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N(A)-ternary semigroups

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In this paper, the terms, 'A-potent', 'left A-divisor', 'right A-divisor', 'A-divisor' elements, 'N(A)-ternary semigroup' for an ideal A of a ternary semigroup are introduced. If A is an ideal of a ternary semigroup T then it is proved that (1) $A \subseteq N_2(A) \subseteq N_1(A) \subseteq N_0(A)$ (2) $N_0(A) = A_2$, $N_1(A)$ is a semiprime ideal of T containing A, $N_2(A) = A_4$ are equivalent, where $N_o(A)$ = The set of all A-potent elements in T, $N_1(A)$ = The largest ideal contained in $N_o(A)$, $N_2(A)$ = The union of all A-potent ideals. If A is a semipseudo symmetric ideal of a ternary semigroup then it is proved that $N_0(A) = N_1(A) = N_2(A)$. It is also proved that if A is an ideal of a ternary semigroup such that $N_0(A)$ = A then A is a completely semiprime ideal. Further it is proved that if A is an ideal of ternary semigroup T then R(A), the divisor radical of A, is the union of all A-divisor ideals in T. In a N(A)- ternary semigroup it is proved that R(A) = $N_1(A)$. If A is a semipseudo symmetric ideal of a ternary semigroup T then it is proved that S is an N(A)- ternary semigroup iff R(A) = $N_0(A)$. It is also proved that if M is a maximal ideal of a ternary semigroup T containing a pseudo symmetric ideal A then M contains all A-potent elements in T or T\M is singleton which is A-potent.

Key words: Pseudo symmetric ideal, semipseudo symmetric ideal, prime ideal, semiprime ideal, completely prime ideal, completely semiprime ideal, semisimple element, A-potent element, A-potent Γ-ideal, A-divisor, N(A)- ternary semigroup.

INTRODUCTION

The theory of ternary algebraic systems was introduced by Lehmer (1932), but earlier such structure was studied by Kerner (2000) who gives the idea of *n*-ary algebras. Lehmer (1932) investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semi groups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to Banach who is credited with example of a ternary semigroup which cannot be reduce to a semigroup. Los (1955) studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Sioson (1965) introduced the ideal theory in ternary semigroup. He also introduced the notion of regular ternary semigroup and characterized them by using the properties of quasi-ideals. Santiago (1990) developed the theory of ternary semigroup and semi heaps. He studied regular and completely regular ternary semigroup. Dixit and Dewan (1995) studied quasi-ideals and bi-ideals in ternary semigroup. Shabir and Khan (2008) studied prime ideals and prime one sided ideals in semigroup. Shabir and Bashir (2009) launched prime ideals in ternary semigroup. In this paper we introduce the notion of N(A)-ternary semigroup and characterize N(A)-ternary semigroup.

PRELIMINARIES

Definition 1

Let T be a non-empty set. Then T is said to be a *ternary*

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semigroup if there exist a mapping from T×T×T to T which maps $(x_{1,}x_{2,}x_3) \to [x_1x_2x_3]$ satisfying the condition:

$$[(x_1x_2x_3)x_4x_5] = [x_1(x_2x_3x_4)x_5] = [x_1x_2(x_3x_4x_5)]$$
$\forall\ x_i \in$ T, $1 \le i \le 5$.

Definition 2

A ternary semigroup T is said to be *commutative* provided for all *a,b,c* $\in$ T, we have *abc* = *bca* = *cab* = *bac* = *cba* = *acb*.

Definition 3

Let T be ternary semigroup. A non empty subset S of T is said to be a *ternary sub-semigroup* of T if *abc* $\in$ S for all *a,b,c* $\in$ S.

Notation 1

A non empty subset S of a ternary semigroup T is a ternary sub-semigroup if and only if SSS $\subseteq$ S.

Definition 4

A nonempty subset A of a ternary semigroup T is said to be *left ternary ideal* or *left ideal* of T if *b, c* $\in$ T, *a* $\in$ A implies *bca* $\in$ A.

Definition 5

A nonempty subset of a ternary semigroup T is said to be a *lateral ternary ideal* or simply *lateral ideal* of T if *b, c* $\in$ T, *a* $\in$ A implies *bac* $\in$ A (Iampan, 2007).

Definition 6

A nonempty subset A of a ternary semigroup T is a *right ternary ideal* or simply *right ideal* of T if *b, c* $\in$ T , *a* $\in$ A implies *abc* $\in$ A

Definition 7

An ideal A of a ternary semigroup T is said to be a *completely prime ideal* of T provided *x, y, z* $\in$ T and *xyz* $\in$ A implies either *x* $\in$ A or *y* $\in$ A or *z* $\in$ A.

Definition 8

An ideal A of a ternary semigroup T is said to be a *prime ideal* of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A or Z $\subseteq$ A.

Definition 9

An ideal A of a ternary semigroup T is said to be a *completely semiprime ideal* provided *x* $\in$ T, $x^n \in$ A for some odd natural number *n* >1 implies *x* $\in$ A.

Definition 10

An ideal A of a ternary semigroup T is said to be *semiprime ideal* provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number *n* implies X ⊆ A.

Definition 11

An ideal A of a ternary semigroup T is said to be *pseudo symmetric* provided *x, y*, *z* $\in$ T, *xyz* $\in$ A implies *xsytz* $\in$ A for all *s*, *t* $\in$ T.

Theorem 1

Every completely semiprime ideal A in a ternary semigroup T is a pseudo symmetric ideal.

Definition 12

An ideal A in a ternary semigroup T is said to be *semipseudo symmetric* provided for any odd natural number *n*, *x* ∈ T, x^n ∈ A ⇒ $< x >^n$ ⊆ A.

Notation 2

If A is an ideal of a ternary semigroup T, then we associate the following four types of sets.

A_1 = The intersection of all completely prime ideals of T containing A.

A_2 = {$x \in T$: $x^n \in A$ for some odd natural numbers *n*}

A_3 = The intersection of all prime ideals of T containing A.

A_4 = {$x \in T$: $< x >^n \subseteq$ A for some odd natural number *n*}

Theorem 2

If A is an ideal of a ternary semigroup T, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Theorem 3

Let A be a semipseudo symmetric ideal of a ternary semigroup T. Then the following are equivalent.

(1) A_1=The intersection of all completely prime ideals of T containing A.

(2) A_1^1= The intersection of all minimal completely prime ideals of T containing A.

(3) A_1^{11}= The minimal completely semiprime ideal of T relative to containing A.

(4) A_2= {$x \in$ T : $x^n \in$A for some odd natural number n}

(5) A_3= The intersection of all prime ideals of T containing A.

(6) A_3^1= The intersection of all minimal prime ideals of T containing A.

(7) A_3^{11}= The minimal semiprime ideal of T relative to containing A.

(8) A_4 = {$x \in$ T : $\langle x\rangle^n \subseteq$A for some odd natural number n}

N(A)-TERNARY SEMIGROUPS

Definition 13

Let A be an ideal in a ternary semigroup T. An element $x \in$T is said to be *A-potent* there exists a odd natural number n such that $x^n \in$A.

Definition 14

Let A be an ideal in a ternary semigroup T. An ideal B of T is said to be *A-potent ideal* provided there exists a odd natural number n such that $B^n \subseteq$A.

Notation 3

If A is an ideal of a ternary semigroup T, then every element of A is a A-potent element of T and A itself an A-potent ideal of T.

Definition 15

Let A be an ideal of a ternary semigroup T. An A-potent element x is said to be a *nontrivial A-potent* element of T if $x \notin$ A.

Notation 4

N_o(A) = The set of all A-potent elements in T, N_1(A) = The largest ideal contained in N_o(A), N_2(A) = The union of all A-potent ideals.

Theorem 4

If A is an ideal of a ternary semigroup T, then $A \subseteq N_2(A) \subseteq N_1(A) \subseteq N_0(A)$.

Proof

Since A is itself an A-potent ideal, and N_2(A) is the union of all A-potent ideals. Therefore A $\subseteq$ N_2(A). Let $x \in$ N_2(A) $\Rightarrow x$ belongs to at least one A-potent ideal $\Rightarrow x$ is an A-potent element. Hence $x \in N_0$(A). Therefore N_2(A) $\subseteq$ N_0(A).

Clearly N_2(A) is an ideal of T. Since N_1(A) is the largest ideal contained in N_o(A), we have $N_2(A) \subseteq N_1(A) \subseteq N_0(A)$. Hence $A \subseteq N_2(A) \subseteq N_1(A) \subseteq N_0(A)$.

Theorem 5

If A is an ideal in a ternary semigroup T, then the following are true.

(1) N_0(A) = A_2.

(2) N_1(A) is a semiprime ideal of S containing A.

(3) N_2(A) = A_4.

Proof

(1) N_o(A) = The set of all A-potent elements= { $x \in$S: $x^n \in$A for some natural number n} = A_2.

(2) Suppose that $\langle x\rangle^2 \subseteq$ N_1(A). Suppose if possible $x \notin N_1$(A). N_1(A), $\langle x\rangle$ are the ideals implies N_1(A) $\cup \langle x\rangle$ is an ideal. Since N_1(A) is the largest ideal in N_0(A), We have N_1(A) $\cup \langle x\rangle \not\subset N_0$(A) $\Rightarrow \langle x\rangle \not\subset N_0$(A). Hence there exists an element y such that $y \in \langle x\rangle \backslash N_0$(A). Now $y^3 \in \langle x\rangle^3 \subseteq N_1$(A) $\subseteq N_0$(A) $\Rightarrow y^3 \in N_0$(A) $\Rightarrow (y^3)^n \in$A for some odd natural number $n \Rightarrow y^{3n} \in$A$\Rightarrow y \in N_0$(A). It is a contradicition. Therefore $x \in N_1$(A). Hence N_1(A) is a semiprime ideal of S containing A.

(3) Let $x \in N_2$(A). Then there exists an A-potent ideal B such that $x \in$B. B is A-potent ideal implies there exists an odd natural number $n \in$ N such that $B^n \subseteq$ A$\Rightarrow \langle x\rangle^n \subseteq B^n \subseteq$ A for some $n \in$ N$\Rightarrow x \in A_4$. Therefore N_2(A) $\subseteq A_4$. Let $x \in A_4 \Rightarrow \langle x\rangle^n \subseteq$ A for some $n \in$ N. So $\langle x\rangle$ is an A-potent ideal in S and hence $\langle x\rangle \subseteq N_2$(A) $\Rightarrow x \in N_2$(A). Therefore $A_4 \subseteq N_2$(A). Hence N_2(A) = A_4.

Notation 5

It is natural to ask whether $N_1(A) = A_3$. This is not true.

Example 1

In the free semigroup S over the alphabet *a, b, c*. For the ideal $A = Sa^3S$, $N_0(A) = \{a\} \cup S^1a^3S^1$ and $N_1(A) = \{a^3\} \cup Sa^3S = S^1a^3S^1$. But Sa^3S is a prime ideal, let A,B,C are three ideals of S such that $ABC \subseteq Sa^3S$, implies all words containing $a^3 \in A$ or all words containing $a^3 \in B$ or all words containing $a^3 \in C \Rightarrow A \subseteq Sa^3S$ or $B \subseteq Sa^3S$ or $C \subseteq Sa^3S$. Therefore Sa^3S is a prime ideal. We have $A_3 = Sa^3S$, so $N_1(A) \neq A_3$. Therefore we can remark that the inclusions in $A_3 \subseteq N_1(A) \subseteq N_0(A) = A_2$ may be proper in an arbitrary ternary semigroup.

Theorem 6

If A is a semipseudo symmetric ideal in a ternary semigroup T, then $N_0(A) = N_1(A) = N_2(A)$.

Proof

Suppose A is a semi pseudo symmetric ideal in a ternary semigroup T. By theorem 3.7, $N_0(A) = A_2$ and $N_2(A) = A_4$. Also by theorem 3, we have $A_2 = A_4$. Hence $N_0(A) = N_2(A)$. By the theorem 4, $A \subseteq N_2(A) \subseteq N_1(A) \subseteq N_0(A)$. We have $N_2(A) \subseteq N_1(A)$. Now let $x \in N_1(A) \Rightarrow x \in N_0(A) \Rightarrow x \in N_2(A)$. Therefore $N_1(A) \subseteq N_2(A)$. Hence $N_1(A) = N_2(A)$. Therefore $N_0(A) = N_1(A) = N_2(A)$.

Theorem 7

For any semipseudo symmetric ideal A in a ternary semigroup T, a nontrivial A-potent element $x(x \notin A)$ cannot be semi simple.

Proof

Since x is a nontrivial A-potent element, there exists a odd natural number n such that $x^n \in A$. Since A is semipseudo symmetric ideal, we have $\langle x\rangle^n \subseteq A$. If x is semisimple, then $\langle x\rangle = \langle x\rangle^3$ and hence $\langle x\rangle = \langle x\rangle^n \subseteq A$, this is a contradiction. Thus x is not semisimple.

Theorem 8

If A is an ideal in a ternary semigroup T, such that $N_0(A) = A$, then A is a completely semiprime ideal and A is a pseudo symmetric ideal.

Proof

Let $x \in T$ and $x^3 \in A$. Since $N_0(A) = A$, $x^3 \in N_0(A)$. Thus there exists a odd natural number n such that $(x^3)^n \in A \Rightarrow x^{3n} \in A \Rightarrow x \in N_0(A) = A$. Therefore A is a completely semiprime ideal. By theorem 1, A is pseudo symmetric ideal. Hence A is completely semiprime and pseudo symmetric ideal.

Theorem 9

If A is a semipseudo symmetric ideal of a ternary semisimple semigroup the $A = N_0(A)$.

Proof

Clearly $A \subseteq N_0(A)$. Let $a \in N_0(A)$. If $a \notin A$ then a is a nontrivial A-potent element. By theorem 7, a cannot be semisimple. It is a contradiction. Therefore $a \in A$ and hence $N_0(A) \subseteq A$. Thus $N_0(A) = A$.

Definition 16

Let A be an ideal in a ternary semigroup T. An element $x \in T$ is said to be a *left A-divisor* provided there exists an element $y,z \in T \backslash A$ such that $xyz \in A$.

Definition 17

Let A be an ideal in a ternary semigroup T. An element $x \in T$ is said to be a *right A-divisor* provided there exists an element $y,z \in T \backslash A$ such that $yzx \in A$.

Definition 18

Let A be an ideal in a ternary semigroup T. An element $x \in T$ is said to be an ***A-divisor*** if x is both a left A-divisor and a right A-divisor element.

Definition 19

Let A be an ideal in a ternary semigroup T. An ideal B in

T is said to be a ***left A-divisor ideal*** provided every element of B is a left A-divisor element.

Definition 20

Let A be an ideal in a ternary semigroup T. An ideal B in T is said to be a *right A-divisor ideal* provided every element of B is a right A-divisor element.

Definition 21

Let A be an ideal in a ternary semigroup T. An ideal B in T is said to be an *A-divisor ideal* provided if it is both a left A-divisor ideal and a right A-divisor ideal of a semigroup T.

Notation 6

$R_l(A)$ = The union of all left A-divisor ideals in T.
$R_r(A)$ = The union of all right A-divisor ideals in T.
$R(A) = R_l(A) \cap R_r(A)$. We call R(A), the divisor radical of T.

Theorem 10

If A is any ideal of a ternary semigroup T, then $N_1(A) \subseteq R(A)$.

Proof

Let $x \in N_1(A)$. Since $N_1(A) \subseteq N_0(A)$, we have $x \in N_0(A) \Rightarrow x^n \in A$ for some odd natural number *n*. Let *n* be the least odd natural number such that $x^n \in A$. If $n = 1$ then $x \in A$ and hence $x \in R(A)$.

If $n > 1$, then $x^n = x^{n-2}.x \in A$, where $x^{n21} \in T\backslash A$. Hence *x* is an A-divisor element.

Thus $x \in R(A)$. Therefore $N_1(A) \subseteq R(A)$.

Theorem 11

If A is an ideal in a ternary semigroup T, then R(A) is the union of all A-divisor ideals in T (Anjaneyulu, 1980).

Proof

Suppose A is an ideal in a ternary semigroup T. Let B be an A-divisor ideal in T. Then B is both a left A-divisor and a right A-divisor ideal in T. Thus $B \subseteq R_l(A)$ and $B \subseteq R_r(A) \Rightarrow B \subseteq R_l(A) \cap R_r(A) = R(A) \Rightarrow B \subseteq R(A)$. Therefore R(A) contains the union of all A-divisor ideals in T. Let $x \in R(A)$. Then $x \in R_l(A) \cap R_r(A)$. So $\langle x \rangle \subseteq R_l(A) \cap R_r(A)$. Hence $\langle x \rangle$ is an A-divisor ideal. So R(A) is contained in the union of all divisor ideals in T. Thus R(A) is the union of all divisor ideals of T.

Corollary 1

If A is a pseudo symmetric ideal in a semigroup T, then R(A) is the set of all A-divisor elements in T.

Proof

Suppose A is a pseudo symmetric ideal in T. Let *x* be an A-divisor element in T. Then $xyz \in A$, where $y, z \in S\backslash A$. $xyz \in A$, A is pseudo symmetric $\Rightarrow \langle x \rangle \langle y \rangle \langle z \rangle \subseteq A \Rightarrow \langle x \rangle$ is an A-divisor ideal $\Rightarrow \langle x \rangle \subseteq R(A) \Rightarrow x \in R(A)$. Hence R(A) is the set of all A-divisor elements in T.

Definition 22

Let A be an ideal in a ternary semigroup T. T is said to be a *N(A)-ternary semigroup* provided every A-divisor is A-potent.

Notation 7

Let T be a semigroup with zero. If A={*0*}, then we write R for R(A) and N for $N_0(A)$ and N- ternary semigroup for N(A)-ternary semigroup.

Theorem 12

If T is an N(A)-ternary semigroup, then $R(A) = N_1(A)$.

Proof

Suppose S is an N(A)-ternary semigroup. By theorem 3.21, $N_1(A) \subseteq R(A)$. Let $x \in R(A) \Rightarrow x$ is an A-divisor $\Rightarrow x$ is an A-potent $\Rightarrow x \in N_1(A)$. $\therefore R(A) \subseteq N_1(A)$.
Hence $N_1(A) = R(A)$.

Theorem 13

Let A be a semipseudo symmetric ideal in a ternary semigroup T. Then T is an N(A)-ternary semigroup iff $R(A) = N_0(A)$.

Proof

Since A is a semipseudo symmetric ideal, by theorem 3.10, $N_0(A) = N_1(A) = N_2(A)$. If T an N(A)-ternary semigroup, then by theorem 3.26, $R(A) = N_1(A)$. Hence $R(A) = N_0(A)$. Conversely suppose that $R(A) = N_0(A)$. Then clearly every A-divisor element is an A-potent element. Hence T is an N(A)-ternary semigroup.

Corollary 2

Let A be a pseudo symmetric ideal in a ternary semigroup T. Then T is an N(A)-ternary semigroup if and only if $R(A) = N_0(A)$.

Proof

Since every pseudo symmetric ideal is a semipseudo symmetric ideal, by theorem 3.27, $R(A) = N_0(A)$.

Corollary 3

Let T be a semigroup with 0 and < 0 > is a pseudo symmetric ideal. Then R = N iff T is an N-ternary semigroup.

Proof

The proof follows from the theorem 12.

Theorem 14

If M is a maximal ideal in a ternary semigroup T containing a pseudo symmetric ideal A, then M contains all A-potent elements in T or T\M is singleton which is A-potent (Anjaneyulu, 1981).

Proof

Suppose M does not contain all A-potent elements. Let $a \in T\backslash M$ be any A-potent element and b be any element in T\M. Since M is a maximal ideal, $M \cup \langle a \rangle = M \cup \langle b \rangle \Rightarrow \langle a \rangle = \langle b \rangle$. Since $b \notin M$, we have $b \in \langle a \rangle$. Let n be the least positive odd integer such that $a^n \in A$. Since A is a pseudo symmetric ideal then A is a semipseudo symmetric ideal and hence $\langle a \rangle^n \subseteq A$. Therefore $b^n \in A$ and hence b is an A-potent element. Thus every element in T\M is A-potent. Similarly we can show that if m is the least positive odd integer such that $b^m \in A$, then $a^m \in A$. Therefore there exists an odd natural number p such that $a^p \in A$ and $a^{P-2} \notin A$ for all $a \in T\backslash M$. Let $a, b, c \in T\backslash M$. Since M is maximal ideal, we have $\langle a \rangle = \langle b \rangle = \langle c \rangle$.

So $b,c \in \langle a \rangle \Rightarrow b = sat$, $c = uav$. So $a \in \langle b \rangle$ and hence $a = sbt$ for some $s, t \in T^1$.

Now since A is a pseudo symmetric ideal, we have $(abc)^{p-2} = (abc)^{p-3}abc = (abc)^{p-3}a(sat)(uav) \in A \Rightarrow abc \in M$. If $b \neq a$ then $s, t \in T$. If $s, t \in M$ then $sat \in M \Rightarrow b \in M$. This is not true. In both the cases we have a contradiction. Hence $a = b$. Similarly we show that $c = a$.

Corollary 4

If M is a nontrivial maximal ideal in a ternary semigroup T containing *a* pseudo symmetric ideal A. Then $N_0(A) \subseteq M$.

Proof

Suppose in $N_0(A) \not\subseteq M$. Then by above theorem 3.30, M is trivial ideal. It is a contradiction. Therefore $N_0(A) \subseteq M$.

Corollary 5

If M is a maximal ideal in a semisimple ternary semigroup T containing *a* semipseudo symmetric ideal A. Then $N_0(A) \subseteq M$.

Proof

By theorem 9, A is pseudo symmetric ideal. If $a \in T\backslash M$ is A-potent, then a cannot be semisimple. It is a contradiction. Therefore $N_0(A) \subseteq M$.

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