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Full Length Research Paper

Estimates for contraction oscillatory integral operators on L^p – spaces.

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In this study, we shall prove that the oscillatory integral operator can be reduced to the study of another contraction oscillatory integral operator, with a fixed point which leads to a sharp estimate.

Key words: Oscillatory integrals operator, bilinear forms, Fourier transform, non singular matrix, intertwining operators.

INTRODUCTION

We give a solution to a problem about the L^2 boundedness of certain oscillatory integral operators, Ricci and Stein (1987), Stein and Weiss (1971). Which was proposed by Phong and Stein (1987),

The operators we study here are of the form

$$(Tf)(x) = \int_{R^n} e^{i(Bx,y)} K\,(x-y)f(y)dy \tag{1}$$

where (Bx, y) is a real bilinear form, and rank(B) = k, K is a function which is smooth away from the origin, homogeneous of degree – (n – k). For operators,

$$f \to p.v. \int_{R^n} e^{i(Bx,y)} K(x-\ y)f(y)dy, \tag{2}$$

where K is C^∞ away from the origin, coincides with a homogeneous function of degree - μ for large $|x|$, with a homogeneous function of degree – n for small $|x|$, and satisfies the cancellation condition.

$$\int_{|x|=\varepsilon} K(x)d\sigma(x) = 0 \tag{3}$$

for ε small, Phong and Stein showed that if $\mu > n$ - *rank*(B), then these operators are bounded on $L^2(R^n)$. Clearly, the kernel functions are homogeneous of critical degree that is, $u = n$ - *rank*(B). In fact, when *rank*(B) = 0$(\mu = n)$, these operators are simply the classical singular integral operators, by the theorem of Calderon and Zygmund, they are bounded on L^2 if and only if

$$\int_{|x|=1} K(x)d\sigma(x) = 0. \tag{4}$$

In the other extreme case, one has *rank*(B) = n, namely the bilinear form has full rank, and K is homogeneous of degree 0. We notice that as a special case, when K is a constant, the operator becomes the Fourier Theorem, which is known to be bounded on L^2. Furthermore, Phong and Stein showed that, for general function K which is homogeneous of degree 0, the operator is still bounded on L^2. Determine the L^2 boundedness of those operators, when 0 < *rank*(B) < n. Their solution is as follows.

Theorem 1

(i) If Range(B) = Range(B^t), where B^t is the transpose of B, and Range$(B)^\perp$ denotes the orthogonal complement of Range(B), then the necessary and sufficient condition

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for T to be bounded on $L^2(R^n)$ is that $K|_{Range(B)^{\perp}}$, as a function on the space $Range(B)^{\perp}$, has vanishing mean value on the unit sphere.

(ii) If $Range(B) \neq Range(B^t)$, the, for all K which is smooth away from the origin and homogeneous of degree – (n – k), T is a bounded operator on $L^2(R^n)$. The fundamental difference can be best seen in the following two operators on R^2.

$$T_1: T_1 f(x_1,x_2) = \int_{R_2} e^{ix_1y_1} \frac{1}{|x-y|} f(y_1,y_2) dy_1\, dy_2,$$

$$T_2: T_2 f(x_1,x_2) = \int_{R_2} e^{ix_1y_1} \frac{1}{|x-y|} f(y_1,y_2) dy_1\, dy_2.$$

Now, to explain the condition given in Theorem 1 that $K|_{Range(B)^{\perp}}$, has vanishing mean value on the unit sphere, we let $E = Range(B)^{\perp}$, and L be a linear transformation which maps R^{n-k} to E, the $(n - k)$ dimensional subspace of R^n, that is, $L(R^{n-k}) = E$. The definition for $K|_E$ to have vanishing mean value is that K satisfies the following

$$\int_{x''\in R^{n-k}, |x''|=1} K(Lx'')\, d\sigma_{n-k}(x'') = 0. \tag{5}$$

Since K is homogeneous of degree – (n – k), it can be checked that this definition does not depend on the choice of L. When the matrix associated with the bilinear form B is of the form

$$\begin{pmatrix} A_1 & B_1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

where A_1 is a nonsingular $k \times k$ real matrix, B_1 is $k \times (n - k)$ real matrix, we can see that condition (5) is exactly the following

$$\int_{x''\in R^{n-k}, |x''|=1} K(0, x'')\, d\sigma(x'') = 0. \tag{7}$$

The following simple lemma says that any matrix B which is of rank k can be normalized as in Equation (6).

Lemma 1

Suppose that B is a $n \times n$ real matrix, rank(B) = k. Then there exists a $n \times n$ nonsingular matrix P such that;

$$P^tBP = \begin{pmatrix} A_1 & B_1 \\ 0 & 0 \end{pmatrix}$$

where A_1 is a $k \times k$ matrix,

$|A_1| \neq 0, B_1$ is a $k \times (n-k)$ matrix.

Now Yibiao (1991) prove Theorem 1 in the model case. In fact, let

$$(T_1 f)(x) = \int_{R^n} e^{iy^tP^tBPx} K_1(x-y) f(y)\, dy$$

where $K_1(x) = K(Px)$, P is the matrix in Lemma 1. Then we get $Tf(Px) = T_1(f_1)(x)$ and $f_1(x) = f(Px)$. The L^2 boundedness of T is therefore equivalent to that of T_1. But now the matrix in the bilinear form in T_1 is of the form

$$\begin{pmatrix} A_1 & B_1 \\ 0 & 0 \end{pmatrix}$$

The condition $Range(B) = Range(B^t)$ ($Range(B) \neq Range(B^t)$) is now simply $B_1 = 0$ ($B_1 \neq 0$). Now we will assume in the following that all the bilinear forms are as in (6).

Theorem 2

Let x = (x', x") denote a point in R^n, where

$x' \in R^k, x'' \in R^{n-k}$,

$$Tf(x', x'') = \int_{R^n} e^{i(Bx,y)} K(x-y) f(y)\, dy$$

$$= \int_{R^n} e^{i\left(y'^t A_1 x' + y'^t B_1 x''\right)} K(x'-y', x''-y'') f(y', y'')\, dy'\, dy'' \tag{8}$$

Where A_1 is a $k \times k$ nonsingular matrix and B_1 is an arbitrary real $k \times (n-k)$ matrix and K satisfies (7); then the operator T is bounded on $L^2(R^n)$ to itself.

Lemma 2

Suppose that K is given as above. Let

$$K_1(x) = K_1(x', x'') = \int_{R^{n-k}} e^{i\xi''x''} K(x', \xi'')\, d\xi'' \tag{9}$$

That is, K_1 is the Fourier Transform of K in the x" variables. Then for any

$\alpha, \alpha = (\alpha_1 \dots, \alpha_k), \alpha_j$ integer, and $\alpha_j \geq 0, |\alpha| = \alpha_1 + \cdots + \alpha_k$, we have

$$\left|\frac{\partial^\alpha}{(\partial x')^\alpha} K_1(x', x'')\right| \leq C_\alpha |x'|^{-|\alpha|} \tag{10}$$

where C_α is a constant independent of x".

Proof

(i) For $\alpha, |\alpha| > 0$, we have

$$\left|\frac{\partial^\alpha}{(\partial x')^\alpha} K_1(x', x'')\right| = \left|\int_{R^{n-k}} e^{i\xi'' x''} \frac{\partial^\alpha}{(\partial x')^\alpha} K(x', \xi'') d\xi''\right|$$

$$\leq C_\alpha \int_{R^{n-k}} \frac{d\xi''}{(|x'|^2 + |\xi''|^2)^{(|\alpha|+n-k)/2}} \leq C_\alpha |x'|^{-|\alpha|}$$

(ii) *For* α = 0, we need to prove that

$$\left|\int_{R^{n-k}} e^{ix''.\xi''} K(x', \xi'') dx''\right| \leq C$$

where *C* is independent of *x′*, *x″*.

When $x' = 0, K(0, x'')$ is homogeneous of degree – (*n* – *k*) and has vanishing mean value in *x″*, its Fourier Transform is bounded; that is,

$$\left|\int_{R^{n-k}} e^{ix''.\xi''} K(0, \xi'') d\xi''\right| \leq C \tag{11}$$

for some absolute constant C. Now we assume that

$x' \neq 0$. Let $R = |x'|$.

$$\int_{R^{n-k}} e^{ix''.\xi''} K(x', \xi'') d\xi'' = \int_{R^{n-k}} e^{iRx''.\eta''} K\left(\frac{x'}{R}, \eta''\right) d\eta''$$

$$= \int_{R^{n-k}} e^{iRx''.\eta''} \left(K\left(\frac{x'}{R}, \eta''\right) - K(0, \eta'')\right) d\eta''$$

$$+ \int_{R^{n-k}} e^{iRx''.\eta''} K(0, \eta'') d\eta''.$$

By (11), we have

$$\left|\int_{R^{n-k}} e^{ix''.\xi''} K(x', \xi'') d\xi''\right| \leq C + \left|\int_{R^{n-k}} e^{iRx''.\eta''} \left(K\left(\frac{x'}{R}, \eta''\right) - K(0, \eta'')\right) d\eta''\right|.$$

Notice that,

$$\left|K\left(\frac{x'}{R}, \eta''\right) - K(0, \eta'')\right| \leq C \frac{1}{|\eta''|^{n-k+1}}.$$

Let ϕ be a C^∞ function which is radial, and satisfies

$$\phi(x) \equiv 1 \quad \text{if} \quad |x| \leq \frac{1}{2} \quad \text{and} \quad \phi(x) \equiv 0 \quad \text{if} \quad |x| > 1.$$

Breaking the integral into two parts, we get

$$\left|\int_{R^{n-k}} e^{ix''.\xi''} K(x', \xi'') d\xi''\right|$$
$$\leq C + \left|\int_{R^{n-k}} e^{iRx''.\eta''} \left(K\left(\frac{x'}{R}, \eta''\right) - K(0, \eta'')\right) \phi(|\eta''|^2) d\eta''\right|$$
$$+ \left|\int_{R^{n-k}} e^{iRx''.\eta''} \left(K\left(\frac{x'}{R}, \eta''\right) - K(0, \eta'')\right) (1 - \phi(|\eta''|^2)) d\eta''\right|.$$

The second integral is

$$\leq C \int_{|\eta''| \geq 1/\sqrt{2}} \frac{1}{|\eta''|^{n-k+1}} d\eta'' \leq C.$$

We also have

$$\left|\int_{R^{n-k}} e^{iRx''.\eta''} K\left(\frac{x'}{R}, \eta''\right) (\phi|\eta''|^2) d\eta''\right| \leq \int_{|\eta''| \leq 1} \frac{d\eta''}{(1 + |\eta''|^2)^{(n-k)/2}} \leq C.$$

Similarly, we can get the estimate

$$\left|\int_{R^{n-k}} e^{iRx''.\eta''} K(0, \eta'') \phi(|\eta''|^2) d\eta''\right| \leq C.$$

This concludes the proof of the lemma. Now, Yibiao (1991) prove Theorem 2. In (8), taking Fourier Transform in the *x″* variables, we get

$$\int_{R^{n-k}} e^{ix''.\xi''} (Tf)(x', \xi'') d\xi''$$
$$= \int_{R^k} \int_{R^{n-k}} \int_{R^{n-k}} e^{i(y'^t A_1 x' + x''\xi// + y'^t B_1 \xi'')} K(x' - y', \xi'' - y'') f(y', y'') dy' dy'' d\xi''$$
$$= \int_{R^k} e^{iy'^t A_1 x'} g(y', x'') K_1(x' - y', x'' + B_1^t y') dy'$$

Where

$$g(y', x'') = \int_{R^{n-k}} e^{ix''^t y''} (f(y', y'') e^{iy'^t B_1 y''}) dy''$$

Clearly, we have

$$\int_{R^{n-k}} |g(y',x'')|^2 dx'' = (2\pi)^{n-k} \int_{R^{n-k}} |f(y',x'')|^2 dx''$$

RESULTS

Theorem 3

If $f_{m+1} = Sf_m \ , m = 0,1,2,\ldots$. For

$$F(f)(z) = \int_{R^k} e^{-iz^t A_1 \eta} f(\eta) d\eta,$$

where A_1 is a non singular matrix. Ahmed (2009). Show that;

(i) $F(f_{m+1})(x) = \int_{R^k} e^{-ix^t A_1 \eta} (S^m f_0)(\eta) d\eta$ for suitable f_0, and

(ii) $\|F(f_{m+1})\|_{L^2} = (2\pi)^{k/2} |A_1|^{-1/2} \|f_{m+1}\|_{L^2}$

(iii) If S is a contraction then $\|F(S^m f_0)\|_{L^2} \le C \|f_0\|_{L^2}$

We define the operator S on $C_0^\infty(R^k)$ by

$$(Sf)(x') = \int_{R^k} e^{iy'^t A_1 x'} K_1(x' - y', x'' + B_1^t y') f(y') dy' \qquad (12)$$

for $f \in C_0^\infty(R^k)$, and x'' is arbitrary in R^{n-k}.

Pan (1991) prove that

$$\|S(f)\|^2_{L^2(R^k)} \le C \|f\|^2_{L^2(R^k)} \qquad (13)$$

for some constant C which is independent of x'', then, for $f \in C_0^\infty(R^n)$,

$$\left\| \int_{R^{n-k}} e^{ix''\xi''} (Tf)(x',\xi'') d\xi'' \right\|^2_{L^2(R^k, dx')} = \|S(g(\cdot,x''))\|^2_{L^2(R^k, dx')}$$

$$\le C \int_{R^k} |g(y',x'')|^2 dy'.$$

By Plancherel's Theorem

$$\|T(f)\|^2_{L^2(R^n)} = \int_{R^k} dx' \int_{R^{n-k}} \left| \int_{R^{n-k}} e^{ix''\xi''} (Tf)(x',\xi'') d\xi'' \right|^2 dx''$$

$$\le C \int_{R^{n-k}} dx'' \int_{R^k} |g(y',x'')|^2 dy' = C \|f\|^2_{L^2(R^n)}.$$

He has reduced the study of the operator T on $L^2(R^n)$ to the study of an operator S on $L^2(R^k)$. For $f \in C_0^\infty(R^k), x \in R^k$, write

$$(Sf)(x) = \int_{R^k} e^{iy^t A_1 x} K_1(x - y, \quad \xi + B_1^t y) f(y) dy$$
$$= (S_1 f)(x) + (S_2 f)(x)$$

Where

$$(S_1 f)(x) = \int_{R^k} e^{iy^t A_1 x} K_1(x - y, \xi + B_1^t y)(1 - \phi)(x - y) f(y) dy$$

and

$$(S_1 f)(x) = \int_{R^k} e^{iy^t A_1 x} K_1(x - y, \xi + B_1^t y) \phi(x - y) f(y) dy.$$

By Lemma 2, $\|K_1(\cdot, \quad \xi + B^t y\|_\infty \le C$, and we get $|(S_2 f)| \le C \int_{R^k} |\phi(x - y)| |f(y)| dy$ which implies the boundedness of S_2 on $L^2(R^k)$. Now Yibiao (1991) proved that

$$\|S_1 f\|_{L^2(R^k)} \le C \|f\|_{L^2(R^k)}. \qquad (14)$$

Let

$$F(f)(z) = \int_{R^k} e^{-iz^t A_1 \eta} f(\eta) d\eta.$$

So we have that $\|F(f)\|_{L^2} = (2\pi)^{k/2} |A_1|^{-1/2} \|f\|_{L^2}$.

But

$$F(S_1 f)(x) = \int_{R^k} e^{-ix^t A_1 \eta} (S_1 f)(\eta) d\eta$$

$$= \int_{R^k} \int_{R^k} e^{-ix^t A_1 \eta} e^{iy^t A_1 \eta} K_1(\eta - y, \xi + B_1^t y)(1 - \phi)(\eta - y) f(y) dy\, d\eta$$

$$= \int_{R^k} e^{ix^t A_1 y} F((1 - \phi) K_1(\cdot, \xi + B_1^t y))(x - y)(e^{iy^t A_1 y} f(y))\, dy.$$

Let $F_{\xi,y}(u) = F\big((1-\phi)K_1(\cdot, \xi + B_1^t y)\big)(u)$.

We claim that

$$\left|F_{\xi,y}(x)\right| \le C_N \frac{1}{|x|^N} \tag{15}$$

for $N > k$, and

$$\left|F_{\xi,y}(x)\right| \le C \frac{1}{|x|^k} \tag{16}$$

The constants C_N and C do not depend on ξ or y. To prove this, we let $A_1^{-1} = (a^{ij})_{1 \le i,j \le k}$ be the inverse matrix of A_1, and $D_u = \frac{1}{-i} \sum_{s=1,j=1}^{k} \frac{x_s}{|x|^2} a^{js} \frac{\partial}{\partial u_j}$

So $D_u\left(e^{-ix^t A_1 u}\right) = e^{-ix^t A_1 u}$. Integrating by parts, we get

$$|F_{\xi,y}(x)| = \left|\int_{R^k} e^{-ix^t A_1 u}(-D_u)^N((1-\phi)(u)K_1(u, \xi + B_1^t y))du\right|$$

$$\le C_N \frac{1}{|x|^N} \int_{|u| \ge 1/2} \frac{du}{|u|^N} \le C_N \frac{1}{|x|^N}$$

if $N > k$, which proves (15).

To prove (16), we break the integral into two parts,

$$|F_{\xi,y}(x)| \le \left|\int_{R^k} e^{-ix^t A_1 u}(1-\phi)(u)K_1(u, \xi + B_1^t y)\phi(|x|^2|u|^2)du\right|$$

$$+\left|\int_{R^k} e^{-ix^t A_1 u}(1-\phi)(u)K_1(u, \xi + B_1^t y)(1-\phi)(|x|^2|u|^2)du\right|$$

$= I_1 + I_2$.

By Lemma 2.

$$|I_1| \le C \int_{|u| \le 1/|x|} du = C \frac{1}{|x|^k} .$$

To estimate I_2, take $N > k$, and use integration by parts,

$$|I_2| = \left|\int_{R^k} e^{-ix^t A_1 u}(1-\phi)(u)(-D_u)^N\big(K_1(u, \xi + B_1^t y)(1-\phi(|x|^2|u|^2))\big)du\right|.$$

In light of (15), we need only to be concerned with those x which are in $|x| < 1/4$. For $1 - \phi(|x|^2|u|^2) \ne 0$, we must have $|u| > 1$, and $(1-\phi)(u) \equiv 1$. Using integration by parts,

$$|I_2| = \left|\int_{R^k} e^{-ix^t A_1 u}(-D_u)^N\big(K_1(u, \xi + B_1^t y)(1-\phi(|x|^2|u|^2))\big)du\right|$$

$$\le \frac{C}{|x|^N} \sum_{|\alpha|+|\beta|=N} \int_{R^k} \left|\frac{\partial^\alpha}{\partial u^\alpha} K_1(u, \xi + B_1^t y)\right| \left|\frac{\partial^\beta}{\partial u^\beta}\big(1-\phi(|x|^2|u|^2)\big)\right| du.$$

Those terms with $\beta \ne 0$ are bounded by

$$\le C \frac{1}{|x|^N} \int_{1/2 < |x|^2|u|^2 < 1} \frac{1}{|u|^{|\alpha|}} |x|^{|\beta|} du \le C \frac{1}{|x|^k}.$$

For terms with $\beta = 0$, we have

$$\le C \frac{1}{|x|^N} \int_{|u| > 1/(\sqrt{2}x)} \frac{1}{|u|^{|N|}} du \le C \frac{1}{|x|^k}$$

for $N > k$. This proves (16).

Now take another function $\Psi \in C_0^\infty(R^k)$,

where $\Psi \equiv 1$ for $|x| \le 1/2$

and supp $\Psi \subset \{|x| \le 1\}$

We write $F(S_1 f)$ as the sum of two parts:

$F(S_1 f) = S_3\tilde{f} + S_4\tilde{f}$ where $f(y) = e^{iy^t A_1 y} f(y)$

and

$$(S_3 f)(x) = \int_{R^k} e^{-ix^t A_1 y} F\big((1-\phi)K_1(\cdot, \xi + B_1^t y)\big)(x-y)(1-\Psi)(x-y)f(y)dy$$

$$(S_4 f)(x) = \int_{R^k} e^{-ix^t A_1 y} F\big((1-\phi)K_1(\cdot, \xi + B_1^t y)\big)(x-y)\Psi(x-y)f(y)dy.$$

By (15), the kernel of S_3 is bounded by

$$C_N \frac{1}{(1+|x-y|)^N}.$$

So it is obvious that S_3 is bounded from $L^2(R^k)$ to itself. In order to study the operator S_4, we introduce another operator S_5, defined by

$$(S_5 f)(x) = \int_{R^k} F((1-\phi)K_1(\cdot,\xi + B_1^t y))(x-y)\psi(x-y)f(y)dy.$$

Once we prove that

$$\|S_5 f\|_{L^2(R^k)} \leq C\|f\|_{L^2(R^k)} \qquad (17)$$

We can get the estimate for S_4 as follows.

Let $R^k = \bigcup_{\alpha\in Z^k} Q_\alpha$,

$$Q_\alpha = \{x \in R^k | -1/2 < x_j - \alpha_j \leq 1/2, j = 1, \dots, k\}.$$

We have

$$(S_4 f\chi_{Q_\alpha})(x) = \int_{R^k} e^{ix^t A_1 y} F((1-\phi)K_1(\cdot,\xi + B_1^t y))(x-y)\psi(x-y)f(y)\chi_{Q_\alpha}(y)dy.$$

So it is easily seen that supp S_4 $(f\chi_{Q_\alpha}) \subset Q_\alpha^*$ Where

$$Q_\alpha^* = \{x \in R^k \| x_j - \alpha_j| \leq 3/2, \quad j = 1, \dots, k\},$$

and we can see that

$$\left|\sum_{\alpha\in Z^k} \chi_{Q_\alpha^*}\right| \leq C.$$

Write

$$(S_4 f\chi_{Q_\alpha})(x) = \int_{R^k} e^{ix^t A_1 y} F((1-\phi)K_1(\cdot,\xi + B_1^t))(x-y)\psi(x-y)f(y)\chi_{Q_\alpha}(y)dy.$$

$$= e^{ix^t A_1 \alpha} \int_{R^k} e^{i(x-y)^t A_1 (y-\alpha)} F((1-\phi)K_1(\cdot,\xi + B_1^t))(x-y)$$

$$\times \psi(x-y)(e^{iy^t A_1(y-\alpha)} f(y)\chi_{Q_\alpha}(y))dy.$$

Applying (16)

$$|(S_4 f\chi_{Q_\alpha})(x) - e^{ix^t A_1 \alpha}(S_5(e^{iy^t A_1(y-\alpha)}\chi_{Q_\alpha} f))(x)|$$

$$\leq C\int_{R^k} |e^{-i(x-y)^t A_1(y-\alpha)} - 1||\psi(x-y)||f(y)|\frac{1}{|x-y|^k}\chi_{Q_\alpha}(y)dy$$

$$\leq C\int_{R^k} \frac{|\psi(x-y)|}{|x-y|^{k-1}} |f(y)|\chi_{Q_\alpha}(y)dy. \qquad (18)$$

By (17) and (18) we get

$$\|S_4(f\chi_{Q_\alpha})\|_{L^2(R^k)} \leq C\|f\chi_{Q_\alpha}\|_{L^2(R^k)} \qquad (19)$$

Uniformly in $\alpha \in Z^k$. Now

$$|S_4 f(x)|^2 = \left|\sum_{\alpha\in Z^k} S_4(f\chi_{Q_\alpha}(x)\chi_{Q_\alpha^*}(x)\right|^2$$

$$\leq \left(\sum_{\alpha\in Z^k} |S_4(f\chi_{Q_\alpha})(x)|^2\right)\left(\sum_{\alpha\in Z^k}\chi_{Q_\alpha^*}(x)\right)$$

$$\leq C\sum_{\alpha\in Z^k} |S_4(f\chi_{Q_\alpha})(x)|^2.$$

By (19) we obtain

$$\|S_4 f\|^2_{L^2(R^k)} \leq C\sum_{\alpha\in Z^k}\|f\chi_{Q_\alpha}\|^2_{L^2(R^k)} = C\|f\|^2_{L^2(R^k)}$$

Now we prove (17), that is, the boundedness of S_5. Observe that

$$\int_{R^k} e^{ix^t A^t z} F(f)(z)dz = (2\pi)^k |A_1|^{-1} f(x)$$

where $|A_1|$ is the determinant of A_1. Let

$$F^{-1}(g)(x) = (2\pi)^k |A_1| \int_{R^k} e^{ix^t A^t z} g(z)dz.$$

We have $F^{-1}(\phi\psi) = (F^{-1}\phi) * (F^1\psi)$ for $\phi, \psi \in S$

Now let $\Psi(x) = F^{-1}(\psi)(x)$

By the definition, we have

$$F^{-1}(S_5 f)(x) = \int_{R^k} F^{-1}((F-\phi)K(\cdot,\xi + B_1^t y))\psi)(x)e^{ix^t A_1^t y} f(y)dy$$

$$= \int_{R^k} e^{ix^t A_1^t y}((1-\phi)K_1(\cdot,\xi + B_1^t y) * \Psi)(x)f(y)dy$$

$$= \int_{R^k} \Psi(u)S_6(e^{iu^t A_1^t y} f(y))(x-u)du \qquad (20)$$

Where the operator S_6 is defined by

$$(S_6 f)(x) = (1-\phi(x)) \int_{R^k} e^{ix^t A_1^t y} K_1(x, \xi + B_1^t y) f(y) dy.$$

If we can prove that

$$\|S_6 f\|_{L^2(R^k)} \leq C \|f\|_{L^2(R^k)}, \qquad (21)$$

Then by (20) we would have

$$\|S_5 f\|_{L^2(R^k)} \leq C \int_{R^k} |\Psi(u)| \|S_6(e^{iu^t A_1^t y} f)\|_{L^2(R^k)} du \leq C \|f\|_{L^2(R^k)}$$

Since $\Psi \in S$. To prove (21), we shall discuss the following two cases separately.

(i) If $B_1 = 0$, we have

$$(S_6 f)(x) = (1-\phi(x)) K_1(x, \xi) \int_{R^k} e^{iy^t A_1 x} f(y) dy.$$

By our estimates on K_1, one gets

$$|S_6 f(x)| \leq C \left| \int_{R^k} e^{iy^t A_1 x} f(y) dy \right|.$$

So by Plancherel's theorem, we see that (21) holds.

(ii) If $B_1 \neq 0$, then there exists two nonsingular matrices M and N; M is $k \times k$, N is $(n-k) \times (n-k)$, such that

$$MB_1 N = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$$

where we have assumed that rank $(B_1) = m$, $and\ 0 < m \leq \min\{k, (n-k)\}$.

Write $S_6 f(x)$ as

$$(S_6 f)(x) = (1-\phi(x)) \int_{R^k} e^{ix^t A_1^t y} \int_{R^{n-k}} e^{i(\xi^t + y^t B_1)\eta} K(x, \eta) f(y) d\eta\, dy$$

Where A_1 is nonsingular. We have

$$(S_6 f)(x) = (1-\phi(x)) \int_{R^{n-k}} e^{i\xi^t N\eta} K(x, N\eta) |N| d\eta \int_{R^k} e^{iy^t(A_1 x + B_1 N\eta)} f(y) dy$$

and

$$(S_6 f)(A_1^{-1} M^{-1} x) = (1-\phi(A_1^{-1} M^{-1} x)) \int_{R^{n-k}} e^{i\xi^t N\eta} K(A_1^{-1} M^{-1} x, N\eta) |N| d\eta$$

$$\times \int_{R^k} e^{iy^t(x + MB_1 N\eta)} f(M^t y) |M| dy.$$

Let

$$g(x) = \int_{R^k} e^{iy \cdot x} f(M^t y) dy.$$

Clearly, we have

$$\|g\|_{L^2(R^k)} = (2\pi)^{k/2} |M|^{-1/2} \|f\|_{L^2(R^k)}$$

and

$$|(S_6 f)(A_1^{-1} M^{-1} x)| \leq C \int_{R^{n-k}} \frac{1}{(|x|^2 + |\eta|^2)^{(n-k)/2}} g(x + MB_1 N\eta)\, d\eta$$

As

$$g(x + MB_1 N\eta) = g(x_1 + \eta_1, \dots, x_m + \eta_m, x_{m+1}, \dots, x_k)$$

for some m such that $0 < m \leq k, n-k,$ we have

$$|(S_6 f)(A_1^{-1} M^{-1} x)| \leq C \int_{R^m} g(x_1 + \eta_1, \dots, x_m + \eta_m, x_{m+1}, \dots, x_k) d\eta_1 \dots d\eta_m$$

$$\times \int_{R^{n-k-m}} \frac{1}{(|x|^2 + |\eta|^2)^{(n-k)/2}} d\eta_{m+1} \dots d\eta_{n-k}$$

Since $n-k > n-k-m$

$$\leq C \int_{R^m} \frac{g(x_1 + \eta_1, \dots, x_m + \eta_m, x_{m+1}, \dots, x_k)}{(|x|^2 + \eta_1^2 + \dots + \eta_m^2)^{m/2}} d\eta_1 \dots d\eta_m$$

$$\leq C \int_{R^m} \frac{g(x_1 + \eta_1, \dots, x_m + \eta_m, x_{m+1}, \dots, x_k)}{(|x| + |\eta_1|)(\cdots)(|x_m| + |\eta_m|)} d\eta_1 \dots d\eta_m$$

We recall that the operator on $L^2(R^1)$

$$f(x) \to \int_{R^1} \frac{f(x+t)}{|x| + |t|} dt, \quad x \in R^1$$

is the Hilbert integral operator, and it is well known that this operator is bounded from L^2 to itself (Phong and Stein, 1986), (Pan, 1989). By induction we have

$$\int_{R^m} |S_6 f(A_1^{-1}M^{-1}x)|^2 dx_1 \dots dx_m$$

$$\leq C \int_{R^m} |g(x_1, \dots, x_m, x_{m+1}, \dots, x_k)|^2 dx_1 \dots dx_m$$

Integrating with respect to $x_{m+1}, \dots, x_k,$ we get

$$\int_{R^k} |(S_6 f)(A_1^{-1}M^{-1}x)|^2 dx \leq C\|f\|_{L^2(R^k)}$$

Which implies that $\|S_6 f\|_{L^2(R^k)} \leq C\|f\|^2_{L^2(R^k)}$. So (21) is proven.

Proof of the main result

(i) For $m = 0, 1, 2, \dots$, the iterated sequence will give

$$F(f_1)(x) = \int_{R^k} e^{-ix^t A_1 \eta} f_1(\eta) d\eta = \int_{R^k} e^{-ix^t A_1 \eta} (Sf_0)(\eta) d\eta, \quad m = 0$$

$$F(f_2)(x) = \int_{R^k} e^{-ix^t A_1 \eta} f_2(\eta) d\eta = \int_{R^k} e^{-ix^t A_1 \eta} (Sf_1)(\eta) d\eta, \quad m = 0$$

$$= \int_{R^k} e^{-ix^t A_1 \eta} (S^2 f_0)(\eta) d\eta,$$

Thus, generally

$$(f_{m+1})(x) = \int_{R^k} e^{-ix^t A_1 \eta} f_{m+1}(\eta) d\eta = \int_{R^k} e^{-ix^t A_1 \eta} (S^m f_0)(\eta) d\eta.$$

(ii) In (i) taking the supremum over all

$\|x\| = 1$ & $\eta(x) = x$ we can find

$$\|F(f_{m+1})\|_{L^2} = (2\pi)^{k/2} |A_1|^{-1/2} \|f_{m+1}\|_{L^2}$$

(iii) Letting $C \equiv (2\pi)^{k/2} |A_1|^{-1/2}$ & $\|S^m\| \leq 1$

$$\|F(S^m f_0)\|_{L^2} = C\|S^m f_0\|_{L^2} \leq C\|f_0\|_{L^2}.$$

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