}$$

(J ' being another discrete index set). It is in this form that the definition of a frame is conventionally given (Duffin and Schaeffer, 1952; Fornasier and Rauhut, 2005) such that Equation is an obvious generalization.

Let $\sigma(A)$ be the spectrum of the self-adjoint (positive) operator A and let $m(A)$ and $M(A)$ be its infimum and supremum, respectively,

$$m(A) = \inf_{\|\phi\|=1} \langle \phi | A\phi \rangle \neq 0$$
$$M(A) = \sup_{\|\phi\|=1} \langle \phi | A\phi \rangle \neq 0 \tag{10}$$

$\phi \in H$, so that $m(A), M(A) \in \sigma(A)$ and

$$\sigma(\mathrm{A}) \subset [m(\mathrm{A}), \mathrm{M}(\mathrm{A})] \tag{11}$$

It is then clear that $\forall \phi \in H$ (Equation 7), implies the usual frame condition:

$$m(A)\|\phi\|^2 \leq \sum_{i=1}^{n} \int_X \left|\langle \eta_x^i | \phi \rangle\right|^2 d\nu(x) \leq M(A)\|\phi\|^2 \tag{12}$$

In other words, $m(A)$ and $M(A)$ are the frame bounds of common parlance. Furthermore, $M(A)^{-1}$ and $m(A)^{-1}$ are the infimum and supremum of $\sigma(A^{-1})$ and both $M(A)^{-1}, m(A)^{-1} \in \sigma(A^{-1})$, with

$$\sigma(A^{-1}) \subset \left[M(A)^{-1}, m(A)^{-1}\right] \tag{13}$$

Defining

$$\eta_x'^i = A^{-1}\eta_x^i \tag{14}$$

$i = 1,2, \cdots, n,\ x \in X,$ we easily verify that

$$\sum_{i=1}^{n}\int_X \left|\eta_x'^i\right\rangle\left\langle\eta_x'^i\right| d\nu(x) = A^{-1}, \tag{15}$$

Satisfying the frame condition

$$M(A)^{-1}\|\phi\|^2 \le \sum_{i=1}^{n}\int_X \left|\left\langle\eta_x'^i \middle| \phi\right\rangle\right|^2 d\nu(x) \le m(A)^{-1}\|\phi\|^2 \tag{16}$$

$\forall\phi \in H$. The frame $F\{\eta_x'^i, A^{-1}, n\}$ is said to be the dual frame of $F\{\eta_x^i, A, n\}$. The width or snugness of the frame $F\{\eta_x^i, A, n\}$ is as follows:

$$w(F) = \frac{\mathrm{M}(\mathrm{A}) - m(\mathrm{A})}{\mathrm{M}(\mathrm{A}) + m(\mathrm{A})} \tag{17}$$

Obviously, $0 \le w(F) < 1$ and $w(F)$ measure the spectral width of the operator A. If $w(F) = 0$, that is, if $A = \lambda I$ ($\lambda > 0$ and $I =$ the identity operator on H), then the frame F is called tight. Note that a frame $F\{\eta_x^i, A, n\}$ and its dual $F\left\{\eta_x^{i'}, A^{-1}, n\right\}$ have the same width and the frame is self-dual iff $A = I$ associated naturally to the frame $F\{\eta_x^i, A, n\}$, there is a self-dual, tight frame $F\{\overline{\eta}_x^i, I, n\}$, with:

$$\overline{\eta}_x^i = A^{-1/2}\eta_x^i, \tag{18}$$

In fact, if T is any operator $GL(H)$ and T^* its adjoint, then writing

$$\overline{\eta}_x^i = T\eta_x^i, \qquad \overline{A} = TAT^*, \tag{19}$$

We see that

$$\sum_{i=1}^{n}\int_X \left|\overline{\eta}_x^i\right\rangle\left\langle\overline{\eta}_x^i\right| d\nu(x) = \overline{A} \tag{20}$$

Such that we obtain $F\{\overline{\eta}_x^i, \overline{A}, n\}$. In particular, if T is a unitary operator $(TT^* = T^*T = I)$, then $F\{\eta_x^i, A, n\}$ and $F\{\overline{\eta}_x^i, \overline{A}, n\}$ are said to be unitarily equivalent frames. In this case, of course, $\sigma(A) = \sigma(\overline{A})$ and the frame widths, $w(F\{\eta_x^i, A, n\}) = w(F\{\overline{\eta}_x^i, \overline{A}, n\})$. Note, however, that Equation 18 is not the only way to obtain a self-dual tight frame from $F\{\eta_x^i, A, n\}$. Indeed, if U is any unitary operator on H, then since we can always write $A = A^{1/2}U^*UA^{1/2}$, we see that with:

$$\overline{\eta}_x^i = UA^{-1/2}\eta_x^i \tag{18a}$$

Another self-dual tight frame $F\{\overline{\eta}_x^i, I, n\}$ is obtained which is unitarily equivalent to $F\{\eta_x^i, I, n\}$.

There is a sense in which any two frames $F\{\eta_x^i, A, n\}$ and $F\{\overline{\eta}_x^i, \overline{A}, n\}$, related by Equation 19, are equivalent and we proceed to study this point a little more closely. If we introduce the positive operator:

$$F(x) = \sum_{i=1}^{n}\left|\eta_x^i\right\rangle\left\langle\eta_x^i\right|, \tag{21}$$

For each $x \in X$, Equation 7 assumes the form:

$$\int_X F(x)d\nu(x) = \mathrm{A} \tag{22}$$

Of course, for each x there is more than one choice of linearly independent vectors η_x^i for which Equation 21 is satisfied. The arbitrariness is quantified by Ali et al., (1993) and Friedberg et al., (2003).

Theorem 1. *If* $\{\overline{\eta}_x^i, i = 1,2,\ldots\ldots, n\}$ is linearly independent set of vectors for which

$$F(x) = \sum_{i=1}^{n}\left|\overline{\eta}_x^i\right\rangle\left\langle\overline{\eta}_x^i\right|, \tag{23}$$

if there exists an $n \times n$ unitary matrix $U(x)$, with elements $U_{ij}(x)$, such that

$$\overline{\eta}_x^i = \sum_{j=1}^{n} U_{ij}(x)\eta_x^j,\ i = 1,2,\ldots\ldots\ldots, n. \tag{24}$$

Proof

It is clear from the unitary of $U(x)$ (that is, from $\sum_{j=1}^{n}\overline{U_{ij}(x)}U_{kj}(x)=\delta_{ik}$) that if $\overline{\eta}_x^i$ and η_x^i are related by Equation 24, then Equation 23 holds. On the other hand, assume that $\overline{\eta}_x^i$ are linearly independent and satisfy Equation 23. Then, for all $\phi \in H$,

$$\langle \phi | F(x) \phi \rangle = \sum_{i=1}^{n} \left| \langle \eta_x^i | \phi \rangle \right|^2 = \sum_{i=1}^{n} \left| \langle \overline{\eta}_x^i | \phi \rangle \right|^2 . \tag{25}$$

Setting

$$z_i = \langle \eta_x^i | \phi \rangle \in C$$
$$z_i' = \langle \overline{\eta}_x^i | \phi \rangle \in C \tag{26}$$

This equation becomes

$$\sum_{i=1}^{n} |z_i|^2 = \sum_{i=1}^{n} |z_i'|^2 . \tag{27}$$

Now let $P(x)$ be the projection operator onto the range of $F(x)$ (and hence $P(x)$ is also its support). Then both $\{\eta_x^i\}_{i=1}^n$ and $\{\overline{\eta}_x^i\}_{i=1}^n$ span the subspace $P(x)H$ of H, and there exists an $n \times n$ invertible matrix $U(x)$ for which

$$\overline{\eta}_x^i = \sum_{j=1}^{n} U_{ij}(x) \eta_x^j .$$

Thus, for all $\phi \in H$,

$$\langle \overline{\eta}_x^i | \phi \rangle = \sum_{j=1}^{n} U_{ij}(x) \langle \eta_x^j | \phi \rangle$$
$$=> z_i' = \sum_{j=1}^{n} U_{ij}(x) z_j ,$$

and, hence by Equation 27, $U(x)$ is unitary.

RESULTS AND DISCUSSION

Applications of continuous frames

Here, we discuss two useful examples on the continuous frame in Hilbert space (Ali et al., 2000, 1993; Gazeau (2016); Rahimi et al., (2017); Friedman A (1970).

Example 1: A toy example - Sea star

Consider the Euclidean plane with Dirac notations

$$\vec{j} \equiv \left| \pi/2 \right\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\langle 0|0 \rangle = 1 = \langle \pi/2 | \pi/2 \rangle, \langle 0 | \pi/2 \rangle = 0$$
$$I = |0\rangle\langle 0| + |\pi/2\rangle\langle \pi/2|$$
$$\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In the Euclidean plane, $\Re^2$ orthonormal basis is defined by the two ket vectors $|0\rangle$ and $|\pi/2\rangle$, where $|\theta\rangle$ denotes the unit vector with polar angle $\theta \in [0, 2\pi)$. This frame is such that

$$\langle 0|0 \rangle = 1 = \langle \pi/2 | \pi/2 \rangle, \langle 0 | \pi/2 \rangle = 0$$

and the resolution of the identity comes through the sum of their corresponding orthogonal projections,

$$I = |0\rangle\langle 0| + |\pi/2\rangle\langle \pi/2|$$
$$\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, coherent state $= \left| \frac{2n\pi}{5} \right\rangle \equiv R\left(\frac{2n\pi}{5}\right)|0\rangle$, where $n = 0,1,2,3,4 \,\mathrm{mod}(5)$ and matrix representation is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

To the unit vector $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|\pi/2\rangle$ corresponds with the orthogonal projector p_θ given by

$$p_\theta = |\theta\rangle\langle\theta| = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$
$$= R(\theta)|0\rangle\langle 0|R(-\theta)$$

Again the resolution of the identity for Sea Star is as follows:

$$\frac{2}{5}\sum_{n=0}^{4}\left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

Here, $X=\{0,1,2,3,4\}\equiv$is the set of orientations $\equiv$ angles $\frac{2\pi n}{5}$ explored by the starfish which is equipped with uniform weight $2/5$. The operator

$p_n = \left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right|$ acts on $H=\Re^2$.

From $p_n = \left|\frac{2\pi n}{5}\right\rangle\left\langle\frac{2\pi n}{5}\right|$ acts on $H=\Re^2$ and given a $n_0\in\{0,1,2,3,4\}$ one derives the probability distribution on $X=\{0,1,2,3,4\}$

$$tr\left(p_{n_0}p_n\right) = \left|\left\langle\frac{2\pi n_0}{5}\middle|\frac{2\pi n}{5}\right\rangle\right|^2 = \cos^2\left(\frac{2\pi(n_0-n)}{5}\right)$$

Choosing $n_0=0$, values are

$$tr\left(p_0 p_n\right) = \left(\frac{1}{2\tau}\right)^2 \approx 0.0955 \text{ for } n=1,4$$

$$tr\left(p_0 p_n\right) = \left(-\frac{\tau}{2}\right)^2 \approx 0.6545 \text{ for } n=2,3$$

with $\tau = (1+\sqrt{5})/2 \approx 1.618$ (golden mean). Check from $\tau^2=\tau+1$ that

$$\frac{2}{5}\left(1+\frac{1}{4\tau^2}+\frac{\tau^2}{4}\right)=1$$

Projectors $n\mapsto p_n$ with Hilbert-Schmidt norm $\|p_n\|^2 = tr\left(p_n p_n^*\right) = tr\left(p_n^2\right)=1$ allow a localization distance on X to be defined:

$$d_{HS}(n,n') = \|p_n - p_{n'}\| = \sqrt{tr\left(p_n - p_{n'}\right)^2}$$
$$= \sqrt{2}\left|\sin\frac{2\pi(n-n')}{5}\right|$$

Similarly, any regular N-fold polygon in the plane have satisfied the resolution of unity by the following way:

$$\frac{2}{N}\sum_{n=0}^{N}\left|\frac{2\pi n}{N}\right\rangle\left\langle\frac{2\pi n}{N}\right| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Finally, if we consider the continuous case, then we have,

$$\frac{1}{\pi}\int_0^{2\pi} d\theta|\theta\rangle\langle\theta| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Example 2: The affine group

In order to obtain a concrete situation where the more general considerations of the results do indeed apply, let us construct a rather unorthodox family of CS for the affine group G_A. The connected affine group consists of transformation of $\Re$ of the following type:

$$x\mapsto ax+b,\ x\in\Re \tag{28}$$

where, $a>0, b\in R$. Writing

$$g=(a,b)\in G_A, \tag{29}$$

We have the multiplication law,

$$g_1 g_2 = (a_1 a_2, b_1 + a_1 b_2). \tag{30}$$

The matrix representation

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}; a\neq 0, b\in R$$

reproduces this composition rule. The inverse is given by the matrix

$$(a,b)^{-1} = \begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ 0 & 1\end{pmatrix} = \left(\frac{1}{a}, -\frac{b}{a}\right)$$

If we take a vector $\begin{pmatrix}x\\ 1\end{pmatrix}$, then trivially the action of the matrix Equation 30 on this vector is given by

$$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix} = \begin{pmatrix}ax+b\\ 1\end{pmatrix}$$

which exactly reproduces the action $x \to ax+b$ on $\Re$. It is clear that this class of matrices constitute a group, called the full affine group which is denoted by G_A but note that, it is not a connected group. So that, if we consider only those matrices from the class of matrices where $a>0$, then the class of matrices forms a subgroup of G_A which is denoted by $G_A{}^{+}$.

On the Hilbert space

$$H_n = L^2(R^+, \mu_n),\ n = \text{integer} \geq 1$$

$$d\mu_n(r) = r^{n+1}dr, \tag{31}$$

G_A has the unitary irreducible representation $g \mapsto U_n(g)$ given by

$$(U_n(a,b)\psi)(r) = a^{n/2}e^{ibr}\psi(ar),\ \psi \in H_n \tag{32}$$

Consider the subgroup H of G,

$$H = \{g \in G_A \mid g = (a,0), a \in \Re^+\}. \tag{33}$$

Then $G_A/H \approx \Re^+,\ \forall(a,b) \in G_A,$

$$(a,b) = (1,b)(a,0), \qquad b \in \Re \tag{34}$$

Also, since $\forall x \in \Re,$

$$(a,b)(1,x) = (1,ax+b)(a,0), \tag{35}$$

On the coset space G_A/H, the action of G_A in $\Re$, can be written as:

$$gx = ax+b, \qquad g = (a,b) \in G_A \tag{36}$$

On G_A/H, so parameterized, we choose the quasi-invariant measure,

$$dv = dx, \tag{37}$$

so that

$$\frac{dv_g}{dv}(x) \equiv \lambda(g,x) = a, g = (a,b) \tag{38}$$

Choosing a (global) section, $\sigma : \Re \to G_A$

$$\sigma(x) = (1,x), \tag{39}$$

we get

$$\lambda(\sigma(x),x) = 1, \tag{40}$$

so that coherent states may be constructed[14] for a suitable choice of $\eta \in H_n$ as

$$\eta_x = \lambda(\sigma(x),x)U_n(\sigma(x))\eta, x \in \Re \tag{41}$$

Thus,

$$\eta_x(r) = e^{ixr}\eta(r),\ r \in \Re^+ \tag{42}$$

The operator,

$$A = \int_R |\eta_x\rangle\langle\eta_x| dx, \tag{43}$$

is easily computed to be a multiplication operator on H_n,

$$(\mathrm{A}\phi)(r) = 2\pi r^n\ |\eta(r)|^2 \phi(x,)\ \forall\phi \in H_n \tag{44}$$

In order for A to be bounded and invertible with $\|A\| = 1,$ we must, therefore, impose on $\eta \in H_n$ the conditions,

$$(i) \quad \sup_{r \in R^+}\left[2\pi r^{n-1}|\eta(r)|^2\right] = 1 \tag{45}$$

$$(ii)\ |\eta(r)|^2 \neq 0, \text{ exceptperhaps at isolated points}$$

$$r \in \Re^+ \tag{46}$$

These conditions, together with the fact that $\eta \in H_n$ that is,

$$\int_{R^+} |\eta(r)|^2 d\mu_n < \infty, \tag{47}$$

imply that A^{-1} is unbounded. In fact, since

$$(\mathrm{A}^{-1}\phi)(r) = \frac{1}{2\pi}\frac{r^{1-n}}{|\eta(r)|^2}\phi(r), \tag{48}$$

ϕ lies in the set $D(A^{-1})$ iff

$$\frac{1}{(2\pi)^2}\int_{R^+}\frac{r^{2-2n}}{|\eta(r)|^4}|\phi(r)|^2 d\mu_n(r)<\infty. \tag{49}$$

Thus, the representation U_n of G_A is square-integrable mod (H,σ) and coherent states (Equation 42) may be constructed for any $\eta \in H_n$ satisfying the admissibility conditions (Equations 45 and 46). However, the coherent states do not define a frame, since A^{-1} is unbounded. In fact from Equations 45 and 47, it follows that none of the vectors η_x is the domain of either $\mathrm{A}^{-1/2}$ or A^{-1}. The map

$$W_\eta : H_n \to L^2(\Re, dx),$$

$$\left(W_\eta\varphi\right)(x)=\langle\eta_x|\varphi\rangle=\int_{\Re^+} e^{irx}\overline{\eta(r)}\varphi(r)\,d\mu_n(r), \tag{50}$$

is clearly bounded and its range in $L^2(R,dx)$ is closed in the norm

$$\|\Phi\|_\eta^2=\left\langle\Phi\middle|A_\eta^{-1}\Phi\right\rangle_{L^2(\Re,dx)}, \tag{51}$$

where A_η^{-1} is the image, on $L^2(\Re,dx)$, of A^{-1} under W_η. The evaluation maps $E_\eta(x): H_\eta \to C$, given by $\Phi \mapsto \Phi(x)$, is easily seen to be continuous (in the Equations 51 norm). However, the reproducing kernel,

$$\mathrm{K}(x,y)=\left\langle\eta_x\middle|\mathrm{A}^{-1}\eta_y\right\rangle=\frac{1}{2\pi}\int_{R^+} e^{i(x,y)r}\,dr, \tag{52}$$

is a distribution which defines a sesquilinear form on H_η.

Conclusion

An introductory-level theory of continuous frames in Hilbert space was studied, focusing primarily on the analysis and ending with its applications to possible physical space. The mathematical construction of frames was illustrated by the examples drawn from a toy example Sea Star and the affine group.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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