Almost n-multiplicative maps

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Let $A$ and $B$ be two linear algebras. A linear map $\varphi: A \to B$ is called an $n$-homomorphism if $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$ for all $a_1, \ldots, a_n \in A$. The continuity of $n$-homomorphisms between Banach algebras as well as the almost multiplicative linear operators has been recently studied. In this note, we have a verification on the behavior of almost $n$-multiplicative linear maps with $n > 2$.

Key words: Almost multiplicative maps, $n$-homomorphism maps, almost $n$-multiplicative maps, stability.

INTRODUCTION

Let $A$ and $B$ be two linear algebras. A linear mapping $\varphi: A \to B$ is called an $n$-homomorphism if $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$ for each $a_1, \ldots, a_n$ in $A$. A $2$-homomorphism is then a homomorphism, in the usual sense (Hejazian et al., 2005). In 1985, Jarosz (1985) introduced the concept of $\varepsilon$-multiplicative maps, where he discussed the continuity of $\varepsilon$-multiplicative linear functionals on Banach algebras. Ansari-Piri and Eghbali (2005) proved that every linear almost multiplicative map from a Banach algebra $A$ into a semi-simple Banach algebra $B$ is continuous. In this paper, we introduce a new concept of perturbations, and we call it almost $n$-multiplicative. Also, we examine the relationship between almost multiplicative maps and almost $n$-multiplicative maps. We will prove that every almost multiplicative linear function is almost $n$-multiplicative (Corllary 3.5), and by introducing an example, we show that, an almost $n$-multiplicative map is not necessarily an almost multiplicative one.

The continuity of almost $n$-multiplicative linear functional on unital Banach algebras as well as the continuity of almost $n$-multiplicative linear maps from a unital Banach algebra into another semi-simple Banach algebra is also investigated, where by a unital Banach algebra means a Banach algebra with a unit element.

PRELIMINARIES

Here, we provide a collection of definitions and related results which are essential and used in this study's discussion.

Definition 2.1

Let $A$ and $B$ be Banach algebras and, $\varphi: A \to B$ a linear map. We say $\varphi$ is an almost multiplicative map if there exists an $\varepsilon > 0$ such that for all $x, y \in A$, $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon \|x\| \|y\| \|$.

Proposition 2.2

Every almost multiplicative linear functional on a Banach algebra is continuous.
Proof

For proof refer to Jarosz (1985).

Theorem 2.3

Every almost multiplicative linear map from a Banach algebra $A$ to a semi-simple Banach algebra $B$ is continuous.

Proof

For proof refer to Ansari-Piri and Eghbali (2005).

Definition 2.4

Let $A$ and $B$ be two linear algebras and $n > 2$ an integer. A linear map $\varphi: A \to B$ is an $n$-homomorphism if for all $a_1, a_2, \ldots, a_n \in A$,

$$\varphi(a_1a_2\ldots a_n) = \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n).$$

THE RELATIONSHIP BETWEEN ALMOST $n$-MULTIPLICATIVE AND ALMOST MULTIPLICATIVE MAPS

Here, we give a proposition to emphasize the existence of a linear almost $n$-multiplicative map and then we study the relationship between almost multiplicative and almost $n$-multiplicative maps.

Definition 3.1

Let $A$ and $B$ be Banach algebras and $n > 2$ an integer. A linear map $\varphi: A \to B$ is called an almost $n$-multiplicative map if there exists $\varepsilon > 0$ such that for all $a_1, a_2, \ldots, a_n \in A$,

$$\|\varphi(a_1a_2\ldots a_n) - \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n)\| \leq \varepsilon \|a_1\| \|a_2\| \ldots \|a_n\|.$$

Proposition 3.2

Let $A$ be a Banach algebra, $T$ a linear functional on $A$, and $S$ a linear map such that $\|S\| \leq \varepsilon$. The map $T + S$ is not $n$-homomorphism where it is almost $n$-multiplicative.

Proof

It is straightforward.

Theorem 3.3

Let $A$ and $B$ be Banach algebras, $f: A \to B$ a linear map and $n > 2$ an integer. If $f$ has one of the following properties, then $f$ is an $n$-Jordan map.

1) for all $a_1, a_2, \ldots, a_n \in A$,

$$\|\varphi(a_1a_2\ldots a_n) - \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n)\| \leq \varepsilon \|a_1\| \|a_2\| \ldots \|a_n\|,$$

2) for all $a_1, a_2, \ldots, a_n \in A$,

$$\|\varphi(a_1a_2\ldots a_n) - \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n)\| \leq \varepsilon (\|a_1\| + \ldots + \|a_n\|),$$

3) for all $a_1, a_2, \ldots, a_n \in A$,

$$\|\varphi(a_1a_2\ldots a_n) - \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n)\| \leq \varepsilon (\|a_1\| + \ldots + \|a_n\|),$$

Proof

The proof of all three parts are similar, so we consider only the second property.

If for all $x_1, x_2, \ldots, x_n \in A$,

$$\|\varphi(x_1x_2\ldots x_n) - \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\| \leq \varepsilon \|x_1\| \|x_2\| \ldots \|x_n\|,$$

for every $x \in A$,

$$\|\varphi(x^n) - \varphi(x)^n\| \leq \varepsilon \|x\|^n.$$

Let $x = 2a$. We have

$$\|\varphi(a^n) - \varphi(a)^n\| \leq \varepsilon 2^{-n} \|a\|^n.$$

Put $\|a\| = 1$. So for all $a \in A$ with $\|a\| = 1$ we have

$$\varphi(a^n) = \varphi(a)^n.$$

Therefore for all $a \in A$ if $\|a\| = 1$ it is sufficient that we put $\frac{a}{\|a\|}$. So $f$ is an $n$-Jordan map.

It is easy to check that every bounded linear map between Banach algebras $A$ and $B$ is an almost $n$-multiplicative and there are some elements in $BL(A, B)$ which are not $n$-homomorphism.

Theorem 3.4

Every almost multiplicative linear map from a Banach algebra $A$ to a semi-simple Banach algebra $B$ is...
almost $n$-multiplicative.

**Proof**

Let $\varphi: A \rightarrow B$ be an almost multiplicative linear map. For every $a_1, \ldots, a_n \in A$ we have:

$$|| \varphi(a_1 \ldots a_n) - \varphi(a_1) \varphi(a_2 \ldots a_n) || \leq \epsilon || a_1 || a_2 \| \ldots \| a_n || .$$

Continuing this process, we get:

$$|| \varphi(a_1 \ldots a_n) - \varphi(a_1) \varphi(a_2 \ldots a_n) || \leq \epsilon (|| || \varphi|| + || || \varphi^2 || + \ldots + || || \varphi^{n-1} ||) || a_1 || \ldots || a_n ||,$$

and the boundedness of $\varphi$ (Theorem 2.3) completes the proof.

**Corollary 3.5**

Every almost multiplicative linear functional on a Banach algebra $A$ is an almost $n$-multiplicative one. Here, we give an example to show that an almost $n$-multiplicative linear map is not necessarily an almost multiplicative one.

**Example 3.6**

Let $(X; || . ||)$ be the normed algebra of all polynomials defined on $[0,1]$ and $f$ be a linear unbounded functional on $X$. Let also

$$A = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in X \right\} \quad \text{with} \quad || A || = \| x \|_1$$

and

$$B = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in X \right\}$$

with Euclidean norm be two Banach algebras with the usual matrix operations for addition, scaler multiplication, and product. Define $\varphi: A \rightarrow B$ with

$$\varphi \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} f(x) \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Since $A^2 = B^3 = 0$, $\varphi$ is a 3-homomorphism and so is a 3-multiplicative linear map. On the other hand:

$$\varphi \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} f'(x) \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Hence, there is no $\epsilon > 0$ such that for all $a \in A$, we have:

$$|| \varphi \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) || \leq \epsilon || \varphi \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) ||^2,$$

and therefore $\varphi$ is not almost multiplicative.

The next Theorems give some more information on the relationship between almost $n$-multiplicative linear maps and almost multiplicative.

**Theorem 3.7**

Let $A$ and $B$ be two unital Banach algebras and $\varphi: A \rightarrow B$ a linear almost $n$-multiplicative map. If $\varphi(1) \in Inv(B)$ then, $\psi(x) = \varphi(1)^{-1} \varphi(x)$ is an almost multiplicative map.

**Proof**

Suppose $a_1, \ldots, a_n \in A$. Then,

$$|| \varphi(a_1 \ldots a_n) - \varphi(a_1) \varphi(a_2 \ldots a_n) || \leq \epsilon || \varphi(1)^{-n} \varphi(a_1 \ldots a_n) - \varphi(a_1) \varphi(a_2 \ldots a_n) || \leq \epsilon || \varphi(1)^{-n} \varphi(a_1 \ldots a_n) - \varphi(a_1) \varphi(a_2 \ldots a_n) || \leq \epsilon || \varphi(1)^{-n} || (1 \| a_1 || \ldots || a_n || + \epsilon || a_1 || \ldots || a_n ||) = \epsilon || \varphi(1)^{-n} || (1 \| a_1 || \ldots || a_n || + \epsilon || a_1 || \ldots || a_n ||)$$

and is almost $n$-multiplicative.

Now, $\psi(1) = 1$ and for $a, b \in A$ we have:

$$|| \varphi(ab) - \varphi(a) \varphi(b) || \leq || \varphi(ab) - \varphi(a) \varphi(b) \psi(1) \div \psi(1) || \leq \delta || a || || b || || b ||^{-1},$$

with $\delta = \epsilon || \varphi(1)^{-n} || (1 \| a_1 || \ldots || a_n || + 1)$.}

**Theorem 3.8**

Let $A$ be a unital Banach algebra and $\varphi$ an almost $n$-multiplicative linear function with $\varphi(1) \neq 0$. Then, $\varphi$ is an almost multiplicative one.
Proposition 3.9

Let $A$ and $B$ be two Banach algebras without unit and $\phi: A \rightarrow B$ a linear almost multiplicative map. Then, $\phi$ has a usual extension $\psi: A \rightarrow B$ such that $\psi$ is an almost multiplicative map.

Proof

Define $\psi: A \rightarrow B$ with $\psi((a, \lambda)(b, \gamma)) = (\phi(ab) + \lambda \phi(b) + \gamma \phi(a), \lambda \gamma)$. We have:

\[ \| \psi((a, \lambda)(b, \gamma)) - \psi((a, \lambda)(b, \gamma)) \| \leq \| \phi(ab) + \lambda \phi(b) + \gamma \phi(a), \lambda \gamma - (\phi(ab)(b) + \phi(a)) \lambda \gamma \|. \]

So $\psi$ is linear almost multiplicative map.

Proposition 3.9 does not hold for almost n-multiplicative maps, because, if we suppose $A$ and $B$ be two Banach algebras without unit and $\phi: A \rightarrow B$ a linear almost n-multiplicative map.

The extension map $\psi: A \rightarrow B$ is again almost n-multiplicative; then, since $\psi(0,1) = (0,1) \in \text{Inv}(B)$, by Theorem 3.6, $\psi$ must be almost multiplicative and then, $\phi$ will be almost multiplicative, where the example 3.6 shows that this is not true.

The Continuity of Almost n-Multiplicative Maps

Here, we prove the continuity of almost n-multiplicative maps.

Proposition 4.1

Let $A$ be a unital Banach algebra. If $\tau$ is a linear almost $n$-multiplicative functional on $A$, then $T$ is continuous.

Proof

If $T(1) = 0$, then

$|T(x)| = |T(x.1...1) - T(x)T(1)...T(1)| \leq \varepsilon \| x \| = 1 \|^{n-1}$,

so $T$ is continuous. In the other case by Theorem 3.8 and Proposition 2.2 the proof is complete.

Theorem 4.2

Every linear almost $n$-multiplicative map from a unital Banach algebra $A$ to a unital semi-simple Banach algebra $B$ is continuous.

Proof

Suppose that $T: A \rightarrow B$ is a linear and almost $n$-multiplicative map. For any multiplicative linear functional $F$ on $B$, $FoT$ is a linear functional and,

$|FoT(a) - Fo(a)| \leq M\|F\|\|a\|\|T(a)\|$.

Therefore $FoT$ is a linear almost $n$-multiplicative functional. So by Proposition 4.1, it is continuous. Let $\{a_n\}$ be a sequence with $a = \lim_{n \rightarrow \infty} a_n$, and $b = \lim_{n \rightarrow \infty} T(a_n)$. We show that $b = T(a)$. For any multiplicative linear functional $F$ we have

$F(b) = F(\lim_{n \rightarrow \infty} T(a_n)) = \lim_{n \rightarrow \infty} F(T(a_n)) = FoT(a)$.

So $F(b - T(a)) = 0$. Since $B$ is semi-simple we get $b = T(a)$. Consequently, $T$ is continuous by the closed graph theorem.

REFERENCES

