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Full Length Research Paper

M|G|∞ queue system parameters for a particular collection of service time distributions

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The problems that arise in the computation of the moments of a particular collection of service time distributions, for which the M|G|∞ busy period and busy cycle become very easy to study, are presented in this paper. It is shown how to overcome them. The busy cycle renewal function, the "peak" and the "modified peak" for the M|G|∞ busy period and busy cycle in the case of those service time distributions are also computed.

Key words: Probability distributions, M|G|∞, busy period, busy cycle.

INTRODUCTION

When, in the M|G|∞ queue systems, the service time is a random variable with distribution function belonging to the collection:

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0\leq p<1 \quad (1.1)$$

The busy period duration is exponential with an atom at the origin. And the busy cycle one is the mixture of two exponential distributions (Ferreira, 2005). Although it is so easy to study the busy period and the busy cycle in this situation it is very difficult to compute the service time moments.

Some results are presented, precisely, concerning the moments computation of the random variables with distribution functions given by that collection that generalize those from Ferreira (1998).

Finally formulae that give the busy cycle renewal function, the "peak" and the "modified peak" to the busy period and the busy cycle of the M|G|∞ system for those service time distributions are deduced, generalizing results from Ferreira (2004, 1997, 1999).

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Moments computation

Be $G(t), t\geq 0$ a distribution function and $g(t)=\frac{dG(t)}{dt}$.

The differential equation in $G(.)$, $(1-p)\frac{g(t)}{1-G(t)}-\lambda p-\lambda(1-p)G(t)=\beta$, where $\lambda>0$ and $-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}$, $0\leq p<1$ ($\rho=\lambda\alpha$, being α the mean of $G(t)$) is verified by (1.1) (Ferreira (2005).

If, in (1.1), $G_i(t)$ is the solution associated to ρ_i, $i=1,2,3,4$ it is easy to see that

$$\frac{G_4(t)-G_2(t)}{G_4(t)-G_1(t)}\cdot\frac{G_3(t)-G_1(t)}{G_3(t)-G_2(t)}=\frac{e^{-\rho_4}-e^{-\rho_2}}{e^{-\rho_4}-e^{-\rho_1}}\cdot\frac{e^{-\rho_3}-e^{-\rho_1}}{e^{-\rho_3}-e^{-\rho_2}} \quad (2.1)$$

as it had to happen since the differential equation considered is a Ricatti one.

And computing,

$$\int_0^{\infty}[1-G(t)]dt=\int_0^{\infty}\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda}dt=$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{1}{e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+1}dt=$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}-e^{-\rho}e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}+e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}dt=$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}dt=$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\cdot\frac{-1}{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}\cdot\left[\log\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\right]_0^\infty=$$

$$=-\frac{1}{\lambda}\left(\log e^{-\rho}-\log 1\right)=\frac{-\rho}{-\lambda}=\alpha$$

in accordance to the fact that (1.1) is a collection of positive random variables distribution functions.

The density associated to $G(t)$ given by (1.1) is

$$g(t)=\frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2 e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\lambda\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2},t>0,-\lambda\le\beta\le\frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1},0\le p<1$$

(2.2).

So,

$$\int_0^\infty t^n g(t)dt=\frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2}{\lambda}\cdot\int_0^\infty t^n\frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\cdot$$

But,

$$\int_0^\infty t^n\frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\ge\int_0^\infty t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt=$$

$$=\frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n},\beta\neq-\lambda.$$

And,

$$\int_0^\infty t^n\frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\le$$

$$e^{2\rho}\int_0^\infty t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt=$$

$$=\frac{e^{2\rho}}{\lambda+\frac{\lambda p+\beta}{1-p}}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n},\beta\neq-\lambda.$$

So, calling T the random variable corresponding to $G(t)$:

$$\frac{\left(1-e^{-\rho}\right)e^{-\rho}}{\lambda}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}}\le\mathrm{E}\left[\mathrm{T}^n\right]\le\frac{e^{\rho}-1}{\lambda}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}},$$

$$,-\lambda<\beta\le\frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1},0\le p<1,n=1,2,...$$

(2.3).

Notes

The expression (2.3), giving bounds for $\mathrm{E}\left[\mathrm{T}^n\right]$, guarantees its existence, For $n=1$ the expression (2.3) is useless because $\mathrm{E}[\mathrm{T}]=\alpha$. Note, curiously, that the upper bound is $\frac{e^{\rho}-1}{\lambda}$, the M|G|∞ system busy period mean value, For $n=2$, subtracting to both bounds α^2, from expression (2.3) result bounds for $VAR[\mathrm{T}]$,

For $\beta=-\lambda,\mathrm{E}\left[\mathrm{T}^n\right]=0,n=1,2,...,$ evidently.

See, however, that (1.1) can be put like:

$$G(t)=\frac{1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{1-\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1$$

(2.4)

And, for $\rho<\log 2$,

$$G(t)=\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\cdot\sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^{k}e^{-k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t},$$

$$,t\geq 0,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (2.5).$$

After (2.5) we can derive easily the T Laplace Transform for $\rho<\log 2$. And, so,

- For $\rho<\log 2$

$$\mathrm{E}\left[\mathrm{T}^{n}\right]=-\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right)n!\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n}},-\lambda<\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1,$$

$$,n=1,2,...$$

(2.6).

Notes

$$\mathrm{E}[\mathrm{T}]=-\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right)\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}=\frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(1-e^{\rho}\right)}{k}=$$

$$=\frac{1}{\lambda}\log e^{\rho}=\frac{\rho}{\lambda}=\alpha.$$

For $n\geq 2$ only a finite number of parcels in the infinite sum must be taken. Calling M this number, to get an error lesser than ε it is necessary to have simultaneously

- $\mathrm{M}>\dfrac{1}{\lambda+\dfrac{\lambda p+\beta}{1-p}}-1,$

- $\mathrm{M}>\log_{\left(e^{\rho}-1\right)}\dfrac{\varepsilon e^{\rho}\lambda}{n!\left(\lambda+\dfrac{\lambda p+\beta}{1-p}\right)}-1.$

So it is evident now that this distributions collection moments computation is a complex task. This was already true for the study of Ferreira (1998) where the results presented are a particular situation of these ones for $p=0$. The consideration of the approximation

$$\mathrm{E}_{m}^{n}=\sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^{n}\left[G\left(\frac{k}{m}\right)-G\left(\frac{k-1}{m}\right)\right],-\lambda<\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1,n=1,2,...$$

(2.7)

may be helpful since $\lim_{m\to\infty}\mathrm{E}_{m}^{n}=\mathrm{E}\left[\mathrm{T}^{n}\right],n=1,2,...$ (Ferreira, 1987)) that allow the numerical computation of the moments.

Busy cycle renewal function computation

The busy cycle (an idle period followed by a busy period) renewal function value of the M|G|∞ queue, at the instant t, gives the mean number of busy periods that begin in $[0,t]$ (Ferreira, 2004). If the service is a random variable with distribution function given by a member of the collection (1.1), calling the value of the renewal function at t $R(t)$, it is obtained

$$R(t)=e^{-\rho}(1+\lambda t)+\left(1-e^{-\rho}\right)\frac{\lambda p+\beta}{\lambda+\beta}e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}+\left(1-e^{-\rho}\right)\frac{\lambda p+\beta}{\lambda+\beta},-\lambda\leq$$

$$\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (3.1).$$

For $p=0$ it turns in the result presented in Ferreira (2004).

The "Peak" and the "Modified Peak" for the busy period and for the busy cycle

The M|G|∞ queue busy period "peak" is the Laplace transform of its duration at $1/\alpha$ (Ferreira, 1997). It is a parameter that characterizes the busy period distribution duration and contains information about all its moments. For the collection of service distributions (1.1) the "peak", called pi, is

$$pi=\frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{\lambda\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)},-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1$$

(4.1).

In Ferreira (1997) it is also introduced another measure, the "modified peak" got after the "peak" taking out the terms that are permanent for the busy period in different service distributions and putting over the common part. Call it qi. It is easy to verify that $qi = pi\frac{\rho}{e^{\rho}-\rho-1}+1$ and, so, for the distributions given by collection (1.1)

$$qi = \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p+\beta}{\lambda\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}\frac{\rho}{e^{\rho}-\rho-1}+1, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},$$

$$0 \le p < 1 \qquad (4.2)$$

For the busy cycle of the M|G|∞ queue the "peak" Ferreira (1999), called now pi', for the service distributions given by the collection (1.1) is

$$pi' = \alpha\frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{(\rho+1)\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \qquad (4.3)$$

and the "modified peak", called now qi', given by $pi'\frac{\rho}{e^{\rho}-\rho}+1$. And for the service distributions given by the collection (1.1) it is

$$qi' = \alpha\frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{(\rho+1)\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}\frac{\rho}{e^{\rho}-\rho}+1, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \qquad (4.4).$$

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