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Full Length Research Paper

On three-dimensional generalized sasakian space – forms

S. Yadav* and D. L. Suthar

Department of Mathematics, Alwar Institute of Engineering and Technology, M.I.A. Alwar-301030, Rajasthan, India.

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The object of the present paper is to study locally φ–symmetric three-dimensional generalized Sasakian space forms and such manifolds with Ricci semi symmetric, η-parallel Ricci tensor and cyclic Ricci tensor. Such space forms with non-null concircular vector field are also considered.

Key words: Generalized sasakian space forms, cosymplectic, locally-φ-symmetric, η-parallel ricci tensor, cyclic ricci tensor and non-null concircular vector field.

INTRODUCTION

Generalized Sasakian space forms are considered as special cases of an almost contact metric manifold (Alegre et al., 2004). Three-dimensional LP-Sasakian manifolds and three-dimensional trans-Sasakian manifolds were studied respectively by De and Tripathi (2003), Venkatesha and Bagewadi, 2006). Where as three-dimensional Para-Sasakian manifolds, three-dimensiomal Lorentzian α–Sasakian manifolds and three-dimensional quasi-Sasakian manifolds were studied respectively by Bagewadi and et al. (2007), Yildiz et al. (2009) and De and Sarkar (2009). In this paper we extend same work for three- dimensional generalized Sasakian space forms and obtain some results.

GENERALIZED SASAKIAN SPACE-FORM

A $(2n+1)$-dimensional Riemannian manifold (M,g) is called an almost contact manifold if the following results hold:

(a) $\varphi^2(X) = -X + \eta(X)\xi$, (1)

(b) $\varphi\xi = 0$

(a) $\eta(\xi)=1$, (2)

(b) $g(X,\xi)=\eta(X)$,(c) $\eta(\varphi X) = 0$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (3)$$

(a) $g(\varphi X, Y) = -g(X, \varphi Y)$ (4)

(b) $g(\varphi X, X) = 0$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) \quad (5)$$

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \varphi Y)$$

Where Φ is called the fundamental two-form of the manifold. If ξ is a killing vector field the manifold is called a k-contact manifold. It is well known that a contact metric manifold is k-contact if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on (M,g). An almost contact metric manifold is Sasakian if and only if $(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X$, for any vector fields X,Y .

Blair (1967) introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds.

*Corresponding author. E-mail: prof_sky16@yahoo.com.

Again Olszak (1986) introduced and characterized three-dimensional quasi-Sasakian manifolds. An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\varphi X \quad , \tag{6}$$

for all $X \in TM$ and a function β such that $\xi\beta = 0$

As the consequence of (6), we get

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\varphi X, Y) \tag{7}$$

$$(\nabla_X \eta)(\xi) = -\beta g(\varphi X, \xi) = 0 \tag{8}$$

Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$ .It is known that for a three-dimensional quasi-Sasakian manifold the Riemannian curvature tensor satisfies

$$R(X,Y)\xi = \beta^2\{\eta(Y)X - \eta(X)Y\} + d\beta(Y)\varphi X - d\beta(X)\varphi Y \, , \tag{9}$$

For a $(2n+1)$ -dimensional generalized Sasakian-space-form, we have

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ &+ f_3\left\{\begin{matrix} \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \end{matrix}\right\} \end{aligned} \tag{10}$$

$$R(X,Y)\xi = (f_1 - f_2)\{\eta(Y)X - \eta(X)Y\}, \tag{11}$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{12}$$

$$g(R(\xi, X)Y, \xi) = (f_1 - f_3)g(\varphi X, \varphi Y), \tag{13}$$

$$R(\xi, X)\xi = (f_1 - f_3)\varphi^2 X \tag{14}$$

$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) \\ &- (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \end{aligned} \tag{15}$$

$$Q\xi = 2n(f_1 - f_3)\xi \tag{16}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + 2n(f_3 - f_1)\eta(X)\eta(Y) \tag{17}$$

The Conformal curvature tensor vanishes in three-dimensional Riemannian manifold therefore we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y - \frac{\tau}{2}[g(Y,Z)X - g(X,Z)X] \end{aligned} \tag{18}$$

Here S is the ricci tensor and τ is the scalar curvature tensor of the space-forms. It is known that (2006) a ($2n+1$)-dimensional $(n>1)$ generalized Sasakian-space-form is conformally flat if and only if $f_2 = 0$.

BASIC RESULTS

Theorem 1

In a three-dimensional generalizedSasakian space forms $M(f_1, f_2, f_3)^{2n+1}, (n=1)$ we get

$$\begin{aligned} QX &= \left[\frac{\tau}{2} + (1-2n)(f_1 - f_3)\right]X \\ &+ \left[(4n-1)(f_1 - f_3) - \frac{\tau}{2}\right]\eta(X)\xi \end{aligned} \tag{i}$$

$$\begin{aligned} S(X,Y) &= \left[\frac{\tau}{2} + (1-2n)(f_1 - f_3)\right]g(X,Y) \\ &+ \left[(4n-1)(f_1 - f_3) - \frac{\tau}{2}\right]\eta(X)\eta(Y) \end{aligned} , \tag{ii}$$

$$\begin{aligned} R(X,Y)Z &= \left[\frac{\tau}{2} + 2(1-4n)(f_1 - f_3)\right] \\ &\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \left[(4n-1)(f_1 - f_3) - \frac{\tau}{2}\right]\left\{\begin{matrix} g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi \\ +\eta(Y)\eta(Z)X \\ -\eta(X)\eta(Z)Y \end{matrix}\right\} \end{aligned} \tag{iii}$$

Proof 1

Substituting $Z = \xi$ in (18) and using (2-b) (11) and (15) we get

$$\begin{aligned} (f_1 - f_3)[\eta(Y) - (X)\xi] &= \eta(Y)QX - \eta(X)QY \\ &+ 2n(f_1 - f_3)\eta(Y)X - 2n(f_1 - f_3)\eta(X)Y \\ &- \frac{\tau}{2}[\eta(Y)X - \eta(X)Y] \end{aligned} \tag{19}$$

Again putting $Y = \xi$ in (19) and using (2-b), we have

$$\begin{aligned} QX &= \left[\frac{\tau}{2} - (1-2n)(f_1 - f_2)\right]X \\ &+ \left[(4n-1)(f_1 - f_3) - \frac{\tau}{2}\right]\eta(X)\xi \end{aligned} \tag{20}$$

And

$$S(X,Y)=\left[\frac{\tau}{2}+(1-2n)(f_1-f_3)\right]g(X,Y)$$
$$+\left[(4n-1)(f_1-f_3)-\frac{\tau}{2}\right]\eta(X)\eta(Y) \quad (21)$$

By virtue of (19) and (20) the equations18) reduces as (iii).

This completes the proof of the theorem1.

Corollary 1

A three-dimensional generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with constant function f_1-f_3 is a manifolds of constant curvature if and only if the scalar curvature is $\tau=2(4n-1)(f_1-f_3)$.

Corollary 2

A three-dimensional generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with constant curvature is flat if and only if $f_1=f_3$.

RICCI-SEMISYMMETRIC THREE-DIMENSIONAL GENERALIZED SASAKIAN SPACE -FORMS

Theorem 2

A three-dimensional Ricci semi symmetric generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with constant function f_1-f_3 is a manifold of constant curvature.

Proof 2

We consider three-dimensional generalized Sasakian space form satisfying the condition

$$R(X,Y)\cdot S=0 \quad (22)$$

From (22), it follows that

$$S(R(X,Y)U,V)+S(U,R(X,Y)V=0 \quad (23)$$

Substituting $X=\xi$ in (23) and using (12) we get

$$4n^2(f_1-f_3)^2[g(Y,U)\eta(V)+g(Y,V)\eta(U)]$$
$$-2n(f_1-f_3)[S(Y,V)\eta(U)+S(U,Y)\eta(V)]=0 \quad (24)$$

Let $\{e_i\}$, $i=1,2,3$ be an orthonormal basis of the tagent space at any point of the manifold then putting $Y=U=e_i$ in (24) and taking summation over i , $1\le i\le 3$, we get

$$g(e_i,e_i)\eta(V)[4n^2(f_1-f_3)^2-2nk(f_1-f_3)]=0, \quad (25)$$

This implies that

$$\tau=2(4n-1)(f_1-f_3), \quad (26)$$

where $k=\left[\frac{\tau}{2}-(1-2n)(f_1-f_3)\right]$

and $g(e_i,e_i)\neq 0$

This completes the proof of the theorem 2.

LOCALLY φ-SYMMETRICTHREE-DIMENSIONAL GENERALIZED SASAKIAN SPACE -FORMS

Definition 1

A generalized Sasakian space-forms is said to be locally φ-symmetric if

$$\varphi^2(\nabla_W R)(X,Y)Z=0,$$

for all vector fields X,Y,Z and W orthogonal to ξ. This notion was introduced by Takahashi (1977) for Sasakian manifolds.

Theorem 3

A three-dimensional cosymplectic (non cosymplectic) generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with constant function f_1-f_3 is locally φ –symmetric if and only if the scalar curvature is constant.

Proof 3

Now differentiating (theorem1, iii) covariantly with respect to W we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{d\tau(W)}{2}\left[g(Y,Z)X-g(X,Z)Y\right] \\ &-\frac{d\tau(W)}{2}\begin{bmatrix} g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ \eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y \end{bmatrix} \\ &+\left((4n-1)(f_1-f_3)-\frac{\tau}{2}\right)\begin{bmatrix} g(Y,Z)(\nabla_W\eta)(X)\xi \\ -g(X,Z)(\nabla_W\eta)(Y)\xi \\ +g(Y,Z)\eta(X)\nabla_W\xi \\ -g(X,Z)\eta(Y)\nabla_W\xi \\ +(\nabla_W\eta)(Y)\eta(Z)X \\ +(Y)(\nabla_W\eta)(Z)X \\ -(\nabla_W\eta)(X)\eta(Z)Y \\ -\eta(X)(\nabla_W\eta)(Z)Y \end{bmatrix}\end{aligned} \tag{27}$$

Taking X,Y,Z and W orthogonal to ξ and using (6,) and (7), we have from above

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{d\tau(W)}{2}\left[g(Y,Z)X-g(X,Z)Y\right] \\ &+\left((4n-1)(f_1-f_3)-\frac{\tau}{2}\right)\begin{bmatrix} -\beta g(Y,Z)g(\varphi W,X)\xi \\ +\beta g(X,Z)g(\varphi W,Y)\xi \end{bmatrix},\end{aligned} \tag{28}$$

From (28) it follows that,

$$\varphi^2(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}\left[g(Y,Z)X-g(X,Z)Y\right] \tag{29}$$

This completes the proof of the theorem3.

Corollary 3

A three-dimensional Ricci semi symmetric generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ is locally φ –symmetric if and only if the scalar curvature is constant.

THREE-DIMENSINAL GENERALIZED SASAKIAN SPACE FORMS WITH η-PARALLEL RICI TENSOR

Definition 2

The Ricci tensor S of generalized Sasakian space-forms is said to be η-parallel if it satisfies

$$(\nabla_W S)(\varphi X,\varphi Y)=0.$$

for all vector fields $W, X,$ and Y. The notion of Ricci η-parallelity for Sasakian manifolds was introduced by Kon (1976).

Theorem 4

A three-dimensional generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ has η–parallel Ricci tensor. Then the scalar curvature is constant.

Proof 4

From (21) we get by virtue of (2.-c) (3), that

$$S(\varphi X,\varphi Y)=\left[\frac{\tau}{2}+(1-2n)(f_1-f_3)\right]\{g(X,Y)-\eta(X)\eta(Y)\} \tag{30}$$

Differentiating (30) covariantly along W ,we get

$$\begin{aligned}(\nabla_W S)(\varphi X,\varphi Y) &= \frac{d\tau(W)}{2}\left[g(X,Y)-\eta(X)\eta(Y)\right] \\ &-\left[\frac{\tau}{2}+(1-2n)(f_1-f_3)\right]\begin{Bmatrix}\eta(Y)(\nabla_W\eta)X \\ +\eta(X)(\nabla_W\eta)Y\end{Bmatrix}\end{aligned} \tag{31}$$

By using (7) in (31) we get

$$\begin{aligned}&d\tau(W)\left[g(X,Y)-\eta(X)\eta(Y)\right]+\left[\tau+2(1-2n)(f_1-f_3)\right] \\ &\{\beta g(\varphi W,X)+\beta g(\varphi W,Y)\eta(X)\}=0\end{aligned} \tag{32}$$

Substituting $X=Y=e_i$ in (32) and taking summation over i, $1\le i\le 3$, we get

$$d\tau(W)=0, \text{ for all } Z. \tag{33}$$

This completes the proof of the theorem4.

Corollary 4

A three-dimensional generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with η–parallel Ricci tensor is locally φ–symmetric.

THREE-DIMENSINAL GENERALIZED SASAKIAN SPACE FORMS WITH CYCLIC PARALLEL RICI TENSOR

Gray (1978) introduced two classes of Riemannian manifolds determine by the covariant derivative of the Ricci tensor. The first one is class consisting of all Riemannian manifolds whose Ricci tensor is of Codazzi tensor, that is

$$(\nabla_W S)(X,Y)+(\nabla_X S)(W,Y)=0$$

The second one is the class of consisting of all Riemannian manifold whose Ricci tensor is cyclic parallel, that is

$$(\nabla_W S)(X,Y)+(\nabla_X S)(Y,W)+(\nabla_Y S)(W,X)=0$$

Theorem 5

A three-dimensional non cosymplectic generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with cyclic parallel Ricci tensor is a manifold of constant curvature if and only if function f_1-f_3 is constant.

Proof 5

We suppose that a three-dimensional generalized Sasakian space form satisfies the cyclic Ricci tensor. Then we have

$$(\nabla_W S)(X,Y)+(\nabla_X S)(Y,W)+(\nabla_Y S)(W,X)=0 \quad (34)$$

Taking covariant derivative of (21) along W, we obtain

$$(\nabla_W S)(X,Y)=\frac{d\tau(W)}{2}[g(X,Y)-\eta(X)\eta(Y)]$$
$$+\left[(4n-1)(f_1-f_3)-\frac{\tau}{2}\right][\eta(Y)(\nabla_W\eta)(X) \quad (35)$$
$$+\eta(X)(\nabla_W\eta)(Y)$$

By virtue of (7) and (33) Equation (35) take the form

$$(\nabla_W S)(X,Y)=\left[(4n-1)(f_1-f_3)-\frac{\tau}{2}\right]$$
$$[\beta\eta(Y)g(\varphi W,X)-\beta\eta(X)g(\varphi W,Y)] \quad (36)$$

Taking cyclic permutation of (36) and adding them we get by virtue of (34), we get

$$\left[(4n-1)(f_1-f_3)-\frac{\tau}{2}\right][\beta\eta(Y)g(\varphi W,X)$$
$$-\beta\eta(X)g(\varphi W,Y)$$
$$+\beta\eta(W)g(\varphi X,Y)-\beta\eta(Y)g(\varphi X,W)$$
$$+\beta\eta(W)g(\varphi Y,X)-\beta\eta(X)g(\varphi Y,W)]$$
$$(37)$$

Substituting $Y=W=e_i$ in (37) and taking summation over i, $1\le i\le 3$, we get

$$\tau=2(4n-1)(f_1-f_3) \quad , \quad (38)$$

This completes the proof of the theorem.

Corollary 5

The necessary and sufficient condition for three-dimensional generalized Sasakian space forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ satisfies cyclic condition if the manifold is cosymplectic.

Corollary 6

A three-dimensional non cosymplectic generalized Sasakian space-forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ with cyclic is Ricci semi symmetric.

THREE-DIMENSINAL GENERALIZED SASAKIAN SPACE FORMS ADMITTING A NON-NULL $(|V|^2\neq 0)$ CONCIRCULAR VECTOR FIELD

Definition 3

A vector field V in generalized Sasakian space-forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ is said to be concircular vector field if it satisfies an equation of the form

$$\nabla_X V=\lambda X, \quad (39)$$

For all X ,where λ is a scalar function. In particular if $\lambda=0$, then V is parallel.

Theorem 6

If a three-dimensional generalized Sasakian space-forms $M(f_1,f_2,f_3)^{2n+1},(n=1)$ admit a non- null concircular vector field then the manifold is an Einstein manifold.

Proof 6

Differentiating (39) coraiantly we have

$$\nabla_X\nabla_Y V-\nabla_Y\nabla_X V=d\lambda(X)Y-d\lambda(Y)X, \quad (40)$$

By virtue of Ricci identity we get from (40) that

$$R'(X,Y,V,Z)=d\lambda(X)g(Y,Z)-d\lambda(Y)g(X,Z), \quad (41)$$

where $g(R(X,Y,V)Z)=R'(X,Y,V,Z)$

Replacing $Z=\xi$ and using (2.-b), we get

$$R'(X,Y,V,\xi)=\eta(R(X,Y)V)$$
$$=d\lambda(X)\eta(Y)-d\lambda(Y)\eta(X) \quad (42)$$

Taking inner product of (theorem 1.-iii) with ξ and replacing $Z=V$, we obtain

$$\eta(R(X,Y)V)=[(1-4n)(f_1-f_3)]\begin{Bmatrix} g(Y,V)\eta(X) \\ -g(X,V)\eta(Y)\end{Bmatrix} \quad (43)$$

From (42) and (43), we get

$$d\lambda(X)\eta(Y)-d\lambda(Y)\eta(X)$$
$$=[(1-4n)(f_1-f_3)]\{g(Y,V)\eta(X)-g(X,V)\eta(Y)\} \quad (44)$$

Subsisting $X=\varphi X$ and $Y=\xi$ in (44) and using (2. a, c), we get

$$d\lambda(\varphi X)=-[(1-4n)(f_1-f_3)]g(\varphi X,V) \quad (45)$$

Again putting $X=\varphi X$ in (45) and using (2.), we obtain

$$-d\lambda(X)+\eta(X)d\lambda(\xi)=$$
$$[(1-4n)(f_1-f_3)]\{g(X,V)-\eta(X)\eta(V)\}, \quad (46)$$

Multiplying both sides of (46) by $g(X,V)$, we get

$$-d\lambda(X)g(X,V)+\eta(X)d\lambda(\xi)g(X,V)=$$
$$[(1-4n)(f_1-f_3)]\{g(X,V)-\eta(X)\eta(V)\}g(X,V), \quad (47)$$

By virtue of (11) and (41), we have

$$d\lambda(X)g(Y,V)=d\lambda(Y)g(X,V), \quad (48)$$

Multiplying both of (48) by $\eta(X)\neq 0$ for all X and putting $Y=\xi$, we obtain that

$$d\lambda(X)\eta(V)\eta(X)=d\lambda(\xi)g(X,V)\eta(X) \quad (49)$$

By virtue of (46) and (49) we get

(i) $d\lambda(X)=[(1-4n)(f_1-f_3)]g(X,V)$ (50)
for all X or
(ii) $g(X,V)=\eta(X)\eta(V)$ for all X [contradicts our assumption]

From (41) and (50-i), we have

$$R'(X,Y,V,Z)=[(1-4n)(f_1-f_3)]$$
$$\{g(X,V)g(Y,Z)-g(Y,V)g(X,Z)\} \quad (51)$$

Substituting $X=Z=e_i$, $i=1,2,3$ in (51) and taking summation for i, $1\leq i\leq 3$, we get

$$S(Y,V)=[2(1-4n)(f_1-f_3)]g(Y,V), \quad (52)$$

This completes the proof of the theorem 6.

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