Full Length Research Paper

Some common fixed point theorems in $D^*$-metric space

Veerapandi T.1* and Aji M. Pillai2

1Department of Mathematics, P. M. T. College, Melaneelithanallur 627 953, India.
2Manomaniam Sundaranar University, Tirunelveli, India.

Accepted 9 August, 2011

In this paper we establish some common fixed point theorems for contraction and generalized contraction mappings in $D^*$-metric space which is introduced by Shaban et al. (2007). In what follows, $(X, D^*)$ will denote $D^*$-metric space, $N$, the set of all natural number and $R^+$, the set of all positive real numbers.

Key words: $D^*$-metric contractive mapping, complete $D^*$-metric space, common fixed point theorem.

INTRODUCTION

There have been a number of generalization in generalized metric space (or $D$-Metric space) initiated by Dhage (1992). He proved the existence of unique fixed point theorems of a self map satisfying a contractive condition in complete and bounded $D$-Metric space. Dealing with $D$-Metric space, Ahmad et al. (2001), Dhage (1992, 1999, 2000), Rhoades (1996), Singh and Sharma (2002), and others made a significant contribution in fixed point theory of $D$-Metric space. Unfortunately almost all theorems in $D$-metric space are not valid (Naidu et al., 2004; 2005a, 2005b). Here our aim is to prove some common fixed point theorems using some generalized contractive conditions in $D^*$-Metric space as a probable modification of the definition of $D$-Metric spaces introduced by Dhage (1992).

Definition 1

Let $X$ be a non empty set. A generalized metric (or $D^*$-metric) on $X$ is a function $D^*: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$

(1) $D^*(x, y, z) \geq 0.$

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z.$

(3) $D^*(x, y, z) = D^*(\rho(x, y, z))$ where $\rho$ is permutation.

(4) $D^*(x, y, a) \leq D^*(x, y, z) + D^*(a, z, z).$

The pair $(X, D^*)$ is called generalized metric (or $D^*$-metric) space.

Example 1

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$

Here, $d$ is the ordinary metric on $X$.

If $X = R^n$, then we define

(c) $D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p}$ for every $p \in R^+.$

If $X = R$, then we define

(d) $D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\
\max\{x, y, z\} & \text{other wise,}
\end{cases}$

Max $\{x, y, z\}$ other wise

Remark 1

In $D^*$-metric space $D^*(x, y, y) = D^*(x, x, y)$
Definition 2

A open ball in a $D^*$ - metric space $X$ with centre $x$ and radius $r$ is denoted by $B_{D^*}(x, r)$ and is defined by $B_{D^*}(x, r) = \{y \in X : D^*(x, y) < r\}$.

Example 2

Let $X=\mathbb{R}$ denote $D^*(x, y, z) = |x-y| + |y-z| + |z-x|$ for all $x, y, z \in \mathbb{R}$.

Thus $B_{D^*}(0, 1) = \{y \in \mathbb{R} / |y| < 1\}$

Definition 3

Let $(X, D^*)$ be a $D^*$ - metric space and $A \subseteq X$.

1. If for every $x \in A$, there exist $r > 0$ such that $B_{D^*}(x, r) \subseteq A$, then subset $A$ is called open subset of $X$.

2. Subset $A$ of $X$ is said to be $D^*$ - bounded if there exist $r > 0$ such that $B_{D^*}(x, y, z) < r$ for all $x, y, z \in A$.

3. A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $D^*(x_n, x) = 0$ as $n \to \infty$.

That is, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ implies $D^*(x, x_n) < \epsilon$. This is equivalent for each $\epsilon > 0$, as there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ implies $D^*(x_n, x_n) < \epsilon$. It is also noted that $D^*(x_n, x_n, x)$ $= D^*(x, x_n) < \epsilon$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

4. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The $D^*$ - metric space $(X, D^*)$ is said to complete if every Cauchy sequence is convergent.

Remark 2

1. $D^*$ is continuous function on $X^2$.

2. If sequence $\{x_n\}$ in $X$ converges to $x$, then $x$ is unique.

3. Any convergent sequence in $(X, D^*)$ is a Cauchy sequence.

Definition 4

A point $x$ in $X$ is a fixed point of the map $T : X \to X$ if $Tx = x$.

Definition 5

A point $x$ in $X$ is a common fixed point of the two maps $T_1, T_2 : X \to X$ if $T_1(x) = T_2(x) = x$.

Theorem 1

Let $X$ be a $D^*$ - complete metric space and $T_1, T_2 : X \to X$ be any two maps such that $D^* (T_1 x, T_2 y, z) \leq \alpha D^* (x, y, z)$ for all $x, y, z \in X$ and $0 \leq \alpha < \frac{1}{2}$ Then $T_1$ and $T_2$ have a unique common fixed point.

Proof

Let $x_0 \in X$ be any fixed arbitrary element define a sequence $\{x_n\}$ in $X$ as,

$x_{n+1} = T_1 x_n$ and $x_{n+2} = T_2 x_{n+1}$ for $n = 0, 1, 2, \ldots$

Now $d_{n+1} = D^*(x_{n+1}, x_{n+2})$

$\leq \alpha D^*(x_{n+1}, x_{n+2})$

$\leq \alpha D^*(x_n, x_{n+1}) + \alpha D^*(x_{n+1}, x_{n+2})$

$= \alpha d_n + \alpha d_{n+1}$

$(1-\alpha) d_{n+1} \leq \alpha d_n$

$d_{n+1} \leq \frac{\alpha}{1-\alpha} d_n$

$d_{n+1} \leq k d_n$ for all $n = 0, 1, 2, \ldots$, where $k = \frac{\alpha}{1-\alpha} < 1$

(Since $\alpha < \frac{1}{2}$)

$d_n \leq k d_{n-1}$

$\leq k^n d_0 \to 0$ as $n \to \infty$

Therefore $\lim_{n \to \infty} d_n = 0$

Thus $\lim_{n \to \infty} D^*(x_n, x_{n+1}, x_{n+1}) = 0$

Now we shall prove that $\{x_n\}$ is a $D^*$ - Cauchy sequence in $X$.

Let $m > n > n_0$ for some $n_0 \in \mathbb{N}$.

Now $D^*(x_m, x_m, x_m) \leq D^*(x_m, x_{m}, x_{m+1}) + D^*(x_{m+1}, x_{m+1}, x_m)$

$\leq \sum_{k=m}^{m+1} D^*(x_k, x_k, x_{k+1}) \to 0$ as $m, n \to \infty$

Thus $\lim_{n \to \infty} D^*(x_n, x_n, x_n) = 0$

Therefore $\{x_n\}$ is $D^*$ - Cauchy sequence in $X$.

Since $X$ is $D^*$ - Complete $x_n \to x$ in $X$. we prove that $x$ is a fixed point of $T_1$, suppose $x \neq T_1 x$. 
Then $D^*(T_1x, x, x) = \lim_{n \to \infty} D^* (T_1x, x_{n+2}, x)$

$= \lim_{n \to \infty} D^*(T_1x, T_2x, x_{n+1}, x)$

$\leq \alpha \lim_{n \to \infty} D^*(x, x_{n+1}, x)$

$= 0.$

Therefore $D^*(T_1x, x, x) = 0$. Therefore $T_1x = x$. Similarly we can prove that $T_2x = x$.

Hence $T_1x = T_2x = x$. Thus $x$ is common fixed point of $T_1$ and $T_2$.

**Uniqueness**

Suppose $x \neq y$ such that $T_1y = T_2y = y$.

Then $D^*(x, y, y) = D^*(T_1x, T_2y, y)$

$\leq \alpha D^*(x, y, y)$.

This implies $(1 - \alpha) D^*(x, y, y) \leq 0$.

Since $x \neq y$, we have $D^*(x, y, y) > 0$ for $(1 - \alpha) < 0$.

This implies $\alpha > 1$, which contradiction $\alpha < \frac{1}{2}$.

Thus $T_1$ and $T_2$ have a unique common fixed point.

**Theorem 2**

Let $X$ be a complete $D^*$ metric space and $T_1, T_2, T_3 : X \to X$ be any three maps such that $D^*(T_1x, T_2y, T_3z) \leq \alpha \ D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq \alpha < 1$. Then $T_1, T_2, T_3$ have a unique common fixed point.

**Proof**

Let $x_n \in X$ be any fixed arbitrary element define a sequence $\{x_n\}$ in X as,

$x_{n+1} = T_1x_n$

$x_{n+2} = T_2x_{n+1}$

$x_{n+3} = T_3x_{n+2}$ for $n = 0, 1, 2, \ldots$.

Let $d_n = D^* (x_n, x_{n+1}, x_{n+2})$

$d_1 = D^* (x_1, x_2, x_3)$

$= D^* (T_1x_0, T_2x_1, T_3x_2)$

$\leq \alpha D^* (x_0, x_1, x_2)$

$d_1 \leq \alpha d_0$

$d_2 = D^* (x_2, x_3, x_4)$

$= D^* (T_2x_1, T_3x_2, T_1x_3)$

$\leq \alpha D^* (x_1, x_2, x_3)$

$\leq \alpha d_1.$

$\leq \alpha^2 d_0$.

Continuing in this way, we get $d_n \leq \alpha^n d_0 = 0$ as $n \to \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in $X$.

Let $d_n^* = D^* (x_n, x_{n+1}, x_{n+2})$

Then $d_{n+1}^* = D^* (x_{n+1}, x_{n+2}, x_{n+3})$

$\leq D^* (x_n, x_{n+1}, x_{n+2}) + D^* (x_{n+1}, x_{n+2}, x_{n+3})$

$\leq \alpha d_n^* + \alpha^2 d_{n+1}^*$

$\leq \alpha^2 d_{n+1}^*$

Hence $\{d_n^*\}$ is monotone decreasing sequence of positive real number and it converges to its glb. Let it be $d$. Then $d_n^* \to d$ as $n \to \infty$.

Now we shall prove that $d = 0$. Suppose $d \neq 0$.

Now $d = \lim_{n \to \infty} d_{n+1}^*$

$\leq \lim_{n \to \infty} \{ \alpha d_n^*, d_{n+1}^*\}$

$= d$, which is contradiction. Thus $d = 0$.

Hence $D^*(x_n, x_{n+1}, x_{n+2}) \to 0$ as $m, n \to \infty$.

Therefore $(x_n)$ is a $D^*$ Cauchy sequence in $X$.

Since $X$ is $D^*$ complete $x \in X$, we prove that $x$ is fixed point of $T_1$.

To prove that $T_1x = x$

Suppose $T_1x \neq x$.

Then $D^*(T_1x, x, x) = \lim_{n \to \infty} D^*(T_1x, x_{n+2}, x_{n+3})$

$= D^*(T_1x, T_2x, T_3x)$

$\leq \alpha D^*(x_{n+1}, x_{n+2})$

$\leq \alpha D^*(x, x, x)$

$= 0$.

Thus $T_1x = x$.

Similarly we can prove that $T_2x = T_3x = x$.

Now we prove that $x$ is a unique common fixed point of $T_1, T_2, T_3$.

Suppose $x \neq y$ and $T_1x = T_2x = T_3x = x$ and $T_1y = T_2y = T_3y = y$.

Then $D^*(x,y,y) = D^*(T_1x, T_2y, T_3y)$

$\leq \alpha D^*(x, y, y)$

Thus impulse $(1 - \alpha) D^*(x, y, y) \leq 0$

Since $x \neq y$, we have $D^*(x, y, y) > 0$.

Thus $(1 - \alpha) < 0$.

Thus impulse $\alpha > 1$ is in contradiction. Hence $T_1, T_2$ and $T_3$ have a unique common fixed point.

**Theorem 3**

Let $X$ be a $D^*$ complete metric space and $S, T : X \to X$ be any two maps such that
D*(STx, Tx, y) ≤ α D*(T x, x, y)

For all x, y ∈ X and 0 ≤ α < ½. Then S and T have a unique common fixed point.

**Proof**

Let x₀ ∈ X be any fixed arbitrary element. Define a sequence {xₙ} in X as

\[ x_{n+1} = T xₙ \]

\[ x_{n+2} = S x_{n+1} \]

for n = 0, 1, 2, …………..

Let \( dₙ = D*(xₙ, x_{n+1}, x_{n+1}) \)
\[ d₁ = D*(x₁, x₂, x₂) ≤ α D*(x₀, x₀, x₂) \]
\[ = α D*(x₀, x₁, x₂) \]
\[ ≤ α D*(x₀, x₁, x₁) + α D*(x₁, x₂, x₂) \]
\[ = α d₀ + α d₁ \]
\[ (1 - α) d₁ ≤ α d₀ \]
\[ d₁ ≤ \frac{α}{1 - α} d₀ \]
\[ d₂ ≤ β d₁ \]
where \( β = \frac{α}{1 - α} < 1 \) (Since 0 ≤ α < ½).

Continuing in this way, we get
\[ dₙ ≤ β dₙ₋₁ \]
\[ ≤ β^n d₀ \]
\[ → 0 \text{ as } n → ∞ \] (since \( β = \frac{α}{1 - α} < 1 \)).

Now we shall prove that \( \{xₙ\} \) is a Cauchy sequence in X. Let m > n > n₀ for some n₀ ∈ N.

\[ D*(xₙ, xₙ, xₙ) ≤ \sum_{k=n}^{m-1} D*(xₙ, xₙ, xₙ) \]
\[ = \sum_{k=n}^{m-1} dₖ \]
\[ ≤ \beta^n \frac{1}{1 - β} d₀ \rightarrow 0 \text{ as } n, m → ∞ \]

Therefore \( \{xₙ\} \) is D* Cauchy sequence in X since X is D* complete xₙ → x in X.

Now we prove that T x = x.

Suppose T x ≠ x
\[ D*(T x, x, x) = \lim_{n → ∞} D*(T x, xₙ, x) \]
\[ = \lim_{n → ∞} D*(T x, S T xₙ, x) \]
\[ ≤ α \lim_{n → ∞} D*(x, T xₙ, x) \]
\[ = α \lim_{n → ∞} D*(x, xₙ, xₙ) \]
\[ = 0 \]

Therefore T x = x.

Next to prove that S x = x.
\[ D*(S T x, T x, x) = \lim_{n → ∞} D*(S T x, T xₙ, x) \]
\[ = \lim_{n → ∞} D*(S T x, S T xₙ, x) \]
\[ = α \lim_{n → ∞} D*(x, S T xₙ, x) \]
\[ = α \lim_{n → ∞} D*(x, xₙ, xₙ) \]
\[ = 0 \]

Thus 1 - α < 0 this implies α > 1 which is contradiction.

Therefore x = y.

Hence x is a unique common fixed point.

**Theorem 4**

Let X be a D* complete metric space and R, S, T : X → X be any three maps such that

\[ D*(R S T x, S T x, T x) ≤ α D*(S T x, T x, x) \]

For all x ∈ X and 0 ≤ α < 1. Then R, S and T have a unique common fixed point.

**Proof**

Let x₀ ∈ X be any fixed arbitrary element define a sequence \( \{xₙ\} \) in X as,

\[ x_{n+1} = T xₙ \]

\[ x_{n+2} = S x_{n+1} \]

Continuing in this way, we get
\[ dₙ ≤ β dₙ₋₁ \]
\[ ≤ β^n d₀ \]
\[ → 0 \text{ as } n, m → ∞ \] (since \( β = \frac{α}{1 - α} < 1 \)).

Therefore \( \{xₙ\} \) is a D* Cauchy sequence in X since X is D* complete xₙ → x in X.

Now we shall prove that \( \{xₙ\} \) is a Cauchy sequence in X.

Let m > n > n₀ for some n₀ ∈ N.

\[ D*(xₙ, xₙ, xₙ) ≤ \sum_{k=n}^{m-1} D*(xₙ, xₙ, xₙ) \]
\[ = \sum_{k=n}^{m-1} dₖ \]
\[ ≤ \beta^n \frac{1}{1 - β} d₀ \rightarrow 0 \text{ as } n, m → ∞ \]
Now \( d = d \) Then \( d \)
\[ \leq D^*(x, x, STx) \]
Continuing in this way, we get
\[ \leq D^*(x, x, x) \]
\[ \leq D^*(x_n, x_{n+1}, x_{n+2}) \]
\[ \leq d_{n+1} \] for all \( n \)
Hence \{ \( d_n \) \} is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \). Then \( d_n \to d \) as \( n \to \infty \).
Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).
Now \[ \lim_{n \to \infty} d_n = 0 \] as \( n \to \infty \).
\[ \leq \lim_{n \to \infty} \{ \alpha ^n d_n \} \]
\[ < \lim_{n \to \infty} \{ \alpha ^n d_n \} \]
\[ \alpha \to 1 \]
\[ 0 \] which is a contradiction. Hence \( R, S, T \) have a unique common fixed point.

**Theorem 4**

Let \( X \) be a complete \( D^* \)-metric space and \( T_1, T_2, T_3 : X \to X \) be any three maps such that \( D^* (T_n x, T_{n+1} x, T_{n+2} x) \leq a \) \( \{ D^*(x, y, z) + D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, T_3 z) \} \) for all \( x, y, z \in X \) and \( 0 \leq a < 1/3 \). Then \( T_1, T_2, T_3 \) have a unique common fixed point.

**Proof**

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{ x_n \} \) in \( X \) as
\[ x_{n+1} = T_1 x_n \]
\[ x_{n+2} = T_2 x_{n+1} \]
\[ x_{n+3} = T_3 x_{n+2} \]
\[ \lim_{n \to \infty} D^*(x_n, x_{n+1}, x_{n+2}) = 0. \]
Thus \( x = x \).
Now we prove that \( x \) is a unique common fixed point of \( R, S, T \).
Suppose \( x \neq y \) and \( Tx = Rx = x \) & \( Ty = Sy = y \)
Then \( D^*(x, x, y) \leq 0 \) since \( x \neq y \), we have \( D^*(x, y, z) > 0 \)
Thus \( ((1-\alpha)) < 0 \)
Then impulse \( (1-\alpha) D^*(x, y, y) \leq 0 \)
Thus impulse \( \alpha > 1 \) which is in contradiction. Hence \( R, S, T \) have a unique common fixed point.
Now \( d = d. \) Then \( d = d. \)

Let \( d \leq H \) hence \( D^*(x, x, x) \leq d, \) which is contradiction. Thus \( d = 0. \)

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence in \( X. \)

Let \( d_n^* = D^*(x, x, x, x_n) \)
Then \( d_n^* = D^*(x, x, x, x_n) \)
\( \leq D^*(x, x, x, x_n) + D^*(x, x, x, x_n) \)
\( \leq d_n + d_n^* \)
\( d_n^* - d_n \leq a^\alpha d_n \rightarrow 0 \) as \( n \rightarrow \infty \) (since \( 0 \leq \alpha < 1 \))
\( d_n^* \leq d_n \) for all \( n \)

Hence \( \{d_n^*\} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d. \) Then \( d_n^* \rightarrow d \) as \( n \rightarrow \infty. \)

Now we shall prove that \( d = 0. \) Suppose \( d \neq 0. \)

Now \( d = \lim_{n \rightarrow \infty} d_n^*. \)

\( \lim_{n \rightarrow \infty} \{d_n, d_n^*\} \)

\( \lim_{n \rightarrow \infty} \{b_n, d_n^*\} \)

\( \lim_{n \rightarrow \infty} \{d_n, d_n^*\} \)

\( =d, \) which is contradiction. Thus \( d = 0. \)

Hence \( D^*(x, x, x_n) \rightarrow 0 \) as \( m, n \rightarrow \infty. \)

Therefore \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X. \)

Since \( X \) is a \( D^* \) complete \( x \rightarrow x \) in \( X. \)

Now we prove that \( x \) is a fixed point of \( T. \)

To prove that \( T^2x = x. \) Suppose \( T^2x \neq x. \) Then

\( D^*(T^2x, x, x, x) = \lim_{n \rightarrow \infty} D^*(T^2x, x, x, x_n) \)

\( = D^*(T^2x, x, x, x) \)

\( \leq a \{D^*(x, x, x, x_n) + D^*(x, x, x, x_n) + D^*(x, x, x, x_n)\} \)

\( \leq a \lim_{n \rightarrow \infty} \{D^*(x, x, x, x_n) + D^*(x, x, x, x_n) + D^*(x, x, x, x_n)\} \)

\( \leq a \lim_{n \rightarrow \infty} \{D^*(x, x, x, x_n) + D^*(x, x, x, x_n) + D^*(x, x, x, x_n)\} \)

\( \leq a \lim_{n \rightarrow \infty} D^*(x, x, x, x_n) \leq 0 \) Hence \( (1-a) < 0. \)

Therefore \( a > 1. \) which is contradiction to \( a < 1. \)

Thus \( T^2x = x. \) Similarly we can prove that \( T^2x = T^3x = x. \)

Now we prove that \( x \) is a unique common fixed point of \( T, T_1, T_2, T_3. \)

Suppose \( x \neq y. \) and \( T_1x = T_2x = T_3x = x \) \& \( T_1y = T_2y = T_3y = y. \)

Then \( D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \)
\( \leq a \{D(x, y, y) + D^*(x, y, y) + D^*(y, y, y)\} \)
\( \leq 2a D^*(x, y, y) \)
\( < D^*(x, y, y), \)

which is a contradiction.

Hence \( T, T_1, T_2 \) and \( T_3 \) have a unique common fixed.

**Theorem 5**

Let \( X \) be a complete \( D^* \) metric space and \( T_1, T_2, T_3 : X \rightarrow X \) be any three maps such that \( D^*(T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 D^*(T_1x, T_2y, T_3z) \)
\( + a_3 \{D^*(x, x, x_n) + D^*(x, x, x_n) + D^*(x, x, x_n)\} \) for all \( x, y, z \in X, \)

\( 0 \leq a_1 + 2a_2 + 2a_3 < 1. \) Then \( T_1, T_2, T_3 \) have a unique common fixed point.

**Proof**

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{x_n\} \) in \( X \) as,

\( x_n+1 = T_1x_n \)
\( x_{n+2} = T_2x_n \)
\( x_{n+3} = T_3x_n, \) for \( n = 0, 1, 2, \ldots. \)

Let \( d_n = D^*(x, x_{n+1}, x_{n+2}) \).

Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\( = a_1 D^*(x_{n+1}, x_{n+2}) + a_2 \{D^*(x_{n+1}, x_{n+2}, x_{n+3}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} + a_3 \{D^*(x_{n+1}, x_{n+2}, x_{n+3}) + D^*(x_{n+1}, x_{n+2}, x_{n+3})\} \)
\( \leq (a_1 + a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3}) + (a_2 + a_3) D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)
\( \leq (a_1 + a_2 + a_3) \lim_{n \rightarrow \infty} d_n + (a_2 + a_3) d_{n+1} \)

\( (1 + a_2 + a_3) d_{n+1} \leq (a_1 + a_2 + a_3) d_n \)
\( d_{n+1} \leq (a_1 + a_2 + a_3) d_n \)

Hence \( d_{n+1} \leq a^n d_0 \rightarrow 0 \) as \( n \rightarrow \infty. \)

Now we prove that \( \{x_n\} \) is \( D^* \) Cauchy sequence.

Let \( d_n^* = D^*(x, x, x_n) \)

Then \( d_{n+1}^* = D^*(x, x, x_n) \)
\( \leq D^*(x, x, x_n) + D^*(x, x, x_n) \)
\( \leq d_n^* \).

Hence \( (1-a) D^*(x, T_1x, x) \leq 0 \) Hence \( (1-a) \leq 0. \)

Therefore \( a > 1. \) which is contradiction to \( a < 1. \)

Thus \( T^2x = x. \) Similarly we can prove that \( T^2x = T^3x = x. \)

Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3. \)

Suppose \( x \neq y. \) and \( T_1x = T_2x = T_3x = x \) \& \( T_1y = T_2y = T_3y = y. \)

Then \( D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \)
\( \leq a \{D(x, y, y) + D^*(x, y, y) + D^*(y, y, y)\} \)
\( \leq 2a D^*(x, y, y) \)
\( < D^*(x, y, y), \)

which is a contradiction.

Hence \( T_1, T_2 \) and \( T_3 \) have a unique common fixed.
\[ d_{n+1}^* - d_n^* \leq d_n \leq \alpha^n d_0 \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1) \]
\[ d_{n+1}^* \leq d_n^* \text{ for all } n \]

Hence \((d_n^*)\) is monotonically decreasing sequence of positive real number and it converges to its g.l.b. Let it be \(d\). Then \(d_n^* \to d\) as \(n \to \infty\).

Now we shall prove that \(d = 0\). Suppose \(d \neq 0\).

Now \(d = \lim_{n \to \infty} d_{n+2}^* \leq \lim_{n \to \infty} d_{n+1} \cdot \lim_{n \to \infty} d_{n+1}^*\)

\[ \leq \lim_{n \to \infty} \{ b \cdot d_n \cdot d_{n+1}^* \} \]

\[ \leq \lim_{n \to \infty} \{ b \cdot d_n \cdot d_{n+1}^* \} \to 0 \text{ as } n \to \infty \text{. Hence } D^*(x_n, x_n, y_n) \to 0 \text{ as } n, m \to \infty \text{. Therefore } \{x_n\} \text{ is a } D^* \text{ Cauchy sequence in } X. \]

Since \(X \) is \(D^* \)-complete \(x_n \to x \) in \(X\).

Now we prove that \(x \) is fixed point of \(T_1\).

To prove that \(T_1x = x\)

Suppose \(T_1x \neq x\), Then

\[ D^*(T_1x, x, x) = \lim_{n \to \infty} D^*(T_1x, x_{n+2}, x_{n+3}) \]

\[ = \lim_{n \to \infty} D^*(T_1x, x_{n+1}, x_{n+2}, x_{n+3}) \]

\[ \leq \lim_{n \to \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \{ D^*(x, T_1x, x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}) \} + a_3 \{ D^*(x, x_{n+1}, x_{n+2}, x_{n+3}) \} \}

\[ \leq a_2 D^*(x, T_1x, x, x) \]

\[ < D^*(x, T_1x, x, x) \text{, which is contradiction. Thus } T_1x = x. \]

Similarly we can prove that \(T_2x = T_3x = x\).

Now we prove that \(x \) is a unique common fixed point of \(T_1, T_2, T_3\).

Suppose \(x \neq y \) and \(T_1x = T_2x = T_3x = x \) & \(T_1y = T_2y = T_3y = y \)

Then \(D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \leq a_1 D^*(x, y, y) + a_2 \{ D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3y) \}

\[ + a_3 \{ D^*(x, y, T_2y) + D^*(y, T_3y) \} \]

\[ = a_1 D^*(x, y, y) + a_3 \{ D^*(x, y, y) + D^*(y, y, y) \}

\[ \leq \alpha^n d_0 \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1) \]

Hence \(T_1, T_2 \) and \(T_3 \) have a unique common fixed point.

**Theorem 6**

Let \(X \) be a complete \(D^* \)-metric space and \(T_1, T_2, T_3 : X \to X \) be any three maps such that \(D^* (T_1x, T_2y, T_3z) \leq a \max \{ D^*(x, y, y) \}

\[ \leq a \max \{ D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3y) \}

\[ + a_3 \{ D^*(x, y, T_2y) + D^*(y, T_3y) \} \]

\[ = (a_1 + a_2 + a_3) D^* (x, y, y) \]

\[ < D^* (x, y, y) \text{, which is contradiction. } \]

Hence \(T_1, T_2 \) and \(T_3 \) have a unique common fixed point.

**Proof**

Let \(x_0 \in X \) a fixed arbitrary element and define a sequence \(\{x_n\} \) in \(X \) as

\[ x_{n+1} = T_1x_n \]

\[ x_{n+2} = T_2x_{n+1} \]

\[ x_{n+3} = T_3x_{n+2} \]

for \(n = 0, 1, 2, \ldots \)

Let \(d_n = D^*(x_n, x_{n+1}, x_{n+2}) \).

Then \(d_n = D^*(x_n, x_{n+1}, x_{n+2}) \)

\[ \leq D^*(x_n, x_{n+1}, x_{n+2}) \]

\[ \leq \max \{ D^*(x, y, y), D^*(x, T_1x, T_2y) + D^*(y, T_2y, T_3y) \}

\[ + a_3 \{ D^*(x, y, T_2y) + D^*(y, T_3y) \} \]

\[ = (a_1 + a_2 + a_3) D^* (x, y, y) \]

\[ < D^* (x, y, y) \text{, which is contradiction. } \]

Hence \(T_1, T_2 \) and \(T_3 \) have a unique common fixed point.
\[
\lim_{n_x \to \infty} \left\{ d_n, d_{n+1}^* \right\} = d
\]

Now we prove that \( \{x_n\} \) is D* - Cauchy sequence in X.
For m > n we have,
\[
D^*(x_m, x_n, x_m) \leq D^*(x_n, x_m, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{n+1}, x_{n+1}, x_n) = 0 \quad \text{as } m, n \to \infty
\]
Thus \( \{x_n\} \) is a D* Cauchy sequence in X and X is D* - complete \( x_n \to x \) in X.

Now we shall prove that \( T x = x \)
\[
D^* (T_2 x, x, x) = \lim_{n \to \infty} D^* (T_1 x, x_n, x_n)
\]
\[
= \lim_{n \to \infty} D^* (T_1 x, T_2 x_n, T_3 x_{n+2})
\]
\[
\leq \lim_{n \to \infty} \max \left\{ D^* (x, x_{n+1}, x_{n+2}) , D^* (x, T x, x_{n+1}), D^* (x_{n+1}, x_{n+2}, T_3 x_{n+2}) \right\}
\]
\[
= \lim_{n \to \infty} \max \left\{ D^* (x, x, T x) , D^* (T x, x, x_{n+2}) , D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq \lim_{n \to \infty} \max \left\{ D^* (x, x, x_{n+2}) , D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq \max \left\{ D^* (x, x, x_{n+2}) , D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq \max \left\{ D^* (x, x, x_{n+2}), D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq D^* (x, x, x_{n+2}) + D^* (x_{n+1}, x_{n+2}, x_{n+3})
\]
\[
\leq \alpha \cdot D^* (x, x, x_{n+2}) + a_2 \max \left\{ D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq \alpha \cdot \Delta_0 + a_2 \Delta_0
\]
\[
d_{n+2} \leq \alpha \cdot d_n + a_2 \Delta_0
\]
\[
d_{n+2} \leq \Delta_0 (1 - a_2) d_n
\]
\[
d_{n+1} \leq \Delta_0 (1 - a_2) d_n
\]
\[
d_{n+1} \leq \beta d_n
\]
\[
\lim_{n \to \infty} d_n = \beta d_n \rightarrow 0
\]

Hence \( \{d_n\} \) is a Cauchy sequence in X.

Let \( d_n^* = D^* (x_n, x_{n+1}, x_{n+2}) \)
Then \( d_n^* \leq D^* (x_n, x_{n+1}, x_{n+2}) \)
\[
\leq D^* (x_n, x_{n+1}, x_{n+2}) + a_2 \max \left\{ D^* (x_{n+1}, x_{n+2}, x_{n+3}) \right\}
\]
\[
\leq \alpha \cdot D^* (x, x, x_{n+2}) + a_2 \Delta_0
\]
\[
d_{n+1} \leq \alpha \cdot d_n + a_2 \Delta_0
\]
\[
d_{n+1} \leq \beta d_n
\]
\[
\lim_{n \to \infty} d_n = \beta d_n \rightarrow 0
\]

Hence \( \{d_n\} \) is a Cauchy sequence in X.

Therefore \( \{x_n\} \) is a D* Cauchy sequence in X.

Since X is D* complete \( x_n \to x \) in X, we prove that \( x \) is fixed point of \( T_1 \)
To prove that \( T x = x \)
Suppose \( T x \neq x \), Then
\[
D^* (T x, x, x) = \lim_{n \to \infty} D^* (T x, x_n, x_{n+1})
\]
\[
= \lim_{n \to \infty} D^* (T x, T x_n, T x_{n+1})
\]
\[
\leq \alpha \cdot D^* (T x, x_n, x_{n+1}) + a_2 \max \left\{ D^* (x_n, x_{n+1}, x_{n+2}) \right\}
\]
\[
\leq \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]

Hence \( \{d_n\} \) is a Cauchy sequence in X.

Therefore \( \{x_n\} \) is a D* Cauchy sequence in X.

Since X is D* complete \( x_n \to x \) in X, we prove that \( x \) is fixed point of \( T_1 \)
To prove that \( T x = x \)
Suppose \( T x \neq x \), Then
\[
D^* (T x, x, x) = \lim_{n \to \infty} D^* (T x, x_n, x_{n+1})
\]
\[
= \lim_{n \to \infty} D^* (T x, T x_n, T x_{n+1})
\]
\[
\leq \alpha \cdot D^* (T x, x_n, x_{n+1}) + a_2 \max \left\{ D^* (x_n, x_{n+1}, x_{n+2}) \right\}
\]
\[
\leq \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} d_n = \alpha \cdot d_n + a_2 \Delta_0
\]
\[
\lim_{n \to \infty} \{ a_1 D^*(x, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x, T_1 x, x_{n+2}), D^*(x_{n+3}, x, x_{n+2}) \} \leq 0, \quad \text{for all } x, y, z \in X.
\]

Similarly we can prove that \( T_2x = T_3x = x \).

Now we prove that there is a unique common point of \( T_1, T_2, T_3 \).

Suppose \( x \neq y \) and \( T_1x = T_2x = T_3x = x \) and \( T_1y = T_2y = T_3y = y \).

Then \( D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \leq a_1 \{ D(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y) \} \}
\]

\[< D^*(x, y, y), \text{which is contradiction.} \]

Hence \( T_1, T_2, \) and \( T_3 \) have a unique common fixed point.

\section*{Remark 1}

If we put \( a_2 = 0, \ T_1 = T_2 = T_3 \), then \( a_1 = a \) in the above theorem we get the following theorem as corollary 1.

\section*{Corollary 1}

Let \((X, D^*)\) be a complete \(D^*\)-metric space and \( T: X \to X \) be a map such that \( D^*(Tx, Ty, Tz) \leq \alpha \max \{D^*(x, y, z) \} \) for all \( x, y, z \in X \) and \( 0 \leq \alpha < 1 \). Then \( T \) has a unique fixed point. The above theorem is known as Banach contraction type theorem in \(D^*\)-metric space.

\section*{Remark 2}

If we put \( a_1 = 0, \ T_1 = T_2 = T_3 \), then \( a_2 = a \) in the above theorem 1. We get the following theorem as Corollary 2.

\section*{Corollary 2}

Let \((X, D^*)\) be a complete \(D^*\)-metric space and \( T: X \to X \) be a map such that \( D^*(Tx, Ty, Tz) \leq \alpha \max \{D^*(x, y, z) \} \) for all \( x, y, z \in X \) and \( 0 \leq \alpha < 1 \). Then \( T \) has a unique fixed point.

\section*{Theorem 8}

Let \( X \) be a complete \(D^*\)-metric space and \( T_1, T_2, T_3: X \to X \) be any three maps such that \( D^*(Tx, Ty, Tz) \leq a_1 \max \{D^*(x, y, z), D^*(y, T_2y, T_3z) \} + a_2 \max \{D^*(x, y, T_2y), D^*(y, z, T_3z) \} \) for all \( x, y, z \in X \) and \( 0 \leq a_1 + 2a_2 + 2a_3 < 1 \). Then \( T_1, T_2, \) and \( T_3 \) have a unique common fixed point.

\section*{Proof}

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{x_n\} \) in \( X \) as,

\[
x_{n+1} = T_1 x_n, \quad n = 0, 1, 2, \ldots.
\]

Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \).

Then \( d_{n+1} = D^*(x_{n+1}, x_{n+2}, x_{n+3}) \)

\[
= D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, T_2x_n, T_3x_n) \} + a_3 \max \{D^*(x_n, T_1x_n, T_2x_n) \}
\]

\[= \alpha \max \{D^*(x_n, x_{n+1}, x_{n+2}) \} + a_3 \max \{D^*(x_n, x_{n+1}, x_{n+2}) \}, \quad \text{for \( n = 0, 1, 2, \ldots \)}
\]

\[< (a_1 + a_2 + a_3) D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) d_{n+1}
\]

\[< \alpha d_n, \quad \text{for all } n \text{ where } \alpha = \{1, a_1+2a_2+2a_3 \} < 1.
\]

Hence \( d_n \to 0 \) as \( n \to \infty \).

Now we prove that \( \{x_n\} \) is \(D^*\)-Cauchy sequence in \( X \).

Let \( d_n = D^*(x_n, x_{n+1}, x_{n+2}) \).

Then \( d_{n+1} = D^*(x_n, x_{n+1}, x_{n+2}) + a_2 \max \{D^*(x_n, T_2x_n, T_3x_n) \} + a_3 \max \{D^*(x_n, T_1x_n, T_2x_n) \}
\]

\[< (a_1 + a_2 + a_3) D^*(x_n, x_{n+1}, x_{n+2}) + (a_2 + a_3) d_{n+1}
\]

\[< \alpha d_n, \quad \text{for all } n \text{ where } \alpha = \{1, a_1+2a_2+2a_3 \} < 1.
\]

Hence \( \{d_n\} \) is monotonically decreasing sequence of positive real number and it converges to its glb. Let it be \( d \).

\( \text{Then } \ d = d_{n*} \to d \) as \( n \to \infty \).

Now we shall prove that \( d = 0 \).

\[\text{Suppose } d \neq 0. \text{Then } \frac{d}{a_1+2a_2+2a_3} = d_{n*} < \alpha d_0 \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1 \).
\]

\[d_{n*} \leq \alpha^2 d_{n+1} \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1 \).
\]

\[d_{n*} \to 0 \text{ for all } n.
\]

\[d_{n*} \to 0 \text{ as } n \to \infty.
\]

\[d = 0, \text{ which is contradiction. Thus } d = 0.
\]

For \( m > n \), we have

\[D^*(x_n, x_m, x_n) \leq D^*(x_n, x_m, x_{n+1}) + D^*(x_{n+1}, x_{n+2}, x_m)
\]

\[\leq D^*(x_n, x_{n+1}, x_m) + D^*(x_{n+1}, x_{n+2}, x_m) + \ldots + D^*(x_m, x_{n+1}, x_m) \to 0 \text{ as } n, m \to \infty.
\]

Hence \( D^*(x_n, x_m) \to 0 \) as \( m, n \to \infty \).
Therefore \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \).

Since \( X \) is \( D^* \) complete, \( x_n \to x \) in \( X \), we prove that \( x \) is fixed point of \( T_1 \).

To prove that \( T_1 x = x \)

Suppose \( T_1 x \neq x \), then

\[
D^*(T_1 x, x) = \lim_{n \to \infty} D^*(T_1 x, x_{n+2}, x_{n+3})
\]

\[
\leq \lim_{n \to \infty} \{ a_1 D^*(x_{n+1}, x_{n+2}) + a_2 \max \{ D^*(x, x_{n+1}, T_2 x_{n+2}), D^*(x_{n+1}, x_{n+2}) \} \}
\]

\[
\leq a_2 \max \{ D^*(x, x_{n+1}, T_2 x_{n+2}), D^*(x_{n+1}, x_{n+2}) \}
\]

\[
< D^*(x, x), \text{ which is contradiction. Thus}, T_1 x = x.
\]

Similarly we can prove that \( T_2 x = T_3 x = x \).

Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \).

Suppose \( x \neq y \) and \( T_1 x = T_2 x = T_3 x = x \) \& \( T_1 y = T_2 y = T_3 y = y \)

Then \( D^*(x, y) = D^*(T_1 x, T_1 y, T_2 y, T_3 y) \)

\[
\leq a_1 D^*(x, y, y) + a_2 \max \{ D^*(x, T_1 x, T_2 y), D^*(y, T_2 y, T_3 y) \}
\]

\[
+ a_3 \max \{ D^*(y, T_2 y, T_3 y), D^*(x, T_2 y, y) \}
\]

\[
= a_1 D^*(x, y, y) + a_2 \max \{ D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, y) \}
\]

\[
+ \max \{ D^*(x, y, y), D^*(y, x, y) \}
\]

\[
= (a_1 + a_2 + a_3) D^*(x, y, y)
\]

\[
< D^*(x, y), \text{ which is contradiction.}
\]

Hence \( T_1, T_2 \) and \( T_3 \) have a unique common fixed point.

**Theorem 9**

Let \( X \) be a complete \( D^* \) - metric space and \( T_1, T_2, T_3 : X \to X \) be any three maps such that \( D^* (T_1 x, T_2 y, T_3 z) \leq \max \{ D^*(x, y, z), 1/2(D^*(x, T_1 x, T_2 y) + D^*(y, T_2 y, T_3 z)) \} \)

\[
1/2 \max \{ D^*(x, y, z) + D^*(y, z, T_3 z) \}
\]

For all \( x, y, z \in X \), and \( 0 \leq a < 1/3 \). Then \( T_1, T_2, T_3 \) have a unique common fixed point.

**Proof**

Let \( x_0 \in X \) a fixed arbitrary element and define a sequence \( \{x_n\} \) in \( X \) as,

\[
x_{n+1} = T_1 x_n
\]

\[
x_{n+2} = T_2 x_{n+1}
\]

\[
x_{n+3} = T_3 x_{n+2}
\]

for \( n = 0, 1, 2, \ldots \)

Let \( d_n = D^* (x_n, x_{n+1}, x_{n+2}) \).

Then \( D_{n+1} = D^* (x_{n+1}, x_{n+2}, x_{n+3}) \)

\[
= D^* (T_1 x_n, T_2 x_{n+1}, T_3 x_{n+2})
\]

\[
\leq a \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, T_1 x_n, T_2 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}) \} \}
\]

\[
= \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \}
\]

\[
\leq \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+2}, x_{n+3}) \} \}
\]

\[
\leq \max \{ d_n \}, 1/2 (d_{n+1} + d_{n+2})
\]

\[
d_{n+1} \leq a (3/2 d_n + 1/2 d_{n+1})
\]

\[
d_{n+1} \leq a (3d_n) \text{ for all } n
\]

Hence \( d_n \to 0 \) as \( n \to \infty \).

Now we prove that \( \{x_n\} \) is \( D^* \) - Cauchy sequence in \( X \).

Let \( d_n^* = D^* (x_n, x_{n+1}) \).

Then \( d_n^* = D^* (x_n, x_{n+1}) \).

\[
\leq \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+1}, x_{n+1}) \} \}
\]

\[
\leq \max \{ d_n \} \leq \alpha^* d_n \to 0 \text{ as } n \to \infty \text{ (since } 0 \leq \alpha < 1 \)
\]

\[
d_n^* \leq \alpha^* d_n \text{ for all } n
\]

Hence \( \{d_n^*\} \) is monotonically decreasing sequence of positive real number and it converges to its glo. Let it be \( d^* \).

Now we shall prove that \( d = 0 \). Suppose \( d \neq 0 \).

\[
\lim_{n \to \infty} \frac{d_{n+2}}{d_n}
\]

\[
\leq \lim_{n \to \infty} \frac{d_{n+1} + d_{n+1}^*}{d_n + d_n^*}
\]

\[
= \lim_{n \to \infty} \{a d_n^*, d_n^*\}
\]

\[
\leq \lim_{n \to \infty} \{a d_n, d_n^*\}
\]

\[
= 0
\]

Now we prove that \( \{x_n\} \) is \( D^* \) - Cauchy sequence in \( X \).

For \( m > n \) we have,

\[
D^*(x_m, x_n) \leq D^*(x_m, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{n+1}, x_{n+1})
\]

\[
\to 0 \text{ as } m, n \to \infty
\]

Thus \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \) and \( X \) is \( D^* \) - complete \( x_n \to x \) in \( X \).

Now we shall prove that \( T_1 x = x \)

\[
D^*(T_1 x, x) = \lim_{n \to \infty} D^*(T_1 x, x_{n+2}, x_{n+3})
\]

\[
= \lim_{n \to \infty} D^*(T_1 x, x_{n+1}, T_3 x_{n+2})
\]

\[
\leq \alpha \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, T_2 x_n, T_3 x_{n+1}) + D^*(x_{n+1}, T_2 x_{n+1}, T_3 x_{n+2}) \} \}
\]

\[
\leq \max \{ D^*(x_n, x_{n+1}, x_{n+2}), 1/2 \{ D^*(x_n, x_{n+1}, x_{n+2}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) \} \}
\]

\[
\leq \max \{ d_n \}
\]

\[
\leq \alpha^* d_n \to 0 \text{ as } n \to \infty
\]

Thus \( \{x_n\} \) is a \( D^* \) Cauchy sequence in \( X \) and \( X \) is \( D^* \) - complete \( x_n \to x \) in \( X \).

Now we shall prove that \( T_1 x = x \)
\[
\lim_{n \to \infty} \max \{ D^*(x_{n+1}, x_{n+2}), D^*(x_{n+2}, x_{n+3}) \} \\
\leq \frac{1}{2} \max \{ D^*(x, T_1x, x), D^*(x, T_2x, x), D^*(x, T_3x, x) \}
\]

\[
< D^*(T_1x, x, x), \text{ which is a contradiction.}
\]

Thus, \( T_1x = x \).

Similarly we can prove that \( T_2x = T_3x = x \).

Now we prove that \( x \) is a unique common fixed point of \( T_1, T_2, T_3 \).

Suppose \( x \neq y \) and \( T_1x = T_2x = T_3x = x \) and \( T_1y = T_2y = T_3y = y \)
Then \( D^*(x, y, y) = D^*(T_1x, T_2y, T_3y) \leq \max \{ D^*(x, y, y), D^*(x, T_2y, T_3y), D^*(y, T_2y, T_3y) \} \)

\[
= \max \{ D^*(x, y, y), \frac{1}{2} D^*(x, y, y), \frac{1}{2} D^*(x, y, y) \}
\]

\[
= D^*(x, y, y), \text{ which is a contradiction.}
\]

Hence \( T_1, T_2 \) and \( T_3 \) have a unique common fixed point.

**Theorem 10**

Let \( X \) be a complete \( D^* \) - metric space and \( T_1, T_2, T_3 : X \to X \) be any three maps such that \( D^*(T_1x, T_2y, T_3z) \leq a_1 D^*(x, y, z) + a_2 \max \{ D^*(x, T_1x, T_2y), D^*(y, T_2y, T_3z) \} \)

\[
+ a_3 \max \{ D^*(x, y, T_2y), D^*(y, z, T_3z) \}. \text{ For all } x, y, z \in X \text{ and } 0 \leq a_1 + 2a_2 + 2a_3 < 1. \text{ Then } T \text{ has a unique fixed point.}
\]

**REFERENCES**


