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*Full Length Research Paper*

# Curve concentration for a singularly perturbed Neumann problem in three dimensional domain

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**In this paper we consider the following problem $\varepsilon^2\Delta\tilde{u}-\tilde{u}+\tilde{u}^p=0,\ \ \tilde{u}>0\ \text{ in }\Omega,\ \ \frac{\partial\tilde{u}}{\partial\nu}=0\ \text{ on }\partial\Omega,$ where $\Omega$ is a bounded domain in $R^3$ with smooth boundary, $\varepsilon$ is a small parameter, $\nu$ denotes the outward normal of $\Omega$ and $p>1$. Let $\Gamma$ be a straight line intersecting with $\partial\Omega$ at exactly two points. We will prove the existence of a solution $u_\varepsilon$ possessing curve concentrating set near $\Gamma$, exponentially small in $\varepsilon$ at any positive distance from the concentrating set, provided $\varepsilon$ is small and away from certain critical numbers.**

**Key words**: Curve concentration, singular perturbation, Neumann problems, spike layer.

## INTRODUCTION

We consider the following problem;

$$\varepsilon^2\Delta\tilde{u}-\tilde{u}+\tilde{u}^p=0,\ \ \tilde{u}>0\text{ in }\Omega\subset R^3,\qquad \frac{\partial\tilde{u}}{\partial\nu}=0\quad\text{on }\partial\Omega, \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^3$ with smooth boundary, $\varepsilon$ is a small parameter, $\nu$ denotes the outward normal of $\Omega$ and $p>1$. Problem (1.1) comes from the shadow system of Gierer-Meinhardt model, which used densities of a chemical activator $U$ and an inhibitor $V$ to describe experiments of regeneration of hydra by the form (Gierer et al., 1972; Ni, 1998, 2004).

In the following, we discuss the existence of some related kinds of concentrated solutions to (1.1). Under the condition that $p$ is subcritical, Lin et al. (1988), Ni et al. (1991, 1993) established the existence of a least-energy solution $U_\varepsilon$ of problem (1.1) and showed that, for $\varepsilon$ sufficiently small, $U_\varepsilon$ has only one local maximum point $P_\varepsilon\in\partial\Omega$. Moreover, $H(P_\varepsilon)\to\max_{P\in\partial\Omega}H(P)$ as $\varepsilon\to 0^+$, where $H(P)$ is the mean curvature of $\partial\Omega$ at the point $P$. Such a solution is called boundary spike-layer.

Since then, many papers investigated further solutions of (1.1) concentrating at one or multiple points of $\bar{\Omega}$. (These solutions are called *spike-layers*.) A general principle is that the location of *interior spike layer* (locating in the interior of $\Omega$) is determined by the distance function from the boundary. We refer the reader to the articles (Bates et al., 2000; Dancer et al., 1999; del Pino et al., 2000; Grossi et al., 2000, Gui and Wei, 1999, 2000, Wei et al., 1998)[2] and references therein. On the otherhand, *boundary spike layers* are related to the mean curvature of $\partial\Omega$. This aspect is discussed in the papers of Bates et al. (1999), Dancer et al. (1999), del Pino et al. (1999), Gui et al. (2000), Li (1998), Wei (1997), Wei et al. (1998) and references therein. A good review of the subject up to 2004 can be found in Ni (2004).

The question of constructing high dimensional concentration sets has been investigated only in recent years. It has been conjectured in Ni (2004) that for any $1\le k\le n-1$, problem (1.1) has a solution $U_\varepsilon$ which concentrates on a $k$-dimensional subset of $\bar{\Omega}$. We mention some results that support such a conjecture.

Malchiodi and Montenegro (2002, 2004) proved that for $n \geq 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (1.1) has a solution $U_{\varepsilon_k}$ which concentrates at the boundary $\partial\Omega$ (or any component of $\partial\Omega$). Malchiodi (2004, 2005) showed the concentration phenomena for (1.1) are also present along a closed non-degenerate geodesic of $\partial\Omega$ in three-dimensional smooth bounded domain $\Omega$. For $(1 \leq k \leq n-2)$, Mahmoudi and Malchiodi (2007) proved a full general concentration of solutions along $k$ -dimensional non-degenerate minimal submanifolds of the boundary for $n \geq 3$ and $1 < p < \frac{n-k+2}{n-k-2}$.

However, for the results discussed in above paragraph, the higher dimensional concentration set is on the boundary. A natural question is that if there are solutions with high dimensional concentration set inside the domain. In this paper we consider problem (1.1) for the existence of solutions with interior concentration layers near a straight line $\Gamma$ intersecting the boundary.

Throughout the paper, our candidate curve $\Gamma \in \overline{\Omega}$ satisfies the following assumptions: The curvature of $\Gamma$ is zero and in the $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ coordinates, $\Gamma$ is contained in the $\tilde{y}_3$ axis. After rescaling, we can always assume $|\Gamma| = 1$. $\Gamma$ intersects $\partial\Omega$ at exactly two points, saying, $\gamma_1 = (0, 0, \frac{1}{2}), \gamma_0 = (0, 0, -\frac{1}{2})$ and at these points $\Gamma \perp \partial\Omega$. We also assume that $\partial\Omega$ can be smoothly represented as $\tilde{y}_3 = \varphi_1(\tilde{y}_1, \tilde{y}_2)$ and $\tilde{y}_3 = \varphi_0(\tilde{y}_1, \tilde{y}_2)$ near $\gamma_1, \gamma_0$ respectively. Hence, there hold

$$\frac{\partial \varphi_0}{\partial \tilde{y}_1}(0,0) = \frac{\partial \varphi_0}{\partial \tilde{y}_2}(0,0) = 0,$$
$$\frac{\partial \varphi_1}{\partial \tilde{y}_1}(0,0) = \frac{\partial \varphi_1}{\partial \tilde{y}_2}(0,0) = 0. \tag{1.2}$$

By defining two matrixes as:

$$A = \begin{pmatrix} \frac{\partial^2 \varphi_1}{\partial \tilde{y}_1 \partial \tilde{y}_1} & \frac{\partial^2 \varphi_1}{\partial \tilde{y}_1 \partial \tilde{y}_2} \\ \frac{\partial^2 \varphi_1}{\partial \tilde{y}_2 \partial \tilde{y}_1} & \frac{\partial^2 \varphi_1}{\partial \tilde{y}_2 \partial \tilde{y}_2} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\partial^2 \varphi_0}{\partial \tilde{y}_1 \partial \tilde{y}_1} & \frac{\partial^2 \varphi_0}{\partial \tilde{y}_1 \partial \tilde{y}_2} \\ \frac{\partial^2 \varphi_0}{\partial \tilde{y}_2 \partial \tilde{y}_1} & \frac{\partial^2 \varphi_0}{\partial \tilde{y}_2 \partial \tilde{y}_2} \end{pmatrix}$$

we also assume further restriction on $\partial\Omega$ at $\gamma_1$ and $\gamma_0$ in the sense that

$$AB = BA \text{ at } (\tilde{y}_1, \tilde{y}_1) = (0,0). \tag{1.3}$$

From the theory of linear algebra, there exists a unitary matrix $Q$ such that

$$Q'AQ = \operatorname{diag}(k_1^1, k_1^2),$$
$$Q'BQ = \operatorname{diag}(k_0^1, k_0^2). \tag{1.4}$$

By defining two *geometric eigenvalue problem*,

$$f_1''(\theta) + \lambda_1 f_1(\theta) = 0, \quad 0 < \theta < 1,$$
$$f_1'(1) + k_1^1 f_1(1) = 0,$$
$$f_1'(0) + k_0^1 f_1(0) = 0, \tag{1.5}$$

$$f_2''(\theta) + \lambda_2 f_2(\theta) = 0, \quad 0 < \theta < 1,$$
$$f_2(1) + k_1^2 f_2(1) = 0,$$
$$f_2'(0) + k_0^2 f_2(0) = 0, \tag{1.6}$$

we say that $\Gamma$ is *non-degenerate* if problem (1.5) and problem (1.6) do not have zero eigenvalues. This is equivalent to:

$$k_0^i - k_1^i + k_0^i k_1^i |\Gamma| \neq 0, \; i = 1, 2. \tag{1.7}$$

Let $w$ be the unique (even) solution of

$$\Delta w - w + w^p = 0 \text{ and } w > 0 \text{ in } R^2,$$
$$\max_{(x,y)\in R^2} w(x,y) = w(0,0),$$
$$w(x,y) \to 0 \text{ as } |(x,y)| \to +\infty, \tag{1.8}$$

and consider the associated linearized eigenvalue problem,

$$\Delta h - h + p w^{p-1} h = \lambda h \quad \text{in } R^2,$$
$$h(x,y) \to 0 \text{ as } |(x,y)| \to +\infty. \tag{1.9}$$

It is well known that this equation possesses a unique positive eigenvalue $\lambda_0$ (the first eigenvalue), with associated even and positive eigenfunction $Z$ in $H^1(R^2)$ which can be normalized in the sense that $\int_{R^2} Z^2 = 1$. Moreover $w_x, w_y$ are eigenfunctions with respect to the zero Eigen values (with $2$-multiplicity). The fourth eigenvalue is negative.

For the special case of dimension $n = 2$, Wei et al. (2007) constructed curve like concentration solutions to problem (1.1) near the nondegenerate segment $\Gamma$, provided that $\varepsilon$ satisfies the gap condition;

$$\left|\tilde{\lambda}_0 - j^2\pi^2\varepsilon^2 / |\Gamma|^2\right| \geq \tilde{c}\varepsilon, \ \forall j \in N, \tag{1.10}$$

with small $\tilde{c} > 0$. $\tilde{\lambda}_0$ is the first eigenvalue of problem (1.9) in one dimensional case (Wei et al., 2008) for clustered concentration solutions.

Now we will extend the result in Wei et al. (2007) to three dimensional case for the existence of curve like concentration solutions.

**THEOREM 1.1**

Assume that the line segment $\Gamma$ satisfies (1.3) and the non-degenerate condition (1.7). Given a small positive constant $\tilde{c}$, there exists $\varepsilon_0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the following gap condition

$$\left|\lambda_0 - j^2\pi^2\varepsilon^2 / |\Gamma|^2\right| \geq \tilde{c}\varepsilon, \ \forall j \in N, \tag{1.11}$$

problem (1.1) has a positive solution $u_\varepsilon$ concentrating along a curve $\Gamma_\varepsilon$ close to $\Gamma$. Near $\Gamma$, $u_\varepsilon$ takes the form;

$$u_\varepsilon(\tilde{y}) = w\left(\frac{\text{dist}(\tilde{y}, \Gamma_\varepsilon)}{\varepsilon}\right)(1 + o(1)). \tag{1.12}$$

Moreover, there exists some number $c_0$, for $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in \Omega$, $u_\varepsilon$ satisfies globally, $u_\varepsilon(\tilde{y}) \leq \exp[-c_0 \,\text{dist}(\tilde{y}, \Gamma_\varepsilon)/\varepsilon]$ and the curve $\Gamma_\varepsilon$ will collapse to $\Gamma$ as $\varepsilon \to 0$.

Let us comment on some related results, the difficulties as well as the main steps in proving Theorem 1.1.

**Remark 1**

The geometric Eigen value problems (1.5) and (1.6) also appeared in the study of transition layer for the following Allen-Cahn equation;

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.13}$$

Using $\Gamma-$ convergence, Kohn and Sternberg, 1989 constructed local minimizers to (1.13) with transition layer at straight line segment contained in $\Omega$ which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$. Later, Kowalczyk 2004, 2005 extended the construction to non-minimizing line segments. More precisely, assuming that $\Gamma$ satisfies (1.7), he constructed a solution $u_\varepsilon$ whose zero set $\Gamma_\varepsilon$ converges to $\Gamma$, for *all* $\varepsilon$ sufficiently small. Pacard and Ritore, 2003 constructed transition layer solutions to (1.13) near minimal submanifold.

**Remark 2**

As for the results in Malchiodi et al. (2002, 2004a,b, 2005), Mahmoudi, (2007) del Pino et al. (2006, 2007), Wei et al. (2007, 2008), existence results are proved only for small $\varepsilon$ satisfying a similar gap condition like (1.11). This is caused by a resonance phenomenon (to be described in the following), which also appears in some geometric problems (Pacard et al., 2003).

**Remark 3**

To explain in a few words the difficulties we have encountered, assume for the moment that $\Omega \subset R^3$ is an infinite strip as;

$$\Omega = R^2 \times (0,1).$$

In terms of the stretched coordinates $(s,t,z) = \varepsilon^{-1}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, the equation would look near the curve approximately like

$$v_{ss} + v_{tt} + v_{zz} - v + v^p = 0 \ \text{in } S,$$
$$\partial v / \partial z = 0 \quad \text{on } \partial S.$$

where $S = R^2 \times (0, 1/\varepsilon)$. The effect of curvature and of the boundary conditions are here neglected. The linearization of this problem around the profile $w(s,t)$ becomes

$$\phi_{zz} + \phi_{ss} + \phi_{tt} - \phi + pw^{p-1}\phi = 0, \quad (s,t,z) \in S, \qquad \partial\phi / \partial z = 0 \quad \text{on } \partial S.$$

Functions of the form

$$\phi^1 = w_s(s,t)\cos(k\pi\varepsilon z),$$

$$\phi^2 = w_t(s,t)\cos(k\pi\varepsilon z),$$

$$\phi^3 = Z(s,t)\cos(k\pi\varepsilon z),$$

are eigenfunctions associated to eigenvalues respectively $-k^2\varepsilon^2$ and $\lambda_0 - k^2\varepsilon^2$. Many of these numbers are small and thus "near non-invertibility" of the linear operator occurs. These effects, combined in principle orthogonally because of the $L^2$-orthogonality of $Z$ and $w_s, w_t$, are actually coupled through the smaller order terms neglected.

In Alikakos et al. (2000), Kowalczyk (2004, 2005), Pacard et al. (2003), related singular perturbation problems, involving the Allen-Cahn equation (1.13), the translation effect $\phi^1$ have been successfully treated through successive improvements of the approximation and fine spectral analysis of the actual linearized operator. In [26, 27] Malchiodi et al. (2004, 2005) resonance phenomena similar to the "$\phi^3$-effect" has been faced in the Neumann problem involving whole boundary concentration. In Mahmoudi et al. (2007), Malchiodi (2004, 2005) the boundary concentration on a $k-$ dimensional minimal surface of the boundary, involving both $\phi^1$ and $\phi^3$ effects, has been treated via arbitrary high order approximations.

The main difficulty in this paper, as well as Wei et al. (2007, 2008), will come from not only the coupling of $\phi^1, \phi^2$ and $\phi^3$, but also the boundary condition. In [8], the error term is of the order $O(\varepsilon^2)$, while here the error term is $O(\varepsilon)$ since the stretching of the boundary conditions gives $\frac{\partial\phi}{\partial z} + O(\varepsilon)$. However, the spectrum gap in (1.11) is also $O(\varepsilon)$ which creates additional difficulty. Worse than that, the spectrum gap caused by $\phi^3$ and the boundary corrections are strongly coupled. We overcome these difficulties by first using successive improvements of the approximation and then perform the infinite-dimensional reduction in [8] to reduce the problem to coupled nonlinear ODEs. The reduced ODEs involve coefficients of both fast and slow variables (See section 6). A careful analysis of Fourier modes is needed to ensure the invertibility.

**Remark 4**

A new ingredient is present in this paper: $\phi^1$ and $\phi^2$ has strong coupling on the boundary, which calls for the symmetric condition (1.3) to decompose these two effects. In fact, under condition (1.3), the terms (of order $O(\varepsilon)$ in (2.10) and (2.11)) involving $tu_s, su_t$ disappear on the boundary $\partial S$. Moreover, we will use the technique in section 5 of del Pino et al. 2007 to find a boundary layer to get further improvement of approximation, see also Wei et al. (2007). It is interesting to construct solutions with twisted concentration set in higher dimensional case with a weaker restriction like (1.3).

The remaining part of this paper is devoted to the complete proof of Theorem 1.1. The organization is as follows: In Section 2, after setting up the problem in stretched variables $(s,t,z)$, we introduce a local approximation by $w(s-f_1, t-f_2)$ in which the parameters $f_1$ and $f_2$ are used to characterize the location of the concentration set. Then we find an improvement of the approximation to cancel all error terms of order $O(\varepsilon)$ on the boundary. In Section 3, a gluing procedure, as in del Pino et al. (2007), reduces the nonlinear problem (1.1) to a projected problem on the infinite strip $S$, while in Section 4 and 5, we show that the projected problem has a unique solution $\phi$ for given parameters $f_1, f_2, e$ in a chosen region. The final step is to adjust the parameters $f_1, f_2, e$ such that problem (1.1) has a real concentrating solution, which is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the functions $f_1, f_2, e$ with suitable boundary conditions. This is done in sections 6 and 7.

## Setting up the problem and approximation

Let us make some notations in what follows as

$$S = \{(x,y,z) : x \in R, y \in R, 0 < z < 1/\varepsilon\}, \qquad (2.1)$$

$$\partial_1 S = \{(x,y,z) : x \in R, y \in R, z = 1/\varepsilon\},$$
$$\partial_0 S = \{(x,y,z) : x \in R, y \in R, z = 0\}. \qquad (2.2)$$

**SETTING UP THE PROBLEM**

Now, we turn to the procedure of setting up the problem near $\Gamma$. Globally in $R^3$, making scaling

$$Y \equiv (y_1, y_2, y_3) = (\tilde{y}_1/\varepsilon,\ \tilde{y}_2/\varepsilon,\ \tilde{y}_3/\varepsilon), \tag{2.3}$$

denote $\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$ and $\nu_\varepsilon$ is the outward normal of $\Omega_\varepsilon$. The problem (1.1) becomes

$$\begin{aligned}&\frac{\partial^2 u}{\partial y_1^2}+\frac{\partial^2 u}{\partial y_2^2}+\frac{\partial^2 u}{\partial y_3^2}-u+u^p=0\\ &\quad \text{and } u>0 \quad \text{in } \Omega_\varepsilon,\\ &\quad \frac{\partial u}{\partial \nu_\varepsilon}=0 \quad \text{on } \partial\Omega_\varepsilon.\end{aligned} \tag{2.4}$$

Introducing new coordinates near $\Gamma_\varepsilon$

$$(s,t,z)=\left( y_1, y_2, \frac{y_3-\varphi_0\left(\varepsilon(y_1,y_2)Q'\right)/\varepsilon}{\varphi_1\left(\varepsilon(y_1,y_2)Q'\right)-\varphi_0\left(\varepsilon(y_1,y_2)Q'\right)} \right), \tag{2.5}$$

where $-\delta_0 < s,t < \delta_0$ for all small $\delta_0$, and then using the assumptions (1.2) - (1.4) to make Taylor expansion, we get that in a neighborhood of $\Gamma_\varepsilon$ problem (2.4) takes the form

$$\begin{aligned}&u_{ss}+u_{tt}+u_{zz}+B_1(u)-u+u^p=0,\\ &-\delta_0<\varepsilon s, \varepsilon t<\delta_0, 0<z<1/\varepsilon,\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\overline{D}_1(u)+\overline{D}_0(u)+u_z=0,\\ &-\delta_0<\varepsilon s, \varepsilon t<\delta_0, 0<z<1/\varepsilon,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\underline{D}_1(u)+\underline{D}_0(u)+u_z=0,\\ &-\delta_0<\varepsilon s, \varepsilon t<\delta_0, 0<z<1/\varepsilon,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}B_1(u)=&-\varepsilon 2k_0^1 s u_{sz}-\varepsilon 2k_0^2 t u_{tz}-\varepsilon(k_0^1+k_0^2)u_z\\ &+\varepsilon^2(a_1 s^2+\varepsilon^2 a_2 st+\varepsilon^2 a_3 t^2)u_{zz}\\ &+\varepsilon^2(a_6 s^2+a_7 st+a_8 t^2+\alpha_1(z)s)u_{sz}\\ &+\varepsilon^2(a_{11} s^2+a_{12} st+a_{13} t^2+\alpha_2(z)t)u_{tz}\\ &+\varepsilon^2(a_{15} s+a_{16} t+\alpha_3(z))u_z+B_0(u),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\overline{D}_1(u)=&-\varepsilon k_1^1 s u_s-\varepsilon^2[b_3 s^2+b_4 st+b_5 t^2]u_s\\ &-\varepsilon k_1^2 t u_t-\varepsilon^2[\frac{1}{2}b_4 s^2+2b_5 st+b_7 t^2]u_t,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\underline{D}_1(u)=&-\varepsilon k_0^1 s u_s-\varepsilon^2[b_{10} s^2+b_{11} st+b_{12} t^2]u_s\\ &-\varepsilon k_0^2 t u_t-\varepsilon^2[\frac{1}{2}b_{11} s^2+2b_{12} st+b_{14} t^2]u_t.\end{aligned} \tag{2.11}$$

The constants $b_j, j=1,\cdots,14,$ are the derivatives (from second order up to third order) of $\varphi_1$ and $\varphi_0$ at the point $(0,0)$.

$$\begin{aligned}&\alpha_1(z)=2(k_0^1-k_1^1)z,\\ &\alpha_2(z)=2(k_0^2-k_1^2)z,\\ &\alpha_3(z)=(k_0^1+k_0^2-k_1^1-k_1^2)z.\end{aligned} \tag{2.12}$$

The constants $a_i, i=1,\cdots,16,$ depend on $b_j, j=1,\cdots,14.$ Note that $B_0(u), \overline{D}_0(u)$ and $\underline{D}_0(u)$ are of size $O(\varepsilon^3)$.

Supposing that the location of the concentration set $\Gamma_\varepsilon$ is characterized by the twisted curve $(f_1(\varepsilon z), f_2(\varepsilon z), z)$, introduce new variables

$$x=s-f_1(\varepsilon z),\ y=t-f_2(\varepsilon z),\ \eta=z, \tag{2.13}$$

and then

$$u_s = u_x, \quad u_t = u_y, \quad u_z = u_\eta - \varepsilon f' u_x,$$
$$u_{ss} = u_{xx}, \quad u_{tt} = u_{yy},$$
$$u_{st} = u_{xy}, \quad u_{sz} = u_{\eta x} - \varepsilon f_1' u_{xx} - \varepsilon f_2' u_{yx},$$
$$u_{tz} = u_{\eta y} - \varepsilon f_1' u_{xy} - \varepsilon f_2' u_{yy},$$
$$u_{zz} = u_{\eta\eta} - 2\varepsilon f_1' u_{x\eta} - 2\varepsilon f_2' u_{y\eta} + \varepsilon^2 (f_1')^2 u_{xx} + 2\varepsilon^2 f_1' f_2' u_{xy} + \varepsilon^2 (f_2')^2 u_{yy} - \varepsilon^2 f_1'' u_x - \varepsilon^2 f_2'' u_y.$$

Therefore, after writing the variable $\eta$ back to $z$ again, we can consider the problem in the infinite strip $S$ as the following

$$S(u) \equiv u_{xx} + u_{yy} + u_{zz} + B_3(u) - u + u^p = 0, \tag{2.14}$$

with boundary conditions

$$u_z + \overline{D}_3(u) + \overline{D}_0(u) = 0 \quad \text{on } \partial_1 S,$$
$$u_z + \underline{D}_3(u) + \underline{D}_0(u) = 0 \quad \text{on } \partial_0 S, \tag{2.15}$$

where

$$\begin{aligned}
B_3(u) &= \varepsilon[-2k_0^1 (x + f_1) - 2 f_1' ]u_{zx} \\
&+ \varepsilon[-2k_0^2 (y + f_2) - 2 f_2' ]u_{zy} \\
&- \varepsilon(k_0^1 + k_0^2) u_z - \varepsilon^2 f_1'' u_x - \varepsilon^2 f_2'' u_y \\
&+ \varepsilon^2 (k_0^1 + k_0^2) f_1' u_x + \varepsilon^2 (k_0^1 + k_0^2) f_2' u_y \\
&+ \varepsilon^2 [a_{15} (x + f_1) + a_{16} (y + f_2) + \alpha_5 (z)] u_z \\
&+ \varepsilon^2 [a_1 (x + f_1)^2 + \varepsilon^2 a_2 (x + f_1)(y + f_2) \\
&\quad + \varepsilon^2 a_3 (y + f_2)^2 ] u_{zz} \\
&+ \varepsilon^2 [(f_1')^2 + 2k_0^1 f_1' (x + f_1)] u_{xx} \\
&+ \varepsilon^2 [2 f_1' f_2' + 2k_0^1 f_2' (x + f_1) \\
&\quad + 2k_0^2 f_1' (y + f_2)] u_{xy} \\
&+ \varepsilon^2 [(f_2')^2 + 2k_0^2 f_2' (y + f_2)] u_{yy} \\
&+ \varepsilon^2 [a_6 (x + f_1)^2 + a_7 (x + f_1)(y + f_2) \\
&\quad + a_8 (y + f_2)^2 \\
&\quad + \alpha_1 (z)(x + f_1) + \alpha_2 (z)(y + f_2)] u_{zx} \\
&+ \varepsilon^2 [a_{11} (x + f_1)^2 + a_{12} (x + f_1)(y + f_2) \\
&\quad + a_{13} (y + f_2)^2 \\
&\quad + \alpha_3 (z)(x + f_1) + \alpha_2 (z)(y + f_2)] u_{zy} \\
&+ B_2 (u),
\end{aligned}$$

and

$$\begin{aligned}
\overline{D}_3(u) &= -\varepsilon[f_1' + k_1^1 (x + f_1)] u_x \\
&- \varepsilon[f_2' + k_1^2 (y + f_2)] u_y \\
&+ \varepsilon^2 [-b_3 (x + f_1)^2 - b_4 (x + f_1)(y + f_2) \\
&\quad - b_5 (y + f_2)^2 ] u_x \\
&+ \varepsilon^2 [-\frac{1}{2} b_4 (x + f_1)^2 - 2b_5 (x + f_1)(y + f_2) \\
&\quad - b_7 (y + f_2)^2 ] u_y,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\underline{D}_3(u) &= -\varepsilon[f_1' + k_0^1 (x + f_1)] u_x \\
&- \varepsilon[f_2' + k_0^2 (y + f_2)] u_y \\
&+ \varepsilon^2 [-b_{10} (x + f_1)^2 - b_{11} (x + f_1)(y + f_2) \\
&\quad - b_{12} (y + f_2)^2 ] u_x \\
&+ \varepsilon^2 [-\frac{1}{2} b_{11} (x + f_1)^2 - 2b_{12} (x + f_1)(y + f_2) \\
&\quad - b_{14} (y + f_2)^2 ] u_y.
\end{aligned} \tag{2.17}$$

Note that $B_2(u)$ is a term of size $O(\varepsilon^3)$. The derivatives in terms $B_0(u)$, $\overline{D}_0(u)$ and $\underline{D}_0(u)$ are expressed in the variables $(x, y, z)$.

**First approximate solution**

We take $u_1 = w(x, y)$ as the first approximate solution of the problem in $S$. The error in $S$ takes the form

$$E_1 \equiv S(w) = B_3(w) = \sum_{i=1}^{4} S_i + B_2(w) \quad \text{in } S, \tag{2.18}$$

where

$$S_1 = \varepsilon^2 [2 f_1' f_2' + 2k_0^1 f_1 f_2' + 2k_0^2 f_2 f_1'] w_{xy}$$

is an odd function in the variable $x$ and $y$

$$S_2 = \varepsilon^2 [(f_1')^2 + 2k_0^1 f_1 f_1'] w_{xx} + \varepsilon^2 [(f_2')^2 + 2k_0^2 f_2 f_2'] w_{yy}$$

is an even function in the variable $x$ and $y$,

$$S_3 = \varepsilon^2 [f_1'' w_x + 2k_0^1 f_1' x w_{xx} + (k_0^1 + k_0^2) f_1' w_x + 2k_0^2 f_1' y w_{xy}]$$

is even in the variable $y$ and odd in the variable $x$,

$$S_4 = \varepsilon^2 [f_2'' w_y + 2k_0^2 f_2' y w_{yy} + (k_0^1 + k_0^2) f_2' w_y + 2k_0^1 f_2' x w_{xy}]$$

is even in the variable $x$ and odd in the variable $y$. On the boundary, the errors can be read

$$\overline{E}_{1b} \equiv \overline{D}_3(w) + \overline{D}_0(w) = \sum_{i=1}^{4} \overline{R}_i + \overline{D}_0(w) \text{ on } \partial_1 S, \tag{2.19}$$

$$\underline{E}_{1b} \equiv \underline{D}_3(w) + \underline{D}_0(w) = \sum_{i=1}^{4} \underline{R}_i + \underline{D}_0(w) \text{ on } \partial_0 S, \tag{2.20}$$

where

$$\overline{R}_1 = -\varepsilon^2 [b_4 f_1(1) + 2b_5 f_2(1)](y w_x + x w_y),$$

is odd in the variables $x$ and $y$;

$$\overline{R}_2 = -\varepsilon k_1^1 x w_x - \varepsilon k_1^2 y w_y - \varepsilon^2 [2b_3 f_1(1) + b_4 f_2(1)] x w_x - \varepsilon^2 [2b_5 f_1(1) + 2b_7 f_2(1)] y w_y,$$

is even in the variables $x$ and $y$;

$$\overline{R}_3 = -\varepsilon [f_1'(1) + k_1^1 f_1(1)] w_x - \varepsilon^2 2b_5 x y w_y - \varepsilon^2 [b_3 x^2 + b_3 f_1^2(1) + b_4 f_1(1) f_2(1) + b_5 f_2^2(1) + b_5 y^2] w_x,$$

is odd in the variables $x$ and even in the variable $y$;

$$\overline{R}_4 = -\varepsilon [f_2'(1) + k_1^2 f_2(1)] w_y - \varepsilon^2 2b_4 x y w_x - \varepsilon^2 [\frac{1}{2} b_4 x^2 + \frac{1}{2} b_4 f_1^2(1) + 2b_5 f_1(1) f_2(1) + b_7 f_2^2(1) + b_7 y^2] w_y,$$

is odd in the variables $y$ and even in the variable $x$;

$$\underline{R}_1 = -\varepsilon^2 [b_{11} f_1(0) + 2b_{12} f_2(0)](y w_x + x w_y),$$

is odd in the variables $x$ and $y$;

$$\underline{R}_2 = -\varepsilon k_0^1 x w_x - \varepsilon k_0^2 y w_y - \varepsilon^2 [2b_{10} f_1(0) + b_{11} f_2(0)] x w_x - \varepsilon^2 [2b_{12} f_1(0) + 2b_{14} f_2(0)] y w_y,$$

is even in the variables $x$ and $y$;

$$\underline{R}_3 = -\varepsilon [f_1'(0) + k_0^1 f_1(0)] w_x - \varepsilon^2 2b_{12} x y w_y - \varepsilon^2 [b_{10} x^2 + b_{10} f_1^2(0) + b_{11} f_1(0) f_2(0) + b_{12} f_2^2(0) + b_{12} y^2] w_x,$$

is odd in the variables $x$ and even in the variable $y$;

$$\underline{R}_4 = -\varepsilon [f_2'(0) + k_0^2 f_2(0)] w_y - \varepsilon^2 b_{11} x y w_x - \varepsilon^2 [\frac{1}{2} b_{11} x^2 + \frac{1}{2} b_{11} f_1^2(0) + 2b_{12} f_1(0) f_2(0) + b_{14} f_2^2(0) + b_{14} y^2] w_y,$$

is odd in the variables $y$ and even in the variable $x$. The terms $\overline{D}_0(w)$ and

$\underline{D}_0(w)$ are some terms of order $O(\varepsilon^3)$.
To cancel the terms of first order of $\varepsilon$ on the boundary we impose the following restrictions for $f_1$ and $f_2$

$$f_1^{'}(1)+k_1^1 f_1(1)=f_1^{'}(0)+k_0^1 f_1(0)=0,$$
$$f_2^{'}(1)+k_1^2 f_2(1)=f_2^{'}(0)+k_0^2 f_2(0)=0. \tag{2.21}$$

Moreover, we need a boundary layer to cancel other terms of order $O(\varepsilon)$ in the error on the boundary $\partial S$, which will be carried out in next subsection.

**The boundary layer problem**

We will construct an improvement in approximation by first solving the following problems

$$L(\Phi_0)\equiv\Delta\Phi_0-\Phi_0+pw^{p-1}\Phi_0=\rho_0(\varepsilon z)Z \quad \text{in } S,$$
$$\Phi_{0,z}(x,y,1/\varepsilon)=0, \quad \Phi_{0,z}(x,y,0)=k_0^1 xw_x+k_0^2 yw_y. \tag{2.22}$$

$$L(\Phi_1)\equiv\Delta\Phi_1-\Phi_1+pw^{p-1}\Phi_1=\rho_1(\varepsilon z)Z \quad \text{in } S,$$
$$\Phi_{1,z}(x,y,0)=0, \quad \Phi_{1,z}(x,y,1/\varepsilon)=k_1^1 xw_x+k_1^2 yw_y. \tag{2.23}$$

**Lemma 2.1:** There exist two functions $\rho_0(\zeta)$ and $\rho_1(\zeta)$ in $L^2(0,1)$ with the bounds

$$\|\rho_0\|_{L^2(0,1)}\le C\varepsilon^{\frac{1}{2}}, \quad \|\rho_1\|_{L^2(0,1)}\le C\varepsilon^{\frac{1}{2}}, \tag{2.24}$$

such that problem (2.34) and problem (2.35) have unique solutions $\Phi_0\in H^2(S)$ and $\Phi_1\in H^2(S)$, which are even in $x$ and $y$ for each $z$. Besides, there is a constant $C>0$ such that for all small $\varepsilon$,

$$\|\Phi_0\|_{H^2(S)}\le C, \quad \|\Phi_1\|_{H^2(S)}\le C. \tag{2.25}$$

In addition there exist constants $\nu<1/4$, $\mu>0$ and $C_\nu>0$ such that the following estimates hold:

$$|\Phi_0(x,y,z)|+|\nabla\Phi_0(x,z)|+|D^2\Phi_0(x,z)|\le C_\nu\, e^{-[(1-\nu)|(x,y)|+\mu z]},$$
$$|\Phi_1(x,y,z)|+|\nabla\Phi_1(x,z)|+|D^2\Phi_1(x,z)|\le C_\nu\, e^{-[(1-\nu)|(x,y)|+\mu(1/\varepsilon-z)]}. \tag{2.26}$$

**Proof:** We will give the proof of this lemma at the end of Section 4.

Let $\Phi_0$ and $\Phi_1$ be the functions defined by Lemma 2.1 and set

$$\phi_1(x,y,z)=\varepsilon\Phi_0(x,y,z)+\varepsilon\Phi_1(x,y,z). \tag{2.27}$$

The next goal is to show that $\phi_1(x,y,z)$ is the boundary layer that we want in previous section. Define the second approximate solution by $u_2=u_1+\phi_1$.
The new error in the interior of $S$ can be computed as the following

$$E_2\equiv S(u_1+\phi_1)=E_1+L(\phi_1)+N(\phi_1)+B_3(\phi_1), \tag{2.28}$$

where

$$N(\phi_1)=(w+\phi_1)^p-w^p-pw^{p-1}\phi_1, \tag{2.29}$$

$$L(\phi_1)\equiv\Delta\phi_1-\phi_1+pw^{p-1}\phi_1$$
$$=\varepsilon\rho_0(\varepsilon z)Z+\varepsilon\rho_1(\varepsilon z)Z. \tag{2.30}$$

The main error term is

$$B_3(\phi_1) = -\varepsilon^2[2k_0^1(x+f_1)+2f_1^{'}](\Phi_0+\Phi_1)_{zx}$$
$$-\varepsilon[2k_0^2(y+f_2)+2f_2^{'}](\Phi_0+\Phi_1)_{zy}$$

$$-\varepsilon(k_0^1+k_0^2)(\Phi_0+\Phi_1)_z + O(\varepsilon^3). \quad 2.31)$$

On the boundary, the error terms are

$$\overline{E}_{2b} = \overline{E}_{1b} + \varepsilon\Phi_{1,z}(x,y,1/\varepsilon) + \varepsilon\Phi_{0,z}(x,y,1/\varepsilon)$$
$$+\overline{D}_3(\phi_1) + \overline{D}_0(w+\phi_1) - \overline{D}_0(w)$$
$$= O(\varepsilon^2) \quad \text{on } \partial_1 S, \quad (2.32)$$

$$\underline{E}_{2b} = \underline{E}_{1b} + \varepsilon\Phi_{1,z}(x,y,1/\varepsilon) + \varepsilon\Phi_{0,z}(x,y,1/\varepsilon)$$
$$+\underline{D}_3(\phi_1) + \underline{D}_0(w+\phi_1) - \overline{D}_0(w)$$
$$= O(\varepsilon^2) \quad \text{on } \partial_0 S. \quad (2.33)$$

Therefore, the following lemma is readily checked.

**Lemma 2.2:** With the notations of previous section we have

$$E_2 \equiv S(u_2) = E_1 + \varepsilon\rho_0(\varepsilon z)Z + \varepsilon\rho_1(\varepsilon z)Z$$
$$+ N(\phi_1) + O(\varepsilon^2).$$

Moreover,

$$\| E_2 \|_{L^2(S)} \le C\varepsilon^{3/2}. \quad (2.34)$$

In addition there is an extension $E_{2b}$ of terms $\overline{E}_{2b}$ and $\underline{E}_{2b}$ to the whole strip $S$ such that

$$\| E_{2b} \||_{H^1(S)} \le C\varepsilon^{3/2}. \quad (2.35)$$

**Proof:** The remaining terms $B_3(\phi_1)$ and $N(\phi_1)$ are easily seen to be smaller then the ones we have just considered. Estimate (2.34) follows immediately from direct computations. Obviously (2.35) is an easy consequence of the construction.

### An improvement of approximation

To improve the approximation for solution still keeping the term of $\varepsilon^2$, we need to introduce a new parameter $e$, additional to $f_1$ and $f_2$, and define our basic approximate solution to the problem near $\Gamma_\varepsilon$ as

$$u_3(x,y,z) = w(x,y) + \phi_1(x,y,z)$$
$$+\varepsilon e(\varepsilon z)Z(x,y). \quad (2.36)$$

In all what follows, we will assume the validity of the following constraints on the parameters $f_1, f_2$ and $e$ as the following

$$\| f_1 \|_a = \| f_1 \|_{L^\infty(0,1)} + \| f_1^{'} \|_{L^\infty(0,1)}$$
$$+\| f_1^{''} \|_{L^2(0,1)} \le \varepsilon^{\frac{1}{2}}, \quad (2.37)$$

$$\| f_2 \|_a = \| f_2 \|_{L^\infty(0,1)} + \| f_2^{'} \|_{L^\infty(0,1)}$$
$$+\| f_2^{''} \|_{L^2(0,1)} \le \varepsilon^{\frac{1}{2}}, \quad (2.38)$$

$$\| e \|_b = \| e \|_{L^\infty(0,1)} + \varepsilon \| e^{'} \|_{L^2(0,1)}$$
$$+\varepsilon^2 \| e^{''} \|_{L^2(0,1)} \le \varepsilon^{\frac{1}{2}}. \quad (2.39)$$

We also impose the periodic boundary condition on $e$ as

$$e(1) = e(0), \quad e^{'}(1) = e^{'}(0). \quad (2.40)$$

We set up the full problem in the form $S(u_3+\phi)=0$, then it can be expanded in the following way

$$S(u_3+\phi) = S(u_3) + L_1(\phi) + B_3(\phi) + N_1(\phi)$$
$$=0 \quad \text{in } S, \quad (2.41)$$

with boundary condition

$$\phi_z + \overline{D}_3(\phi) + \overline{D}_0(u_3+\phi) = -\overline{E}_{3b} + \overline{D}_0(u_3)$$
$$\equiv g_1 \quad \text{on } \partial_1 S, \quad (2.42)$$

$$\phi_z + \underline{D}_3(\phi) + \underline{D}_0(u_3+\phi) = -\underline{E}_{3b} + \underline{D}_0(u_3) \equiv g_0 \quad \text{on } \partial_0 S, \tag{2.43}$$

where

$$L_1(\phi) = \phi_{xx} + \phi_{yy} + \phi_{zz} - \phi + p u_3^{p-1}\phi,$$
$$N_1(\phi_1) = (u_3+\phi)^p - u_3^p - p u_3^{p-1}\phi. \tag{2.44}$$

$\overline{D}_0(u_3+\phi)$ and $\underline{D}_0(u_3+\phi)$ are of order $O(\varepsilon^3)$. Other boundary error terms are

$$\overline{E}_{3b} - \overline{D}_0(u_3) = \overline{E}_{2b} + \varepsilon^2 e' Z + \overline{D}_3(\varepsilon e Z) - \overline{D}_0(u_2) = O(\varepsilon^2) \quad \text{on } \partial_1 S, \tag{2.45}$$

$$\underline{E}_{3b} - \underline{D}_0(u_3) = \underline{E}_{2b} + \varepsilon^2 e' Z + \underline{D}_3(\varepsilon e Z) - \overline{D}_0(u_2) = O(\varepsilon^2) \quad \text{on } \partial_0 S. \tag{2.46}$$

The error of the approximation is

$$E_3 = S(u_3) = S(w+\phi_1) + \varepsilon(\varepsilon^2 e'' Z + \lambda_0 e Z) + B_3(\varepsilon e Z) + (w+\phi_1+\varepsilon e Z)^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1}\varepsilon e Z + \varepsilon p[(w+\phi_1)^{p-1} - w^{p-1}] e Z, \tag{2.47}$$

where $S(w+\phi_1)$ is defined in (2.28). Moreover, we decompose

$$E_3 = E_{31} + E_{32}, \tag{2.48}$$

with $E_{31} = \varepsilon^3 e'' Z + \varepsilon\lambda_0 e Z$ and $E_{32} = E_3 - E_{31}$.

For further reference, it is useful to estimate the $L^2(S)$ norm of $E_3$. From the uniform bound of $e$ in (2.39), it is trivial that

$$\| E_{31} \|_{L^2(S)} \le C\varepsilon^{\frac{1}{2}}. \tag{2.49}$$

Since $\phi_1$ and $\varepsilon e Z$ are of size $O(\varepsilon)$, all terms in $E_{32}$ carry $\varepsilon^2$ in front. We claim that

$$\| E_{32} \|_{L^2(S)} \le C\varepsilon^{\frac{3}{2}}. \tag{2.50}$$

A rather delicate term in $E_{32}$ is the one carrying $f_1''$ and $f_2''$ since we only assume a uniform bound on $\| f_1'' \|_{L^2(0,1)}$ and $\| f_2'' \|_{L^2(0,1)}$. For example, we have a term $\varepsilon^2 f_1''$ in $S(w)$ which has bound like

$$\| \varepsilon^2 f_1'' \|_{L^2(S)} \le C\varepsilon^2.$$

Other terms can be estimated in the similar way. Moreover, for the Lipshitz dependence of the term of error $E_{32}$ on the parameter $f_1, f_2$ and $e$ for the norm defined in (2.37) - (2.39), we have the validity of the estimate

$$\| E_{32}(f_1,f_2,e) - E_{32}(\tilde{f}_1,\tilde{f}_2,\tilde{e}) \|_{L^2(S)} \le C\varepsilon^{3/2}[\| f_1 - \tilde{f}_1 \|_a + \| f_2 - \tilde{f}_2 \|_a + \| e - \tilde{e} \|_b]. \tag{2.51}$$

## THE GLUING PROCEDURE

In this section, we will use the reduction method in del Pino et al. (2007) to reduce the problem (1.1) to a projected problem.

Let $u_3(Y)$ denote the approximate solution constructed near the curve $\Gamma_\varepsilon$ in the coordinates $Y = (y_1, y_2, y_3)$, which was introduced in (2.3) in $R^3$.

Let $\delta < \delta_0/100$ be a fixed number, where $\delta_0$ is a constant defined in (2.5). We consider a smooth cut-off function $\eta_\delta(t)$ where $t \in R_+$ such that

$$\eta_\delta = 1 \text{ if } t < \delta \text{ and } \eta_\delta = 0 \text{ if } t > 2\delta. \tag{3.1}$$

Denote as well $\eta_\delta^\varepsilon(|(s,t)|) = \eta_\delta(\varepsilon|(s,t)|)$ where $|(s,t)|$ is the normal coordinate to $\Gamma_\varepsilon$. We define our first global approximation to be simply

$$W = \eta_{3\delta}^\varepsilon(|(s,t)|)u_3, \tag{3.2}$$

extended globally as $0$ beyond the $6\delta\varepsilon$-neighborhood of $\Gamma_\varepsilon$.

Denote the term $\bar{S}(u) = \square_Y u - u + u^p$ for $u = W + \hat{\phi}$, now $\hat{\phi}$ globally defined in $\Omega_\varepsilon$.

Then $u$ satisfies (2.4) if and only if

$$\tilde{L}(\hat{\phi}) = -\tilde{E} - N(\hat{\phi}) \quad \text{in } \Omega_\varepsilon, \tag{3.3}$$

with boundary condition

$$\frac{\partial \hat{\phi}}{\partial \nu_\varepsilon} + \frac{\partial W}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.4}$$

Where

$$\tilde{E} = \bar{S}(W), \quad \tilde{L}(\hat{\phi}) = \square_y \hat{\phi} - \hat{\phi} + pW^{p-1}\hat{\phi},$$
$$\tilde{N}(\hat{\phi}) = (W + \hat{\phi})^p - W^p - pW^{p-1}\hat{\phi}.$$

We further separate $\hat{\phi}$ in the following

$$\hat{\phi} = \eta_{3\delta}^\varepsilon \phi + \psi,$$

where, in the coordinates $(x, y, z)$ of the form (2.13), we assume that $\phi$ is defined in the whole strip $S$. Obviously, (3.3) - (3.4) is equivalent to the following problem;

$$\eta_{3\delta}^\varepsilon(\square_Y \phi - \phi + pW^{p-1}\phi) = -\eta_\delta^\varepsilon[\tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) + \tilde{E} - pW^{p-1}\psi], \tag{3.5}$$

$$\square_Y \psi - \psi + (1-\eta_\delta^\varepsilon) pW^{p-1}\psi = -\varepsilon^2(\square_Y \eta_{3\delta}^\varepsilon)\phi - 2\varepsilon(\nabla_Y \eta_{3\delta}^\varepsilon)(\nabla_Y \phi) + (1-\eta_\delta^\varepsilon)\tilde{N}(\eta_\delta^\varepsilon \phi + \psi) + (1-\eta_\delta^\varepsilon)\tilde{E}. \tag{3.6}$$

On the boundary, we get

$$\eta_{3\delta}^\varepsilon \frac{\partial \phi}{\partial \nu_\varepsilon} + \eta_\delta^\varepsilon \frac{\partial W}{\partial \nu_\varepsilon} = 0, \tag{3.7}$$

$$\frac{\partial \psi}{\partial \nu_\varepsilon} + (1-\eta_\delta^\varepsilon)\frac{\partial W}{\partial \nu_\varepsilon} + \varepsilon \frac{\partial \eta_{3\delta}^\varepsilon}{\partial \nu_\varepsilon}\phi = 0. \tag{3.8}$$

The observation is that, after solving (3.6) and (3.8), the problem can be changed to the following nonlinear problem involving the parameter $\psi$

$$\tilde{L}(\phi) = -\eta_\delta^\varepsilon[\tilde{N}(\phi+\psi) + \tilde{E} - pW^{p-1}\psi] \quad \text{in } S,$$
$$\frac{\partial \phi}{\partial \nu_\varepsilon} + \eta_\delta^\varepsilon \frac{\partial W}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial S. \tag{3.9}$$

Notice that the operators $\tilde{L}$ in $\Omega_\varepsilon$ and $\frac{\partial}{\partial \nu_\varepsilon}$ on $\partial\Omega_\varepsilon$ may be taken as any compatible extension outside the $6\delta\varepsilon$-neighborhood of the interface $\Gamma_\varepsilon$ in the strip $S$.

Firstly, we solve, given a small $\phi$, problem (3.6) and (3.8) for $\psi$. Assume now that $\phi$ satisfies the following decay property;

$$|\nabla \phi(y)| + |\phi(y)| \le e^{-\gamma\varepsilon} \text{ if } |s| > \delta\varepsilon, \tag{3.10}$$

for certain constant $\gamma > 0$. The solvability can be done in the following way: let us observe that $W$ is exponentially small for $|s| > \delta\varepsilon$, where $s$ is the normal coordinate to $\Gamma_\varepsilon$, then the problem

$$\Delta\psi - [1-(1-\eta_\delta^\varepsilon)pW^{p-1}]\psi = h \quad \text{in } \Omega_\varepsilon,$$
$$\frac{\partial\psi}{\partial\nu_\varepsilon} = -(1-\eta_\delta^\varepsilon)\frac{\partial W}{\partial\nu_\varepsilon} + \varepsilon\frac{\partial\eta_{3\delta}^\varepsilon}{\partial\nu_\varepsilon}\phi \quad \text{on } \Omega_\varepsilon, \tag{3.11}$$

has a unique bounded solution $\psi$ if $\|h\|_\infty \le +\infty$. Moreover, $\|\psi\|_\infty \le C\|h\|_\infty$.

Since $\tilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (3.6) and (3.8) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_{L^\infty} \le C\varepsilon[\|\phi\|_{L^\infty(|s|>\delta\varepsilon)} + \|\nabla\phi\|_{L^\infty(|s|>\delta\varepsilon)} + e^{-\delta\varepsilon}], \tag{3.12}$$

where $|s| > \delta\varepsilon$ denotes the complement of $\delta\varepsilon$-neighborhood of $\Gamma_\varepsilon$. Moreover, the nonlinear operator $\psi$ satisfies a Lipshitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{L^\infty} \le C\varepsilon[\|\phi_1 - \phi_2\|_{L^\infty(|s|>\delta\varepsilon)} + \|\nabla\phi_1 - \nabla\phi_2\|_{L^\infty(|s|>\delta\varepsilon)}]. \tag{3.13}$$

Therefore, from above discussion, the full problem (3.3)-(3.4) has been reduced to solving the following (nonlocal) problem in the infinite strip $S$

$$L_2(\phi) = -\eta_\delta^\varepsilon[\tilde{N}(\phi+\psi(\phi)) + \tilde{E} - pW^{p-1}\psi(\phi)] \text{ in } S, \tag{3.14}$$

$$B(\phi) + \eta_\delta^\varepsilon\frac{\partial W}{\partial\nu_\varepsilon} = 0 \quad \text{on } \partial S, \tag{3.15}$$

for $\phi \in H^2(S)$ satisfies condition (3.10). Here $L_2$ denotes a linear operator that coincides with $\tilde{L}$ and $B$ denotes the outward normal derivatives of $S$ that coincides with outward normal $\frac{\partial}{\partial\nu_\varepsilon}$ of $\Omega_\varepsilon$ on the region $|(s,t)| < 10\delta\varepsilon$. The definitions of these operators can be showed as the following. The operator $\tilde{L}$ for $|(s,t)| < 10\delta\varepsilon$ is given in coordinates $(x,y,z)$ by formula (2.44). We extend it for functions $\phi$ defined in the strip $S$ in terms of $(x,y,z)$ as follows

$$L_2(\phi) = L_1(\phi) + \chi(\varepsilon|(x,y)|)B_3(\phi) \quad \text{in } S, \tag{3.16}$$

where $\chi(r)$ is a smooth cut-off function which equals 1 for $0 \le r < 10\delta$ and vanished identically for $r > 20\delta$ and $L_1$ is the operator in (2.44). Similarly, the boundary conditions can be written as

$$\phi_z + \chi(\varepsilon|(x,y)|)\overline{D}_3(\phi) + \chi(\varepsilon|(x,y)|)\overline{D}_0(W+\phi) = \chi(\varepsilon|(x,y)|)g_1 \quad \text{on } \partial_1 S,$$
$$\phi_z + \chi(\varepsilon|(x,y)|)\underline{D}_3(\phi) + \chi(\varepsilon|(x,y)|)\underline{D}_0(W+\phi) = \chi(\varepsilon|(x,y)|)g_0 \quad \text{on } \partial_0 S, \tag{3.17}$$

where the operators $\overline{D}_3$ and $\underline{D}_3$ are defined in (2.16)-(2.17) and $\overline{D}_0$, $\underline{D}_0$ are defined in (2.15).

Rather than solving problem (3.14) and (3.15), we deal with the following projected problem: given functions $f_1, f_2$ and $e$ satisfying (3.37)-(3.39), finding functions $\phi \in H^2(S)$ and $c_1, c_2, d \in L^2(0,1)$ such that

$$L_2(\phi) = -\chi E_3 - \chi N_2(\phi) + c_1(\varepsilon z)\chi w_x + c_2(\varepsilon z)\chi w_y + d(\varepsilon z)\chi Z \quad \text{in } S, \tag{3.18}$$

$$\phi_z = \chi g_1 - \chi\overline{D}_3(\phi) - \chi\overline{D}_0(W+\phi) \quad \text{on } \partial_1 S, \tag{3.19}$$

$$\phi_z = \chi g_0 - \chi\underline{D}_3(\phi) - \chi\underline{D}_0(W+\phi) \quad \text{on } \partial_0 S, \tag{3.20}$$

$$\int_{R^2} \phi w_x dxdy = \int_{R^2} \phi w_y dxdy$$
$$= \int_{R^2} \phi Z dxdy = 0, \qquad 0 < z < \frac{1}{\varepsilon}, \tag{3.21}$$

where $N_2(\phi) = \tilde{N}(\phi + \psi(\phi)) - pW^{p-1}\psi(\phi)$.

In Proposition 5.1, we will prove that this problem has a unique solution $\phi$ whose norm is controlled by the $L^2$-norm, not of whole $E_3$, but rather of $E_{32}$ and, moreover and that $\phi$ will satisfies (3.10). After this has been done, our task is to adjust the parameter $f_1, f_2$ and $e$ such that the functions $c_1, c_2$ and $d$ are identically zero. Finally, we need to solve a nonlocal, nonlinear coupled second order system of differential equations for the pair $(f_1, f_2, e)$ with boundary conditions. In Section 6, we will see that this system is solvable in a region where the bounds (2.37)-(2.39) hold.

**The invertibility of $L_2$**

Let $L_2$ be the operator defined in $H^2(S)$ by (3.16). Note that the function $\chi(\varepsilon|(x,y)|)$ in the definition of $L_2$ is an even function in $R^2$. In this section, we study the linear problem, for given functions $h \in L^2(S)$, $g \in H^1(S)$, finding functions $\phi \in H^2(S)$ and $c_1, c_2, d \in L^2(0,1)$ such that

$$L_2(\phi) = h + c_1(\varepsilon z)\chi w_x + c_2(\varepsilon z)\chi w_y$$
$$+ d(\varepsilon z)\chi Z \quad \text{in } S, \qquad \frac{\partial \phi}{\partial \nu} = g \quad \text{on } \partial S, \tag{4.1}$$

$$\int_{R^2} \phi w_x dxdy = \int_{R^2} \phi w_y dxdy$$
$$= \int_{R^2} \phi Z dxdy = 0, \quad 0 < z < 1/\varepsilon. \tag{4.2}$$

**Proposition 4.1:** If $\delta$ in the definition of $L_2$ is chosen small enough and $h \in L^2(S)$ and $g \in H^1(S)$, then there exists a constant $C > 0$, independent of $\varepsilon$, such that for all small $\varepsilon$, the problem has a unique solution $(c_1, c_2, d, \phi) = T_2(h, g)$ which satisfies

$$\|\phi\|_{H^2(S)} \le C[\|h\|_{L^2(S)} + \|g\|_{H^1(S)}].$$

Moreover, if $h$ has a support contained in $|(x,y)| \le 20\delta\varepsilon$, then

$$|\phi(x,z)| + |\nabla\phi(x,z)| \le \|\phi\|_{L^\infty} e^{-2\delta\varepsilon}$$
$$\text{for } |(x,y)| > 40\delta\varepsilon. \tag{4.3}$$

For the proof of Proposition 4.1, we need the validity of a priori estimate and existence result for a simpler problem. Given $\tilde{h} \in L^2(S), \tilde{g} \in H^1(S)$, let us consider the problem

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} + \frac{\partial^2 \tilde{\phi}}{\partial z^2} - \tilde{\phi} + pw^{p-1}\tilde{\phi} = \tilde{h} \quad \text{in } S,$$
$$\frac{\partial \tilde{\phi}}{\partial \nu} = g \quad \text{on } \partial S, \tag{4.4}$$

$$\int_{R^2} \tilde{\phi} w_x dxdy = \tilde{\Lambda}_1(z), \quad \int_{R^2} \tilde{\phi} w_y dxdy = \tilde{\Lambda}_2(z),$$
$$\int_{R^2} \tilde{\phi} Z dxdy = \tilde{\Lambda}_3(z), \quad 0 < z < \frac{1}{\varepsilon}, \tag{4.5}$$

where

$$\|\tilde{\Lambda}_i\|_{H^2(0,1/\varepsilon)} \le C, i = 1, 2, 3. \tag{4.6}$$

**Lemma 4.2:** There exists a constant $C > 0$, independent of $\varepsilon$ such that solutions of (4.4)-(4.5) with $\tilde{\Lambda}_1, \tilde{\Lambda}_2, \tilde{\Lambda}_3$ satisfying (4.6) have the estimate

$$\|\tilde{\phi}\|_{H^2(S)} \le C[\|\tilde{h}\|_{L^2(S)} + \|\tilde{g}\|_{H^1(S)}$$
$$+ \sum_{i=1}^{3} \|\tilde{\Lambda}_i\|_{H^2(0,1/\varepsilon)}].$$

**Proof:** Let $\phi_0$ be the solution of

$$\Delta\phi_0 - \phi_0 = 0 \quad \text{in } S, \quad \frac{\partial\tilde{\phi}}{\partial\nu} = g \quad \text{on } S,$$

and set $\tilde{\phi} = \bar{\phi} - \phi_0$, then $\bar{\phi}$ is a solution to a similar problem, except that it has homogeneous Neumann boundary condition, with all nonhomogeneous terms replaced by $\bar{h}, \bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\Lambda}_3$ with bounds like

$$\| \bar{h} \|_{L^2(S)} \le C[\| \tilde{h} \|_{L^2(S)} + \| \tilde{g} \|_{H^1(S)}],$$

$$\| \bar{\Lambda}_i \|_{H^2(0,1/\varepsilon)} \le C[\| \tilde{\Lambda}_i \|_{H^2(0,1/\varepsilon)} + \| \tilde{g} \|_{H^1(S)}], \quad i = 1, 2, 3.$$

T o prove the general case it suffices to apply the following argument with

$$\phi = \bar{\phi} - \frac{\bar{\Lambda}_1(z)}{\int_{R^2} w_x^2} w_x(x, y) - \frac{\bar{\Lambda}_2(z)}{\int_{R^2} w_y^2} w_y(x, y) - \frac{\bar{\Lambda}_3(z)}{\int_{R^2} Z^2} Z(x, y).$$

Then $\phi$ satisfies a problem of the same form with homogeneous Neumann boundary condition and orthogonality condition replaced by $\Lambda_i = 0, i = 1, 2, 3$ as well as $\bar{h}$ replaced by a function $h$ with $L^2(S)$ norm bounded by

$$\| h \|_{L^2(S)} \le C[\| \tilde{h} \|_{L^2(S)} + \| \tilde{g} \|_{H^1(S)} + \sum_{i=1}^{3} \| \tilde{\Lambda}_i \|_{H^2(0,1/\varepsilon)}].$$

Let us consider Fourier series decompositions for $h$ and $\phi$ of the form

$$\phi(x, y, z) = \sum_{k=0}^{\infty} \phi_k(x, y) \cos(\pi k \varepsilon z),$$

$$h(x, y, z) = \sum_{k=0}^{\infty} h_k(x, y) \cos(\pi k \varepsilon z).$$

Then we have the validity of the equations

$$-k^2\pi^2\varepsilon^2\phi_k + L_0(\phi_k) = h_k \quad \text{in } R^2, \tag{4.7}$$

and conditions

$$\int_{R^2} \phi_k w_x \, dxdy = 0, \int_{R^2} \phi_k w_y \, dxdy = 0, \int_{R^2} \phi_k Z \, dxdy = 0, \tag{4.8}$$

for all $k$. We have denoted here

$$L_0(\cdot) = \frac{\partial^2}{\partial x \partial x} + \frac{\partial^2}{\partial y \partial y} - 1 + pw^{p-1}.$$

Let us consider the bilinear form in $H^1(R)$ associated to the operator $L_0$, namely

$$B(\psi, \psi) = \int_{R^2} [|\psi_x|^2 + |\psi_y|^2 + |\psi|^2 - pw^{p-1}|\psi|^2] \, dxdy.$$

Since (4.8) holds uniformly in $k$ we conclude that

$$C[\| \phi_k \|^2_{L^2(R^2)} + \| \phi_{k,x} \|^2_{L^2(R^2)} + \| \phi_{k,y} \|^2_{L^2(R^2)}] \le B(\phi_k, \phi_k), \tag{4.9}$$

for a constant $C > 0$ independent of $k$. Using this fact and equation (4.7) we find the estimate

$$(1 + \pi^4 k^4 \varepsilon^4) \| \phi_k \|^2_{L^2(R^2)} + \| \phi_{k,x} \|^2_{L^2(R^2)} + \| \phi_{k,y} \|^2_{L^2(R^2)} \le C \| h_k \|^2_{L^2(R^2)}.$$

Moreover, we see from (4.7) that $\phi_k$ satisfies an equation of the form

$$\phi_{k,xx} + \phi_{k,yy} - \phi_k = \tilde{h}_k \quad \text{in } R^2$$

where $\| \tilde{h}_k \|_{L^2(R^2)} \le C \| h_k \|_{L^2(R^2)}$. Hence it follows that additionally we have the estimate

$$\| \phi_{k,xx} \|^2_{L^2(R^2)} + \| \phi_{k,yy} \|^2_{L^2(R^2)} \le C \| h_k \|^2_{L^2(R^2)}. \tag{4.10}$$

Adding up estimates (4.9), (4.10) in $k$ we conclude that

$$\| D^2\phi\|^2_{L^2(S)} + \| D\phi\|^2_{L^2(S)} + \|\phi\|^2_{L^2(S)} \le C\| h\|^2_{L^2(S)} .$$

The final estimate of $\tilde{\phi}$ can be easily derived.

We consider now the following problem: given $h\in L^2(S),\ g\in H^1(S)$ finding functions $\phi\in H^2(S)$, $c_1, c_2, d \in L^2(0,1)$ such that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\phi+pw^{p-1}\phi=h+c_1(\varepsilon z)\chi w_x +c_2(\varepsilon z)\chi w_y+d(\varepsilon z)\chi Z \quad \text{in } S, \tag{4.11}$$

$$\frac{\partial\phi}{\partial\nu}=g \quad \text{on } \partial S, \tag{4.12}$$

$$\int_{R^2}\phi w_x\,dxdy=\Lambda_1(z),\quad \int_{R^2}\phi w_y\,dxdy=\Lambda_2(z),\quad \int_{R^2}\phi Z\,dxdy=\Lambda_3(z),\quad 0<z<\frac{1}{\varepsilon}. \tag{4.13}$$

**Lemma 4.3:** If the functions $h, g,\ \Lambda_1,\Lambda_2,\Lambda_3$ satisfy the conditions in previous lemma, then problem (4.11)-(4.13) possesses a unique solution, denoted by $(c_1,c_2,d,\phi)=T_1(h,g,\Lambda_1,\Lambda_1,\Lambda_3)$.
Moreover,

$$\|\phi\|_{H^2(S)} \le C[\| h\|_{L^2(S)} + \| g\|_{H^1(S)} + \sum_{i=1}^{3}\|\Lambda_i\|_{H^2(0,1/\varepsilon)}].$$

**Proof**. From the argument in Lemma 4.2, it is sufficient to prove this result for the case $\Lambda_1=\Lambda_2=\Lambda_3\equiv 0$ and $g\equiv 0$. For the proof of existence, we write again

$$h(x,y,z)=\sum_{k=0}^{\infty} h_k(x,y)\cos(\pi k\varepsilon z)$$

and consider the problem of finding $\phi_k \in H^1(R^2)$, and constants $c_{1,k}$, $c_{2,k}$, $d_k$ such that

$$-k^2\pi^2\varepsilon^2\phi_k + L_0(\phi_k) = h_k + c_{1,k}w_x + c_{2,k}w_y + d_k Z \quad \text{in } R^2,$$

and

$$\int_{R^2}\phi_k\, w_x\, dxdy=0,\quad \int_{R^2}\phi_k\, w_y\, dxdy=0,\quad \int_{R^2}\phi_k\, Z\, dxdy=0.$$

Fredholm's alternative yields that this problem is solvable with the choices

$$c_{1,k} = -\frac{\int_{R^2} h_k\chi w_x\,dxdy}{\int_{R^2} w_x^2\,dxdy},\quad c_{2,k} = -\frac{\int_{R^2} h_k\chi w_y\,dxdy}{\int_{R^2} w_y^2\,dxdy},\quad d_k = -\frac{\int_{R^2} h_k\chi Z\,dxdy}{\int_{R^2} Z^2\,dxdy}.$$

Observe in particular that

$$\sum_{k=0}^{\infty}| c_{1,k}|^2 \le C\varepsilon\| h\|^2_{L^2(S)},\quad \sum_{k=0}^{\infty}| c_{2,k}|^2 \le C\varepsilon\| h\|^2_{L^2(S)},\quad \sum_{k=0}^{\infty}| d_k|^2 \le C\varepsilon\| h\|^2_{L^2(S)} . \tag{4.14}$$

Finally define

$$\phi(x,y,z)=\sum_{k=0}^{\infty}\phi_k(x,y)\cos(\pi k\varepsilon z),$$

and correspondingly

$$c_1(\zeta)=\sum_{k=0}^{\infty} c_{1,k}\cos(\pi k\zeta),$$

$$c_2(\zeta)=\sum_{k=0}^{\infty} c_{2,k}\cos(\pi k\zeta),$$

$$d(\zeta)=\sum_{k=0}^{\infty} d_k\cos(\pi k\zeta).$$

Estimate (4.14) gives that terms $c_1(\varepsilon z)w_x$, $c_2(\varepsilon z)w_y$ and $d(\varepsilon z)Z$ have their $L^2(S)$ norm controlled by that of $h$. The a priori estimates of the previous lemma tell us that the series for $\phi$ is convergent in $H^2(S)$ and defines a unique solution for the problem with the desired bounds.

**Proof of proposition 4.1:** As the argument in Lemma 4.1, it suffices to consider the case of homogeneous boundary condition, that is, $g=0$. The problem can be written as;

$$\Delta\phi-\phi+pw^{p-1}\phi=-p(W^{p-1}-w^{p-1})\phi-\chi B_3(\phi)+h$$
$$+c_1(\varepsilon z)\chi w_x+c_2(\varepsilon z)\chi w_y+d(\varepsilon z)\chi Z \quad \text{in } S, \qquad (4.15)$$

$$\frac{\partial\phi}{\partial\nu}=0 \quad \text{on } \partial S, \qquad (4.16)$$

$$\int_{R^2}\phi w_x\,dxdy=\int_{R^2}\phi w_y\,dxdy$$
$$=\int_{R^2}\phi Z\,dxdy=0, \quad 0<z<\frac{1}{\varepsilon}. \qquad (4.17)$$

Let

$$\varphi=T_1\left(h-p(W^{p-1}-w^{p-1})\phi-\chi B_3(\phi),0,0,0,0\right), \qquad (4.18)$$

where $T_1$ is the bounded operator defined by Lemma 4.3. The point is that the operator

$$B_4(\phi)=-\chi B_3(\phi)-p(W^{p-1}-w^{p-1})\phi,$$

is small in the sense that

$$\|B_4(\phi)\|_{L^2(S)}\le C\delta\|\phi\|_{H^2(S)}.$$

Hence, the results can be derived by the invertibility conclusion of Lemma 4.3 if we choose $\delta$ sufficiently small. Since $\chi$ is supported on $|(x,y)|<20\delta/\varepsilon$, then $\phi$ satisfies for $|(x,y)|>20\delta/\varepsilon$ a problem of the form

$$\phi_{zz}+\phi_{xx}+\phi_{yy}-(1+o(1))\phi=0,$$
$$|(x,y)|>20\delta/\varepsilon,\ 0<z<\frac{1}{\varepsilon},$$
$$\frac{\partial\phi}{\partial\nu}=0 \quad \text{on } \partial S. \qquad (4.19)$$

Hence, the validity of formula (4.3) can be showed easily.

As a special case of Lemma 4.3, we give a proof of Lemma 2.1.

**Proof of Lemma 2.1:** We only give the proof for the existence of problem (2.35). From the linear theory just developed in Lemma 4.3, the problem

$$\Delta\Phi_1-\Phi_1+pw^{p-1}\Phi_1=c_1(\varepsilon z)w_x$$
$$+c_2(\varepsilon z)w_y+\rho_1(\varepsilon z)Z \quad \text{in } S, \qquad (4.20)$$

$$\Phi_{1,z}(x,y,1/\varepsilon)=-k_1^1 xw_x-k_1^2 yw_y,$$
$$\Phi_{1,z}(x,y,0)=0, \qquad (4.21)$$

$$\int_{R^2}\Phi_1(x,y,z)w_x\,dxdy=\int_{R^2}\Phi_1(x,y,z)w_y\,dxdy$$
$$=\int_{R^2}\Phi_1(x,y,z)Z\,dxdy=0, \qquad (4.22)$$

has a solution $(c_1,c_2,\rho_1,\Phi_1)\in H^2(S)$. Careful checking the proof of Lemma 4.3 will give the bound of $\rho_1$. On the other hand, uniqueness of the problem and evenness of the functions $xw_x(x,y)$ and $yw_y(x,y)$ in the variables $x$ and $y$ imply that $\Phi_1(x,y,z)$ is even in $x$ and $y$ for each $z$ and $c_1(\varepsilon z)$ and $c_2(\varepsilon z)$ are identically zero. Besides, $\|\Phi_1\|_{H^2(S)} \le C\|g\|_{H^1(S)}$ where $g$ is any $H^1$-extension of the boundary condition. Let us take for instance $g(x,z) = e^{-z}[k_1^1 x w_x + k_1^2 y w_y]\eta(2\varepsilon z)$, with a suitable cut-off function $\eta$, in such a way that $\|g\|_{H^1(S)} \le C$ with $C$ independent of $\varepsilon$. Thus we get

$$\|\Phi_1\|_{H^2(S)} \le C, \tag{4.23}$$

as desired. We will establish the decay estimates (2.38). We observe first that since

$$\int_{R^2}\Phi_1(x,y,z)w_x dxdy = \int_{R^2}\Phi_1(x,y,z)w_y dxdy = \int_{R^2}\Phi_1(x,y,z)Zdxdy = 0,$$

hence

$$\int_{R^2}[|\Phi_{1,x}|^2 + |\Phi_{1,y}|^2 + |\Phi_1|^2 - pw^{p-1}|\Phi_1|^2]dxdy \ge \lambda_2\int_{R^2}|\Phi_1(x,y,z)|^2 dxdy, \tag{4.24}$$

where $\lambda_2 > 0$ is the third eigenvalue of the operator

$$L_0(\psi) = -\psi_{xx} - \psi_{yy} + \psi - pw^{p-1}\psi \quad \text{in } R^2.$$

Consider function

$$H(z) = \int_{R^2}|\Phi_1(x,y,z)|^2 dxdy.$$

From (4.24) it follows that $-H_{zz} + \lambda_2 H \le 0$ and from (4.23) we get that $|H_z(0)| \le C$. Clearly we have also $H_z(1/\varepsilon) = 0$ and thus by a comparison argument we get that $|H(z)| \le Ce^{-\mu z}, \quad \mu \le \sqrt{\lambda_2}$.

Using local elliptic estimates we then get $|\Phi_1(x,y,z)e^{\mu z}| \le C \quad \text{in } S$.

From this, passing a suitable barrier we get the estimates in (2.38).

## SOLVING THE NONLINEAR PROJECTED PROBLEM

In this section, we will solve (3.18)-(3.21) in $S$. A first elementary, but crucial observation is the following. The term $E_{31} = \varepsilon^3 e'' Z + \varepsilon\lambda_0 eZ$, in the decomposition of $E_1$, has precisely the form $d(\varepsilon z)Z$ and can be absorbed in that term. Let $g$ be an $H^1(S)$-extension of the boundary terms $\chi g_1$ and $\chi g_0$ defined in (2.42) and (2.43). Let us take for instance

$$g(x,y,z) = e^{z-1/\varepsilon}\chi g_1(x,y)\tilde{\eta}(2\varepsilon(z-1/\varepsilon)) + e^{-z}\chi g_0(x,y)\tilde{\eta}(2\varepsilon z),$$

with a suitable smooth cutoff function $\tilde{\eta}$, in such a way that $g$ is an even function in the variables $x, y$ for each $z$, and satisfies the estimate $\|g\|_{H^1(S)} \le C$, and the boundary constraints $g(x,y,1/\varepsilon) = g_1$, $g(x,y,0) = g_0$, with $C$ independent of $\varepsilon$. Similarly, we make an $H^1(S)$-extension of the nonlinear bounder terms $\chi\overline{D}_3(\phi) - \chi\overline{D}_0(W+\phi)$ and $\chi\underline{D}_3(\phi) - \chi\underline{D}_0(W+\phi)$ and denote it by $G(\phi)$. Then the problem (3.18)-(3.21) is equivalent to the fixed point problem

$$\phi = T_2(-\chi E_{32} - \chi N_2(\phi), g + G(\phi)) \equiv A(\phi). \tag{5.1}$$

where $T_2$ is the bounded operator defined by proposition 4.1.

We collect some useful facts to find the domain of the operator $A$ such that $A$ becomes a contraction mapping. The big difference between terms $E_{31}$ and $E_{32}$ is their sizes. From formulas (2.49) and (2.50), we get

$$\|E_{32}\|_{L^2(S)} \le c_*\varepsilon^{3/2}, \tag{5.2}$$

while $E_{31}$ is only of size $O(\varepsilon^{1/2})$. From proposition 4.1, the operator $T_2$ has a useful property: assume $\hat{h}$ has a support contained in $|(x,y)| \le 20\delta/\varepsilon$, then $\phi = T_2(\hat{h})$ satisfies the estimate

$$|\phi(x,y,z)|+|\nabla\phi(x,y,z)|\le\|\phi\|_{L^\infty}\ e^{-2\delta\varepsilon}$$
$$\text{for } |(x,y)|>40\delta\varepsilon. \tag{5.3}$$

Recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition

$$\|\psi(\phi)\|_{L^\infty}\le C\varepsilon[\||\phi|+|\nabla\phi|\|_{L^\infty(|(x,y)|>20\delta\varepsilon)}+e^{-\delta\varepsilon}], \tag{5.4}$$

and a Lipshitz condition of the form

$$\|\psi(\phi_1)-\psi(\phi_2)\|_{L^\infty}\le C\varepsilon[\||\phi_1-\phi_2| + |\nabla(\phi_1-\phi_2)|\|_{L^\infty(|(x,y)|>20\delta\varepsilon)}]. \tag{5.5}$$

Now, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (3.18)-(3.21). Consider the following closed, bounded subset

$$D=\{\phi\in H^2(S):\|\phi\|_{H^2(Ps)}\le\tau\varepsilon^{3/2}, \tag{5.6}$$

$$\||\phi|+|\nabla\phi|\|_{L^\infty(|(x,y)|>40\delta\varepsilon)}\le\|\phi\|_{H^2(S)}\ e^{-\delta\varepsilon}\}.$$

As the arguments in Wei et al. 2007, we can prove that if the constant $\tau$ is sufficiently large, then the map $A$ defined in (5.1) is a contraction form $D$ into itself. In fact, from the properties of $W$ and $\psi(\phi)$ we obtain

$$\|\chi N_2(\phi)\|_{L^2(S)}\le C(\varepsilon^{3/2}\tau^p+\varepsilon^3\tau^2). \tag{5.7}$$

Using the Lipshitz dependence of $\psi$ on $\phi$, it can be derived

$$\|\chi N_2(\phi_1)-\chi N_2(\phi_2)\|_{L^2(S)} \le C(\varepsilon^{\frac{3}{2}(p-1)}\tau^{p-1}+\varepsilon^{\frac{3}{2}}\tau)\|\phi_1-\phi_2\|_{H^2(S)}. \tag{5.8}$$

Now, we can find the solution of (4.1) in the sequel. Let $\phi\in D$ and $\nu=A(\phi)$, then from (5.2) and (5.7)

$$\|\nu\|_{H^2(S)}\le\|T_2\|[c_*\varepsilon^{3/2}+C\tau^p\varepsilon^{3p/2}+C\tau^2\varepsilon^3].$$

Choosing any number $\tau>C_*\|T_2\|$, we get that for small $\varepsilon$

$$\|\nu\|_{H^2(S)}\le\tau\varepsilon^{3/2}.$$

From (5.3)

$$\||\nu|+|\nabla\nu|\|_{L^\infty(|(x,y)|>40\delta\varepsilon)}\le\|\nu\|_\infty\ e^{\frac{-2\delta}{\varepsilon}} \le\|\nu\|_{H^2(S)}\ e^{\frac{-\delta}{\varepsilon}}.$$

Therefore, $\nu\in D$. $A$ is clearly a contraction thanks to (5.8) and we can conclude that (5.1) has a unique solution in $D$.

The error $E_{32}$ and the operator $T_2$ itself carry the functions $f_1, f_2$ and $e$ as parameters. For future reference, we should consider their Lipshitz dependence on these parameters. (2.51) is just the formula about the Lipshitz dependence of error $E_{32}$ on these two parameters. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show this dependence is indeed Lipschitz with respect to the $H^2$-norm (for all $\varepsilon$). Emphasizing the dependence on $f_1$ and $f_2$ what we find for the linear operator $T_2$ is the Lipschitz dependence

$$\|T_2(f_1,f_2)-T_2(\tilde f_1,\tilde f_2)\| \le C\varepsilon(\|f_1-\tilde f_1\|_a+\|f_2-\tilde f_2\|_a).$$

Moreover, the operator $N_2$ also has Lipshitz dependence on $(f_1,f_2,e)$. It is easily checked that for $\phi\in D$ we have, with obvious notation

$$\|\chi N_{2,(f_1,f_2,e)}(\phi)-\chi N_{2,(\tilde f_1,\tilde f_2,\tilde e)}(\phi)\|_{L^2(S)} \le C\varepsilon^{5/2}[\|f_1-\tilde f_1\|_a+\|f_2-\tilde f_2\|_a+\|e-\tilde e\|_b].$$

Hence, from the fixed point characterization we get that

$$\| \phi(f_1, f_2, e) - \phi(\tilde{f}_1, \tilde{f}_2, \tilde{e}) \|_{H^2(S)} \\ \le C\varepsilon^{3/2} [\| f_1 - \tilde{f}_1 \|_a + \| f_2 - \tilde{f}_2 \|_a + \| e - \tilde{e} \|_b]. \tag{5.9}$$

As a conclusion of this section, we get that

**Proposition 5.1:** There is a number $\tau > 0$ such that for all $\varepsilon$ small enough and all parameters $(f_1, f_2, e)$ satisfying (2.37) - (2.39), problem (3.18) - (3.21) has a unique solution $\phi = \phi(f_1, f_2, e)$ which satisfies

$$\| \phi \|_{H^2(S)} \le \tau\varepsilon^{3/2},$$

$$\| |\phi| + |\nabla \phi| \|_{L^\infty(|(x,y)| > 40\delta/\varepsilon)} \le \| \phi \|_{H^2(S)} \; e^{-\delta/\varepsilon}.$$

Moreover, function $\phi$ depends on Lipshitz continuously on the parameters $f_1, f_2$ and $e$ in the sense of the estimate (5.9).

As we mentioned in Section 3, in the next part of the paper, we will set up equations for the parameters $f_1, f_2$ and $e$ which are equivalent to making the functions $c_1, c_2$ and $d$ in (3.18) - (3.21) are zero. These equations are obtained by simply integrating the equations (only in $x, y$ ) against $w_x, w_y$ and $Z$ respectively. It is therefore of crucial importance to carry out computations of the terms $\int_{R^2} E_3 w_x dxdy$, $\int_{R^2} E_3 w_y dxdy$ and $\int_{R^2} E_3 Z dxdy$ and some other similar terms involving $\phi$.

## ESTIMATES FOR PROJECTIONS

In this section, the main object is to carry out estimates for the terms

$$\int_{R^2} E_3 w_x dxdy, \; \int_{R^2} E_3 w_y dxdy, \; \int_{R^2} E_3 Z dxdy$$

as well as some other similar terms involving $\phi$. For the pair $(f_1, f_2, e)$ satisfying (2.37) - (2.39), denote by $b_{1\varepsilon}$ and $b_{2\varepsilon}$, generic, uniformly bounded continuous functions

$$b_{l\varepsilon} = b_{l\varepsilon}\big(z, f_1(\varepsilon z), f_2(\varepsilon z), e(\varepsilon z), \\ f_1'(\varepsilon z), f_2'(\varepsilon z), \varepsilon e'(\varepsilon z)\big), \quad l = 1, 2, \tag{6.1}$$

where $b_{1\varepsilon}$ is uniformly Lipshitz in its four last arguments.

Firstly, multiplying (2.47) by $w_x$ and integrating over the variables $x, y$, using the decomposition of $E_3$ in (2.48) and the facts that $w$ and $Z$ are even functions in $x, y$, we obtain

$$\int_{R^2} E_3 w_x dxdy \\ = \int_{R^2} S(w+\phi_1) w_x dxdy + \varepsilon \int_{R^2} B_3(eZ) w_x dxdy \\ + \varepsilon p \int_{R^2} [(w+\phi_1)^{p-1} - w^{p-1}] eZ w_x dxdy \\ + \int_{R^2} [(w+\phi_1+\varepsilon eZ)^p - (w+\phi_1)^p \\ - p(w+\phi_1)^{p-1} \varepsilon eZ] w_x dxdy \\ \equiv I_1 + I_2 + I_3 + I_4. \tag{6.2}$$

We calculate these terms as the following. From (2.28), $I_1$ can be rewritten as

$$I_1 = \int_{R^2} S(w+\phi_1) w_x dxdy \\ = \int_{R^2} S(w) w_x dxdy + \int_{R^2} B_3(\phi_1) w_x dxdy \\ + \int_{R^2} [(w+\phi_1)^p - w^p - p w^{p-1} \phi_1] w_x dxdy \\ \equiv I_{11} + I_{12} + I_{13}. \tag{6.3}$$

From formula (2.18), integration by parts and using the symmetric properties of $w$, we get

$$I_{11} = \int_{R^2} S_3 w_x dxdy + \int_{R^2} B_2(w) w_x dxdy \\ = \varepsilon^2 f_1'' \int_{R^2} w_x^2 dxdy + \varepsilon^2 2k_0^1 f_1' \int_{R^2} x w_{xx} w_x dxdy \\ + \varepsilon^2 (k_0^1 + k_0^2) f_1' \int_{R^2} w_x^2 dxdy \\ + \varepsilon^2 2k_0^2 f_1' \int_{R^2} y w_{xy} w_x dxdy + \varepsilon^4 b_{2\varepsilon} f_1'' + \varepsilon^3 b_{1\varepsilon} \\ = \varepsilon^2 \delta f_1'' + \varepsilon^4 b_{2\varepsilon} f_1'' + \varepsilon^3 b_{1\varepsilon}, \tag{6.4}$$

where $\delta_1 = \int_{R^2} w_x^2 dxdy$. From the definitions of $B_3(\phi_1)$ in (2.31), we obtain

$$
\begin{aligned}
I_{12} &= \int_{R^2} B_3(\phi_1) w_x dxdy \\
&= -\varepsilon^2 \int_{R^2} [2k_0^1(x+f_1)+2f_1^{'}](\Phi_0+\Phi_1)_{zx} w_x dxdy \\
&\quad -\varepsilon^2 \int_{R^2} [2k_0^2(y+f_2)+2f_2^{'}](\Phi_0+\Phi_1)_{zy} w_x dxdy \\
&\quad -\varepsilon^2 \int_{R^2} \varepsilon(k_0^1+k_0^2)(\Phi_0+\Phi_1)_z w_x dxdy \\
&\quad +\varepsilon^4 b_{2\varepsilon} f_1^{''} + \varepsilon^3 b_{1\varepsilon} \\
&= \varepsilon^2 \delta_1 \beta_1(z) f_1^{'} + \delta_1 \beta_2(z) f_1 + \varepsilon^4 b_{2\varepsilon} f_1^{''} + \varepsilon^3 b_{1\varepsilon}.
\end{aligned}
$$

where

$$
\beta_1(z) = -\frac{2}{\delta_1} \int_{R^2} (\Phi_0+\Phi_1)_{zx} w_x dxdy,
$$

$$
\beta_2(z) = -\frac{2k_0^1}{\delta_1} \int_{R^2} (\Phi_0+\Phi_1)_{zx} w_x dxdy. \tag{6.5}
$$

The same analysis can be applied to other terms and it can be concluded that

$$
\int_{R^2} E_3 w_x dxdy = \varepsilon^2[\delta_1 f_1^{''} + \delta_1 \beta_1(z) f_1^{'} + \delta_1 \beta_2(z) f_1]
+\varepsilon^3 b_{1\varepsilon}[e+e^{'}+\varepsilon^2 e^{''}] + \varepsilon^3 b_{2\varepsilon}(f_1^{''}+f_2^{''}) + \varepsilon^3 b_{2\varepsilon}. \tag{6.6}
$$

Similarly, we also get the formula

$$
\int_{R^2} E_3 w_y dxdy = \varepsilon^2[\delta_1 f_2^{''} + \delta_1 \beta_3(z) f_2^{'} + \delta_1 \beta_4(z) f_2]
+\varepsilon^3 b_{1\varepsilon}[e+e^{'}+\varepsilon^2 e^{''}] + \varepsilon^3 b_{2\varepsilon}(f_1^{''}+f_2^{''}) + \varepsilon^3 b_{2\varepsilon}. \tag{6.7}
$$

where

$$
\beta_3(z) = -\frac{2}{\delta_1} \int_{R^2} (\Phi_0+\Phi_1)_{zy} w_y dxdy,
$$

$$
\beta_4(z) = -\frac{2k_0^2}{\delta_1} \int_{R^2} (\Phi_0+\Phi_1)_{zy} w_y dxdy. \tag{6.8}
$$

Secondly, multiplying (2.47) by $Z$, integrating over the variables $x$ and $y$, and then using the decomposition of $E_3$ in (2.48), we get

$$
\begin{aligned}
\int_{R^2} E_3 Z dxdy &= \int_{R^2} E_{31} Z dxdy + \int_{R^2} E_{32} Z dxdy \\
&= \varepsilon^3 e^{''} + \varepsilon \lambda_0 e + \int_{R^2} E_{32} Z dxdy,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_{R^2} E_{32} Z dxdy \\
&= \int_{R^2} S(w+\phi_1) Z dxdy + \varepsilon \int_{R^2} B_3(eZ) Z dxdy \\
&\quad + \varepsilon p \int_{R^2} [(w+\phi_1)^{p-1} - w^{p-1}] e Z^2 dxdy \\
&\quad + \int_{R^2} [(w+\phi_1+\varepsilon e Z)^p - (w+\phi_1)^p \\
&\qquad - p(w+\phi_1)^{p-1} \varepsilon e Z] Z dxdy \\
&\equiv J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{6.9}
$$

The computations for these terms are listed in the following. The formula (2.28) gives

$$
\begin{aligned}
J_1 &= \int_{R^2} S(w+\phi_1) Z dxdy \\
&= \int_{R^2} S(w) Z dxdy + \int_{R^2} B_3(\phi_1) Z dxdy \\
&\quad + \int_{R^2} [(w+\phi_1) - w^p - w^{p-1}\phi_1] Z dxdy \\
&\quad + \varepsilon \int_{R^2} (\rho_0(\varepsilon z) + \varepsilon \rho_0(\varepsilon z)) Z^2 dxdy \\
&= J_{11} + J_{12} + J_{13} + \varepsilon(\rho_0(\varepsilon z) + \rho_1(\varepsilon z)).
\end{aligned}
$$

We deal with the components of $J_{13}$ in the sequel. From the formula (2.27)

$$
\begin{aligned}
J_{13} &= \frac{1}{2} p(p-1) \int_{R^2} w^{p-2} \phi_1^2 Z dxdy + \varepsilon^3 b_{1\varepsilon} \\
&= \frac{\varepsilon^2}{2} p(p-1) \int_{R^2} w^{p-2} (\Phi_0+\Phi_1)^2 Z dxdy + \varepsilon^3 b_{1\varepsilon} \\
&= \varepsilon^2 \beta_5(z) + \varepsilon^3 b_{1\varepsilon},
\end{aligned} \tag{6.10}
$$

where

$$
\beta_5(z) = \frac{1}{2} p(p-1) \int_{R^2} w^{p-2} (\Phi_0+\Phi_1)^2 Z dxdy. \tag{6.11}
$$

Since $\phi_1$ is of size $O(\varepsilon)$, then

$$
\begin{aligned}
&J_3 + J_4 \\
&= \varepsilon p(p-1)e\int_{R^2} w^{p-2}\phi_1 Z^2 dxdy \\
&\quad + \frac{p(p-1)}{2}\varepsilon^2 \int_{R^2} (w+\phi_1)^{p-2} e^2 Z^3 dxdy \\
&\quad + \varepsilon^3 b_{1\varepsilon} \\
&= \varepsilon^2 p(p-1)e\int_{R^2} w^{p-2}(\Phi_0+\Phi_1)Z^2 dxdy \\
&\quad + \varepsilon^2 \frac{p(p-1)}{2} e^2 \int_{R^2} w^{p-2} Z^3 dxdy + \varepsilon^3 b_{1\varepsilon} \\
&\equiv \varepsilon^2 \beta_6(z)e + \varepsilon^3 b_{1\varepsilon}.
\end{aligned}
\tag{6.12}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\int_{R^2} E_3 Z dxdy = \varepsilon^3 e'' + \varepsilon\lambda_0 e + \varepsilon^2 \beta_6(z)e \\
&+ \varepsilon(\rho_0(\varepsilon z) + \rho_1(\varepsilon z)) + \varepsilon^2 \beta_5(z) \\
&+ \varepsilon^4 b_{1\varepsilon} e'' + \varepsilon^4 b_{2\varepsilon}(f_1'' + f_2'') + \varepsilon^3 b_{1\varepsilon}.
\end{aligned}
\tag{6.13}
$$

As a final part of this section, we consider the terms that involve $\phi$ in (3.18)-(3.21) integrated against the functions $w_x, w_y$ and $Z$ in $x, y$. For example, concerning $w_x$, we denote by $\Lambda(\phi)$ the sum of these terms with the following estimates
$\| \Lambda(\phi) \|_{L^2(0,1)} \le C\varepsilon^3$.

Moreover, $\Lambda(\phi)$ can be decomposed into components: one defines for fixed $\varepsilon$ a compact operator of the pair $(f_1, f_2, e)$ from $H^2(0,1)$ into $L^2(0,1)$ and the other has Lipschitz dependence on $(f_1, f_2, e)$ of the form

$$
\begin{aligned}
&\| \Lambda(\phi)(f_1, f_2, e) - \Lambda(\phi)(\tilde f_1, \tilde f_2, \tilde e) \|_{L^2(0,1)} \\
&\le C\varepsilon^{3+\frac{1}{2}} [\| f_1 - \tilde f_1 \|_a + \| f_2 - \tilde f_2 \|_a + \| e - \tilde e \|_b ].
\end{aligned}
$$

## THE SYSTEM FOR $(f_1, f_2, e)$ : PROOF OF THE THEOREM

In this section we set up equations relating to $f_1, f_2$ and $e$ such that for the solution $\phi$ of (3.18) - (3.21) predicted by proposition 5.1, one has that the coefficients $c_1(\varepsilon z), c_2(\varepsilon z), d(\varepsilon z)$ are identically zero. To achieve this, we multiply first the equation against $w_x$ and integrate only in $x$ and $y$, then the equation $c_1 = 0$ is equivalent to the relation;

$$
\begin{aligned}
&\int_{R^2} \chi E_3 w_x dxdy + \int_{R^2} [\chi N_2(\phi) + \chi B_3(\phi) \\
&\qquad + p(W^{p-1} - w^{p-1})\phi] w_x dxdy = 0.
\end{aligned}
$$

Similarly, $c_2 = 0$ and $d = 0$ if and only if

$$
\begin{aligned}
&\int_{R^2} \chi E_3 w_y dxdy + \int_{R^2} [\chi N_2(\phi) + \chi B_3(\phi) \\
&\qquad + p(W^{p-1} - w^{p-1})\phi] w_y dxdy = 0, \\
&\int_{R^2} \chi E_3 Z dxdy + \int_{R^2} [\chi N_2(\phi) + \chi B_3(\phi) \\
&\qquad + p(W^{p-1} - w^{p-1})\phi] Z dxdy = 0.
\end{aligned}
$$

Using the estimates in previous sections, we find that the relations above are equivalent to the following nonlinear, nonlocal system of differential equations for $(f_1, f_2, e)$.

$$
\begin{aligned}
&L_1^*(f_1) \equiv f_1''(\theta) + \beta_1(\theta\varepsilon) f_1'(\theta) + \beta_2(\theta\varepsilon) f_1(\theta) \\
&= \varepsilon M_{1\varepsilon}, \quad 0 < \theta < 1,
\end{aligned}
\tag{7.1}
$$

$$
\begin{aligned}
&L_2^*(f_2) \equiv f_2''(\theta) + \beta_3(\theta\varepsilon) f_2'(\theta) + \beta_4(\theta\varepsilon) f_2(\theta) \\
&= \varepsilon M_{2\varepsilon}, \quad 0 < \theta < 1,
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
&L_3^*(e) \equiv \varepsilon^2 e''(\theta) + \varepsilon\beta_6(\theta\varepsilon)e(\theta) + \lambda_0 e(\theta) \\
&= \varepsilon\beta_5(\theta\varepsilon) + \rho_0(\theta) + \rho_1(\theta) + \varepsilon^2 M_{3\varepsilon}, 0 < \theta < 1,
\end{aligned}
\tag{7.3}
$$

with the boundary conditions

$$
f_1'(1) + k_1^1 f_1(1) = 0, \quad f_1'(0) + k_0^1 f_1(0) = 0,
\tag{7.4}
$$

$$
f_2'(1) + k_1^2 f_1(1) = 0, \quad f_2'(0) + k_0^2 f_2(0) = 0,
\tag{7.5}
$$

$$
e'(1) = e'(0), \quad e(1) = e(0),
\tag{7.6}
$$

where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ are smooth functions defined in (6.5), (6.8) and (6.11) and (6.12) respectively.

The functions $\rho_0$ and $\rho_1$ are defined by Lemma 2.1. The operators $M_j^i, j=1,2,3, i=0,1$ are some terms of order $O(\varepsilon^{\frac{1}{2}})$. The operators $M_{1\varepsilon}, M_{2\varepsilon}, M_{3\varepsilon}$ can be decomposed in the following form $M_{l\varepsilon}(f_1,f_2,e)=A_{l\varepsilon}(f_1,f_2,e)+K_{l\varepsilon}(f_1,f_2,e), l=1,2,3,$ where $K_{l\varepsilon}$ is uniformly bounded in $L^2(0,1)$ for $(f_1,f_2,e)$ satisfying (2.37) -(2.39) and is also compact. The operator $A_{l\varepsilon}$ is Lipshhitz in this region,

$$\| A_{l\varepsilon}(f_1,f_2,e)-A_{l\varepsilon}(\tilde{f}_1,\tilde{f}_2,\tilde{e})\|_{L^2(0,1)} \tag{7.7}$$

$$\leq C\varepsilon[\| f_1-\tilde{f}_1\|_a+\| f_2-\tilde{f}_2\|_a+\| e-\tilde{e}\|_b], \quad l=1,2,3.$$

Some basic facts are derived to solve above system. Firstly, we consider the following problems;

$$\begin{aligned} f_1''(\theta)+\beta_1(\theta/\varepsilon)f_1'(\theta)+\beta_2(\theta/\varepsilon)f_1 &= h_1(\theta) \quad \text{in } (0,1),\\ f_1'(1)+k_1^1 f_1(1)=0, \quad f_1'(0)+k_0^1 f_1(0)=0. \end{aligned} \tag{7.8}$$

$$\begin{aligned} f_2''(\theta)+\beta_3(\theta/\varepsilon)f_2'(\theta)+\beta_4(\theta/\varepsilon)f_2 &= h_2(\theta) \quad \text{in } (0,1),\\ f_2'(1)+k_1^2 f_2(1)=0, \quad f_2'(0)+k_0^2 f_2(0)=0. \end{aligned} \tag{7.9}$$

**Lemma 7.1:** If $h_1,h_2\in L^2(0,1)$ then there is a constant $\varepsilon_0$, depending on $\tilde{c}$ in (1.4), for each $0<\varepsilon<\varepsilon_0$, the problem (7.8) and problem (7.9) have unique solutions $f_1,f_2\in H^2(0,1)$ which satisfy

$\| f_1\|_a \leq C\| h_1\|_{L^2(0,1)}, \ \| f_2\|_a\leq C\| h_2\|_{L^2(0,1)}$.

**Proof:** The key point is that we can show a priori estimates for all solutions to problems (7.8) and (7.9) in that the terms $\beta_1(\theta/\varepsilon)$, $\beta_2(\theta/\varepsilon)$, $\beta_3(\theta/\varepsilon)$ and $\beta_4(\theta/\varepsilon)$ are very small in the sense that if we projected them onto the basis spanned by all eigenfunctions of the eigenvalue problems corresponding to (7.8) and (7.9) respectively (Wei et al., 2007).

Secondly, we consider the following problem;

$$\begin{aligned} \varepsilon^2 e''(\theta)+\varepsilon\beta_6(z)e+\lambda_0 e(\theta) &= g(\theta) \quad \text{in } (0,1),\\ e'(1)=e'(0), \quad e(1)=e(0). \end{aligned} \tag{7.10}$$

**Lemma 7.2:** If $g\in L^2(0,1)$ then for $\varepsilon$ satisfying (1.11) there is a unique solution $e\in H^2(0,1)$ to problem (7.10) which satisfies

$\| e\|_b \leq c\,\varepsilon^{-1}\| g\|_{L^2(0,1)}$.

Moreover, if $g\in H^2(0,1)$ then

$\varepsilon^2\| e''\|_{L^2(0,1)}+\| e'\|_{L^2(0,1)}+\| e\|_{L^\infty(0,1)}\leq c\| g\|_{L^2(0,1)}$.

**Proof:** Consider the following Eigen value problem corresponding to problem (7.10)

$$\begin{aligned} e''(\theta)+\zeta e(\theta)=0 \quad \text{in } (0,1),\\ e'(1)=e'(0), \quad e(1)=e(0). \end{aligned} \tag{7.11}$$

It is standard that the eigenvalue problem has an infinite sequence of eigenvalues $\{\zeta_n\}_{n=0}^\infty$ and eigenfunctions $\{y_n\}_{n=0}^\infty$, which forms a complete basis in $L^2$. Moreover, $\zeta_n$ has the asymptotic expression (Levitan et al., 1991)

$$\zeta_n=(2n\pi)^2+O(\frac{1}{n^3}), \tag{7.12}$$

The condition (1.11) shows that $\frac{\lambda_0}{\varepsilon^2}\neq\zeta_n, \forall n\in N$. Hence the proof of a priori estimate follows from the smallness of the term $\beta_6(\theta/\varepsilon)$. The reader can refer to Lemma 8.1 of Wei et al., 2007 for more details.

For completeness of the paper, we now prove Theorem 1.1 in the following.

**Proof Theorem 1.1:** Let $\hat{e}$ solves

$$\begin{aligned} L_2^*(\hat{e})=\varepsilon\beta_5(\theta/\varepsilon)+\rho_1(\theta)+\rho_0(\theta) \quad \text{in } (0,1),\\ \hat{e}'(1)=\hat{e}'(0), \quad \hat{e}(1)=\hat{e}(0). \end{aligned} \tag{7.13}$$

By Lemma 7.2 and Lemma 7.1, we get

$\| \hat{e}\|_b\leq C\varepsilon^{1/2}$.

Setting $e=\hat{e}+\tilde{e}$, the system (7.1) - (7.6) keeps the same form except that the term $\varepsilon\beta_5+\rho_1+\rho_0$ disappear.

By Lemma 7.1 and 7.2, the linear problem $L(f_1, f_2, e) \equiv (L_1^*(f_1), L_2^*(f_2), L_3^*(e)) = (h_1, h_2, g)$ with suitable boundary conditions is invertible and has the following priori estimate

$$\| f_1 \|_a + \| f_2 \|_a + \| e \|_b \le c[\| h_1 \|_{L^2(0,1)} + \| h_2 \|_{L^2(0,1)} + \| g \|_{L^2(0,1)}].$$

As the method in Wei et al., 2007, we can solve (7.1)-(7.5) by the contraction mapping principle and Schauder's fixed point theorem. By Proposition 5.1 and the lines followed, we complete the proof of Theorem 1.1.

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