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Applications of Cauchy-Schwarz inequalities in the mapping structure of linear operator

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By applications of Cauchy-Schwarz inequalities, several sufficient conditions in terms of hypergeometric inequalities were found such that the linear operator $H^{a,b,c}_{\mu,\delta}$ preserves and transforms certain well known subclasses of univalent functions to another classes. Relevant connections of our work with the earlier work is pointed out.

Key words: Analytic function, subordination, starlike function, convex function, hypergeometric function, Cauchy-Schwarz inequality.

INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of functions of the form (Equation 1) which are also univalent in $\mathbb{U}$ was denoted by $\mathcal{S}$. A function $f \in \mathcal{A}$ is said to be in k-UCV, the class of k-uniformly convex functions $(0 \le k < \infty)$ if $f \in \mathcal{S}$ along with the property that for every circular arc γ contained in $\mathbb{U}$, with centre ξ where $|\xi| \le k$, the image curve $f(\gamma)$ is a convex arc (Kanas and Wisniowska, 1999). It is well known that (Kanas and Wisniowska, 1999) $f \in k-$UCV if and only if the image of the function p, where

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U}),$$

is a subset of the conic region:

$$\Omega_k = \{w = u + iv : u^2 > k^2(u-1)^2 + k^2v^2,\ 0 \le k < \infty\}. \qquad (2)$$

The class $k-\mathcal{ST}$, consisting of k-starlike functions, is

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defined via $k-\mathrm{UCV}$ by the usual Alexander's relation as:

$$f\in k-\mathrm{ST}\Leftrightarrow g\in k-\mathrm{UCV}\quad \text{where}\quad g(z)=\int_0^z\frac{f(t)}{t}dt.$$

For $k=0$, the classes $k-\mathrm{UCV}$ and $k-\mathrm{ST}$ reduce to the classes of convex and starlike functions studied by Robertson (1936) and Silverman (1975) and for $k=1$, the aforementioned classes reduce to the classes of uniformly convex and uniformly starlike functions in U studied by Goodman (1991a; b).

Let $\phi(z)$ be an analytic function with positive real part in U with $\phi(0)=1,\ \phi'(0)>0$, which is starlike with respect to 1 and is also symmetric with respect to the real axis. For such functions ϕ, Bansal (2011; 2013) introduced a class $\mathrm{R}_\gamma^\tau(\phi)$ of functions satisfying the condition:

$$\mathrm{R}_\gamma^\tau(\phi):=\{f\in\mathrm{A}:1+\frac{1}{\tau}\{f'(z)+\gamma zf''(z)-1\}\prec\phi(z),\ z\in\mathrm{U}\},\qquad(3)$$

where $0\le\gamma\le1,\ \tau\in\mathrm{C}\setminus\{0\}$ and $\prec$ denote the subordination between analytic functions.

Taking $\phi(z)=\dfrac{1+Az}{1+Bz}\ (-1\le B<A\le1;\ z\in\mathrm{U})$ in (Equation 3), we observe that a function $f\in\mathrm{R}_\gamma^\tau\left(\dfrac{1+Az}{1+Bz}\right)=\mathrm{R}_\gamma^\tau(A,B)$ if and only if the following condition is satisfied:

$$\left|\frac{f'(z)+\gamma zf''(z)-1}{(A-B)\tau-B(f'(z)+\gamma zf''(z)-1)}\right|<1.\qquad(4)$$

By giving appropriate values to the parameters τ,γ,A and B, we get various subclasses of S studied by different researchers. By taking

$\gamma=0$, the class $\mathrm{R}_0^\tau(A,B)$ reduces to the class $\mathrm{R}^\tau(A,B)$ introduced and studied by Dixit and Pal (1995); $A=1-2\beta\ (0\le\beta<1),\ B=-1$, the class $\mathrm{R}_\gamma^\tau(1-2\beta,-1)$ reduces to the class $\mathrm{R}_\gamma^\tau(\beta)$ studied by Swaminathan (2010);

$\gamma=0,\ \tau=e^{-i\eta}cos\eta\ (|\eta|<\frac{\pi}{2}),\ A=1-2\beta\ (0\le\beta<1),\ B=-1$, the class $\mathrm{R}_0^\tau(1-2\beta,-1)$ reduces to $\mathrm{R}_\eta(\beta)$ studied by Ponnusamy and RØnning (1998); $\gamma=0,\ \tau=1,\ A=\beta,\ B=-\beta\ (0<\beta\le1)$, the class $\mathrm{R}_0^1(\beta,-\beta)$ reduces to the class $D(\beta)$ studied by Caplinger and Causey (1973) and Padmanabhan (1979).

The Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ given by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n\quad(z\in\mathrm{U})\qquad(5)$$

is the solution of the homogeneous hypergeometric differential equation:

$$z(1-z)w''(z)+[c-(a+b+1)z]w'(z)-abw(z)=0.\qquad(6)$$

Here a,b and c are complex parameters such that $c\ne0,-1,-2,\cdots$, $(a)_0=1$ for $a\ne0$, and for each positive integer $n,\ (a)_n=a(a+1)(a+2)....(a+n-1)$ is the Pochhammer symbol. In the case of $c=-k,\ k=0,1,2,\cdots$, the function ${}_2F_1(a,b;c;z)$ is defined if $a=-j$ or $b=-j$ where $j\le k$. Note that ${}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z)$ and

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad(\Re(c-a-b)>0).\qquad(7)$$

The behaviour of the hypergeometric function ${}_2F_1(a,b;c;z)$ near $z=1$ is classified into three cases according as $\Re(c-a-b)>0,=0,<0$. The function ${}_2F_1(a,b;c;z)$ is bounded if $\Re(c-a-b)>0$ and has pole at $z=1$ if $\Re(c-a-b)\le0$ (Owa and Srivastava, 1987; Whitteaker and Watson, 1927). The hypergeometric function ${}_2F_1(a,b;c;z)$ has been extensively studied by various authors and play an important role in Geometric Function Theory (Carlson and Shaffer, 1984; Cho et al., 2002; Ponnusamy and Viorinen, 2001; Swaminathan, 2010).

The normalized hypergeometric function $z{}_2F_1(a,b;c;z)$ has a series expansion of the form:

$$z{}_2F_1(a,b;c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n.\qquad(8)$$

Using function $z{}_2F_1(a,b;c;z)$, we consider the function

(Tang and Deng, 2014), with p=1

$$\mathrm{J}_{\mu,\delta}(a,b;c;z)=(1-\mu+\delta)[z_2F_1(a,b;c;z)]+(\mu-\delta)z[z_2F_1(a,b;c;z)]'+\mu\delta z^2[z_2F_1(a,b;c;z)]''$$

$$(\mu,\delta\geq 0,\ \mu\geq\delta;\ z\in\mathrm{U}). \qquad (9)$$

Using convolution operator, consider a linear operator $H_{\mu,\delta}^{a,b,c}:\mathrm{A}\to\mathrm{A}$ defined by means of Hadamard product as Panigrahi and El-Ashwah, Unpublished:

$$H_{\mu,\delta}^{a,b,c}(f)(z)=\mathrm{J}_{\mu,\delta}(a,b;c;z)*f(z)$$
$$=z+\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n \quad (z\in\mathrm{U}). \qquad (10)$$

The linear operator $H_{\mu,\delta}^{a,b,c}$ unifies several of such previously well studied operators. For example, by taking $\delta=0$, $H_{\mu,\delta}^{a,b,c}(f)=L_{\mu}(f)$ studied by Kim and Shon (2003); $\delta=\mu=0$, $H_{0,0}^{a,b,c}(f)=\mathrm{I}_{c}^{a,b}(f)$ where $\mathrm{I}_{c}^{a,b}$ is the Hohlov operator studied by Hohlov (Yu, 1978); $\delta=\mu=0, b=1$, $H_{0,0}^{a,1,c}(f)=\mathrm{L}(a,c)(f)$, where $\mathrm{L}(a,c)$ is Carlson-Shaffer operator studied in (Carlson and Shaffer, 1984).

It is well known that the class S and many of its subclasses are not closed under the ring operation of usual addition and multiplication of functions. As such techniques of algebra from group theory, ring theory, etc., and those of functional analysis do not find ready applications in the class S. Therefore, the study of class-preserving and class-transforming operations is an interesting problem in geometric function theory.

PRELIMINARIES LEMMAS

Each of the following lemmas and the concept of Cauchy-Schwarz inequalities will be require for our investigation.

Lemma 1

Let the function f of the form (Equation 1) be a member of S or ST (de Branges, 1985). Then, the sharp estimate

$$|a_n|\leq n \quad (n\in\mathrm{N}\setminus\{1\}) \qquad (11)$$

holds true.

Lemma 2

Let the function $f\in\mathrm{A}$ be of the form (Equation 1) (Bansal, 2013). If

$$\sum_{n=2}^{\infty}n[1+\gamma(n-1)]\,|a_n|\leq\frac{(A-B)|\tau|}{1+|B|} \quad (-1\leq B<A\leq 1,\tau\in\mathrm{C}\setminus\{0\};\ z\in\mathrm{U}), \qquad (12)$$

then $f\in\mathrm{R}_{\gamma}^{\tau}(A,B)$. The result is sharp for the function:

$$f(z)=z+\frac{(A-B)|\tau|}{n[1+\gamma(n-1)](1+|B|)}z^n \quad (n\in\mathrm{N}\setminus\{1\}).$$

Lemma 3

Let (Kanas and Wisniowska, 1999; 2000):

$$P_k(z)=1+p_1(k)z+p_2(k)z^2+\cdots \quad (p_1(k)>0;\ z\in\mathrm{U}) \qquad (13)$$

be the Riemann map of U onto Ω_k where the region Ω_k is defined as in Equation 2 and let the function f be given by Equation 1. If $f\in k-\mathrm{ST}$, then

$$|a_n|\leq\frac{(p_1(k))_{n-1}}{(1)_{n-1}} \quad (n\in\mathrm{N}\setminus\{1\}). \qquad (14)$$

Further, if $f\in k-\mathrm{UCV}$, then

$$|a_n|\leq\frac{(p_1(k))_{n-1}}{(1)_{n}} \quad (n\in\mathrm{N}\setminus\{1\}). \qquad (15)$$

The estimates Equations 14 and 15 are sharp.

Lemma 4

Let the function $f\in\mathrm{A}$ be of the form (Equation 1) (Goodman, 1957). If

$$\sum_{n=2}^{\infty}n\,|a_n|\leq 1, \qquad (16)$$

then $f\in\mathrm{ST}$.

Lemma 5

Let the function f, given by Equation 1 be a member of $\mathrm{R}_{\gamma}^{\tau}(A,B)$ (Bansal, 2013).

Then $|a_n| \leq \dfrac{(A-B)|\tau|}{n[1+\gamma(n-1)]} \quad (n \in \mathrm{N} \setminus \{1\}).$ (17)

The use of Cauchy–Schwarz inequality, known as the Cauchy–Bunyakovsky–Schwarz inequality find a place in various areas of mathematics such as linear algebra, analysis, probability theory, vector algebra and many more. It is considered to be one of the most important inequalities in mathematics. It states that for complex parameters $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$, we have

$$|u_1\overline{v_1} + u_2\overline{v_2} + ... + u_n\overline{v_n}|^2 \leq (|u_1|^2 + |u_2|^2 + ...|u_n|^2)(|v_1|^2 + |v_2|^2 + ...|v_n|^2)$$

that is,

$$|\sum_{i=1}^{n} u_i \overline{v_i}|^2 \leq |\sum_{j=1}^{n} u_j|^2 |\sum_{k=1}^{n} v_k|^2 .$$

Motivated by Mishra and Panigrahi, 2011; Aouf et al., 2016; Bansal, 2013; Mostafa, 2009; Panigrahi and El-Ashwah, Unpublished) Sharma et al., 2013; Sivasubramanian et al., 2011), in this paper by applications of Cauchy-Schwarz inequalities, we find several sufficient conditions in terms of hypergeometric inequalities for the linear operator $H_{\mu,\delta}^{a,b,c}$ defined in (Equation 10) to preserves and transform certain well known subclasses of univalent functions to another class.

MAIN RESULTS

Throughout the paper, we assume that

$$-1 \leq B < A \leq 1,\ 0 \leq \gamma \leq 1,\ \tau \in \mathrm{C} \setminus \{0\},\ \mu, \delta \geq 0 \text{ and } \mu \geq \delta.$$

Theorem 1

Let $a, b \in \mathrm{C} \setminus \{0\}$ and $c \in \mathrm{C}$ satisfy the inequality

$$\Re c > \max\{0, 2\Re a + 5, 2\Re b + 5\}. \quad (18)$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a - 5)\Gamma(\Re c - 2\Re b - 5)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - b)|}[\gamma\mu\delta|(a)_5||(b)_5| + (\mu\delta + \gamma\mu - \gamma\delta + 13\gamma\mu\delta)|(a)_4||(b)_4|$$
$$\{(\Re c - 2\Re a - 5)(\Re c - 2\Re b - 5)\}^{\frac{1}{2}} + (\mu - \delta + \gamma + 9\mu\delta + 8\gamma\mu - 8\gamma\delta + 46\gamma\mu\delta)|(a)_3||(b)_3|$$
$$\{(\Re c - 2\Re a - 5)_2(\Re c - 2\Re b - 5)_2\}^{\frac{1}{2}} + (1 + 5\mu - 5\delta + 5\gamma + 19\mu\delta + 14\gamma\mu - 14\gamma\delta + 46\gamma\mu\delta)$$
$$|(a)_2||(b)_2|\{(\Re c - 2\Re a - 5)_3(\Re c - 2\Re b - 5)_3\}^{\frac{1}{2}} + (3 + 4\mu - 4\delta + 4\gamma + 8\mu\delta + 4\gamma\mu - 4\gamma\delta + 8\gamma\mu\delta)|ab|$$
$$\{(\Re c - 2\Re a - 5)_4(\Re c - 2\Re b - 5)_4\}^{\frac{1}{2}} + \{(\Re c - 2\Re a - 5)_5(\Re c - 2\Re b - 5)_5\}^{\frac{1}{2}}] \quad (19)$$
$$\leq 1 + \frac{(A-B)|\tau|}{1+|B|},$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class S (or ST) into the class $\mathrm{R}_\gamma^\tau(A, B)$.

Proof

Let the function f given by Equation 1 be in the class S or ST. By Equation 10, we have

$$H_{\mu,\delta}^{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)(\mu - \delta + n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n \quad (z \in \mathrm{U}).$$

By virtue of Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n[1 + \gamma(n-1)][1 + (n-1)(\mu - \delta + n\mu\delta)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n\right| \leq \frac{(A-B)|\tau|}{1+|B|}.$$

Since $f \in \mathrm{S}$ (or ST), by making use of Lemma 1, it is again sufficient to show

$$S_1 = \sum_{n=2}^{\infty} n^2[1 + \gamma(n-1)][1 + (n-1)(\mu - \delta + n\mu\delta)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq \frac{(A-B)|\tau|}{1+|B|}. \quad (20)$$

Using elementary inequality

$$|(c)_p| > (\Re c)_p \quad (p \in \mathrm{N}),$$

we have

$$S_1 \leq \sum_{n=1}^{\infty}(n+1)^2[1 + n\gamma][1 + n(\mu - \delta + (n+1)\mu\delta)]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$= \sum_{n=1}^{\infty}(n+1)^2(1 + n\gamma)[1 + n(\mu - \delta) + n(n+1)\mu\delta]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$= \sum_{n=1}^{\infty}(n+1)^2[1 + n(\mu - \delta) + n(n+1)\mu\delta + n\gamma + n^2\gamma(\mu - \delta) + n^2(n+1)\gamma\mu\delta]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$= \sum_{n=1}^{\infty}(n+1)^2\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + \sum_{n=1}^{\infty}n(n+1)^2(\mu - \delta)\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + \mu\delta\sum_{n=1}^{\infty}n(n+1)^3\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$+ \gamma\sum_{n=1}^{\infty}n(n+1)^2\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + \gamma(\mu - \delta)\sum_{n=1}^{\infty}n^2(n+1)^2\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + \gamma\mu\delta\sum_{n=1}^{\infty}n^2(n+1)^3\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$= \sum_{n=1}^{\infty}[n(n-1) + 3n + 1]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + \sum_{n=1}^{\infty}(n+1)^2(\mu - \delta)\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}} + \mu\delta\sum_{n=1}^{\infty}(n+1)^3\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}$$
$$+ \gamma\sum_{n=1}^{\infty}(n+1)^2\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}} + \gamma(\mu - \delta)\sum_{n=1}^{\infty}n(n+1)^2\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}} + \gamma\mu\delta\sum_{n=1}^{\infty}n(n+1)^3\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}$$
$$= \sum_{n=1}^{\infty}[n(n-1) + 3n + 1]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + (\mu - \delta + \gamma)\sum_{n=1}^{\infty}[(n-1)(n-2) + 5(n-1) + 4]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}$$
$$+ \mu\delta\sum_{n=1}^{\infty}[(n-1)(n-2)(n-3) + 9(n-1)(n-2) + 19(n-1) + 8]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}$$

$$+\gamma(\mu-\delta)\sum_{n=1}^{\infty}[(n-1)(n-2)(n-3)+8(n-1)(n-2)+14(n-1)+4]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}$$

$$+\gamma\mu\delta\sum_{n=1}^{\infty}[(n-1)(n-2)(n-3)(n-4)+13(n-1)(n-2)(n-3)$$

$$+46(n-1)(n-2)+46(n-1)+8]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}=\sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}+[3+4(\mu-\delta+\gamma)$$

$$+8\mu\delta+4\gamma(\mu-\delta)+8\gamma\mu\delta]\sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}}+[1+5(\mu-\delta+\gamma)+19\mu\delta+14\gamma(\mu-\delta)+46\gamma\mu\delta]$$

$$\sum_{n=2}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-2}}+[\mu-\delta+\gamma+9\mu\delta+8\gamma(\mu-\delta)+46\gamma\mu\delta]\sum_{n=3}^{\infty}\frac{|(a)_n(b)_n|}{(\Re)_n(1)_{n-3}}$$

$$+[\mu\delta+\gamma(\mu-\delta)+13\gamma\mu\delta]\sum_{n=4}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-4}}+\gamma\mu\delta\sum_{n=5}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-5}}=\left[\sum_{n=0}^{\infty}\frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n}-1\right]$$

$$+[3+4(\mu-\delta+\gamma)+8\mu\delta+4\gamma(\mu-\delta)+8\gamma\mu\delta]\sum_{n=0}^{\infty}\frac{|(a)_{n+1}||(b)_{n+1}|}{(\Re c)_{n+1}(1)_n}+[1+5(\mu-\delta+\gamma)+19\mu\delta$$

$$+14\gamma(\mu-\delta)+46\gamma\mu\delta]\sum_{n=0}^{\infty}\frac{|(a)_{n+2}||(b)_{n+2}|}{(\Re c)_{n+2}(1)_n}+[\mu-\delta+\gamma+9\mu\delta+8\gamma(\mu-\delta)+46\gamma\mu\delta]\sum_{n=0}^{\infty}\frac{|(a)_{n+3}||(b)_{n+3}|}{(\Re c)_{n+3}(1)_n}$$

$$+[\mu\delta+\gamma(\mu-\delta)+13\gamma\mu\delta]\sum_{n=0}^{\infty}\frac{|(a)_{n+4}||(b)_{n+4}|}{(\Re c)_{n+4}(1)_n}+\gamma\mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+5}||(b)_{n+5}|}{(\Re c)_{n+5}(1)_n}.$$

The repeated applications of the relation

$$(e)_m=e(e+1)_{m-1}\quad(e\in\mathbb{C},m\in\mathbb{N})$$

give

$$S_1\le\sum_{n=0}^{\infty}\frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n}+[3+4(\mu-\delta+\gamma)+8\mu\delta+4\gamma(\mu-\delta)+8\gamma\mu\delta]\frac{|ab|}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n||(b+1)_n|}{(\Re c+1)_n(1)_n}$$

$$+[1+5(\mu-\delta+\gamma)+19\mu\delta+14\gamma(\mu-\delta)+46\gamma\mu\delta]\frac{|(a)_2||(b)_2|}{(\Re c)_2}\sum_{n=0}^{\infty}\frac{|(a+2)_n||(b+2)_n|}{(\Re c+2)_n(1)_n}$$

$$+[\mu-\delta+\gamma+9\mu\delta+8\gamma(\mu-\delta)+46\gamma\mu\delta]\frac{|(a)_3||(b)_3|}{(\Re c)_3}\sum_{n=0}^{\infty}\frac{|(a+3)_n||(b+3)_n|}{(\Re c+3)_n(1)_n}+[\mu\delta+\gamma(\mu-\delta)$$

$$+13\gamma\mu\delta]\frac{|(a)_4||(b)_4|}{(\Re c)_4}\sum_{n=0}^{\infty}\frac{|(a+4)_n||(b+4)_n|}{(\Re c+4)_n(1)_n}+\gamma\mu\delta\frac{|(a)_5||(b)_5|}{(\Re c)_5}\sum_{n=0}^{\infty}\frac{|(a+5)_n||(b+5)_n|}{(\Re c+5)_n(1)_n}-1. \quad (21)$$

Applying Cauchy-Schwarz inequality to the individual sums in Equation 21, we get

$$S_1\le\left\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n}{(\Re c)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n}{(\Re c)_n(1)_n}\right\}^{\frac{1}{2}}+[3+4(\mu-\delta+\gamma)+8\mu\delta+4\gamma(\mu-\delta)+8\gamma\mu\delta]$$

$$\frac{|ab|}{\Re c}\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n}{(\Re c+1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n}{(\Re c+1)_n(1)_n}\right\}^{\frac{1}{2}}+[1+5(\mu-\delta+\gamma)+19\mu\delta+$$

$$14\gamma(\mu-\delta)+46\gamma\mu\delta]\frac{|(a)_2||(b)_2|}{(\Re c)_2}\left\{\sum_{n=0}^{\infty}\frac{(a+2)_n(\bar{a}+2)_n}{(\Re c+2)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+2)_n(\bar{b}+2)_n}{(\Re c+2)_n(1)_n}\right\}^{\frac{1}{2}}$$

$$+[\mu-\delta+\gamma+9\mu\delta+8\gamma(\mu-\delta)+46\gamma\mu\delta]\frac{|(a)_3||(b)_3|}{(\Re c)_3}\left\{\sum_{n=0}^{\infty}\frac{(a+3)_n(\bar{a}+3)_n}{(\Re c+3)_n(1)_n}\right\}^{\frac{1}{2}}$$

$$\left\{\sum_{n=0}^{\infty}\frac{(b+3)_n(\bar{b}+3)_n}{(\Re c+3)_n(1)_n}\right\}^{\frac{1}{2}}+[\mu\delta+\gamma(\mu-\delta)+13\gamma\mu\delta]\frac{|(a)_4||(b)_4|}{(\Re c)_4}\left\{\sum_{n=0}^{\infty}\frac{(a+4)_n(\bar{a}+4)_n}{(\Re c+4)_n(1)_n}\right\}^{\frac{1}{2}}$$

$$\left\{\sum_{n=0}^{\infty}\frac{(b+4)_n(\bar{b}+4)_n}{(\Re c+4)_n(1)_n}\right\}^{\frac{1}{2}}+\gamma\mu\delta\frac{|(a)_5||(b)_5|}{(\Re c)_5}\left\{\sum_{n=0}^{\infty}\frac{(a+5)_n(\bar{a}+5)_n}{(\Re c+5)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+5)_n(\bar{b}+5)_n}{(\Re c+5)_n(1)_n}\right\}^{\frac{1}{2}}-1$$

$$=\gamma\mu\delta\frac{|(a)_5||(b)_5|}{(\Re c)_5}\{{}_2F_1(a+5,\bar{a}+5;\Re c+5;1)\}^{\frac{1}{2}}\{{}_2F_1(b+5,\bar{b}+5;\Re c+5;1)\}^{\frac{1}{2}}+[\mu\delta+\gamma(\mu-\delta)+13\gamma\mu\delta]$$

$$\frac{|(a)_4||(b)_4|}{(\Re c)_4}\{{}_2F_1(a+4,\bar{a}+4;\Re c+4;1)\}^{\frac{1}{2}}\{{}_2F_1(b+4,\bar{b}+4;\Re c+4;1)\}^{\frac{1}{2}}+[\mu-\delta+\gamma+9\mu\delta+8\gamma(\mu-\delta)$$

$$+46\gamma\mu\delta]\frac{|(a)_3||(b)_3|}{(\Re c)_3}\{{}_2F_1(a+3,\bar{a}+3;\Re c+3;1)\}^{\frac{1}{2}}\{{}_2F_1(b+3,\bar{b}+3;\Re c+3;1)\}^{\frac{1}{2}}+[1+5(\mu-\delta+\gamma)$$

$$+19\mu\delta+14\gamma(\mu-\delta)+46\gamma\mu\delta]\frac{|(a)_2|(b)_2|}{(\Re c)_2}\{{}_2F_1(a+2,\bar{a}+2;\Re c+2;1)\}^{\frac{1}{2}}\{{}_2F_1(b+2,\bar{b}+2;\Re c+2;1)\}^{\frac{1}{2}}$$

$$+[3+4(\mu-\delta+\gamma)+8\mu\delta+4\gamma(\mu-\delta)+8\gamma\mu\delta]\frac{|ab|}{\Re c}\{{}_2F_1(a+1,\bar{a}+1;\Re c+1;1)\}^{\frac{1}{2}}$$

$$\{{}_2F_1(b+1,\bar{b}+1;\Re c+1;1)\}^{\frac{1}{2}}+\{{}_2F_1(a,\bar{a};\Re c;1)\}^{\frac{1}{2}}\{{}_2F_1(b,\bar{b};\Re c;1)\}^{\frac{1}{2}}-1. \tag{22}$$

Since the condition in Equation 18 is satisfied, using Gauss summation formula in Equation 22, we obtain

$$S_1\le\gamma\mu\delta\frac{|(a)_5||(b)_5|}{(\Re c)_5}\left\{\frac{\Gamma(\Re c+5)\Gamma(\Re c-2\Re a-5)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\Re c+5)\Gamma(\Re c-2\Re b-5)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}+[\mu\delta+\gamma\mu$$

$$-\gamma\delta+13\gamma\mu\delta]\frac{|(a)_4||(b)|_4}{(\Re c)_4}\left\{\frac{\Gamma(\Re c+4)\Gamma(\Re c-2\Re a-4)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\Re c+4)\Gamma(\Re c-2\Re b-4)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}$$

$$+[\mu-\delta+\gamma+9\mu\delta+8\gamma\mu-8\gamma\delta+46\gamma\mu\delta]\frac{|(a)_3||(b)_3|}{(\Re c)_3}\left\{\frac{\Gamma(\Re c+3)\Gamma(\Re c-2\Re a-3)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}$$
$$\left\{\frac{\Gamma(\Re c+3)\Gamma(\Re c-2\Re b-3)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}+[1+5\mu-5\delta+5\gamma+19\mu\delta+14\gamma\mu-14\gamma\delta+46\gamma\mu\delta]\frac{|(a)_2||(b)_2|}{(\Re c)_2}$$
$$\left\{\frac{\Gamma(\Re c+2)\Gamma(\Re c-2\Re a-2)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\Re c+2)\Gamma(\Re c-2\Re b-2)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}+[3+4\mu-4\delta+4\gamma+8\mu\delta+$$
$$4\gamma\mu-4\gamma\delta+8\gamma\mu\delta]\frac{|ab|}{\Re c}\left\{\frac{\Gamma(\Re c+1)\Gamma(\Re c-2\Re a-1)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\Re c+1)\Gamma(\Re c-2\Re b-1)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}$$
$$+\left\{\frac{\Gamma(\Re c)\Gamma(\Re c-2\Re a)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar{a})}\right\}^{\frac{1}{2}}\left\{\frac{\Gamma(\Re c)\Gamma(\Re c-2\Re b)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar{b})}\right\}^{\frac{1}{2}}-1. \quad (23)$$

Since gamma function is symmetric about the real axis, that is,. $\overline{\Gamma(z)}=\Gamma(\bar{z})$, we have from Equation 23,

$$S_1\le\frac{\Gamma(\Re c)\{\Gamma(\Re c-2\Re a-5)\Gamma(\Re c-2\Re b-5)\}^{\frac{1}{2}}}{|\Gamma(\Re c-a)||\Gamma(\Re c-\bar{b})|}[\gamma\mu\delta|(a)_5||(b)_5|+(\mu\delta+\gamma\mu-\gamma\delta+13\gamma\mu\delta)]$$

$$|(a)_4||(b)_4|\{(\Re c-2\Re a-5)(\Re c-2\Re b-5)\}^{\frac{1}{2}}+(\mu-\delta+\gamma+9\mu\delta+8\gamma\mu-8\gamma\delta$$
$$+46\gamma\mu\delta)|(a)_3||(b)_3|\{(\Re c-2\Re a-5)_2(\Re c-2\Re b-5)_2\}^{\frac{1}{2}}+[1+5\mu-5\delta+5\gamma+19\mu\delta+14\gamma\mu$$
$$-14\gamma\delta+46\gamma\mu\delta]|(a)_2||(b)_2|\{(\Re c-2\Re a-5)_3(\Re c-2\Re b-5)_3\}^{\frac{1}{2}}+[3+4\mu-4\delta+4\gamma+8\mu\delta+4\gamma\mu-$$
$$4\gamma\delta+8\gamma\mu\delta]|ab|\{\{(\Re c-2\Re a-5)_4(\Re c-2\Re b-5)_4\}^{\frac{1}{2}}+\{(\Re c-2\Re a-5)_5(\Re c-2\Re b-5)_5\}^{\frac{1}{2}}-1.$$

Therefore, in view of Equation 20, if the hypergeometric inequality Equation 19 is satisfied, then $H_{\mu,\delta}^{a,b,c}(f)\in \mathcal{R}_{\gamma}^{\tau}(A,B)$. This ends the proof of Theorem 1.

Putting $\mu=\delta=\gamma=0$ in Theorem 1, after simplification we get the following result due to Mishra and Panigrahi (2011):

Corollary 1

Let $a,b\in\mathbb{C}\setminus\{0\}$ and $c\in\mathbb{C}$ satisfy (Mishra and Panigrahi, 2011), Theorem 1, p. 55)

$$\Re c>\max\{0,2\Re a+2,2\Re b+2\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c-2\Re a-2)\Gamma(\Re c-2\Re b-2)\}^{\frac{1}{2}}}{|\Gamma(\Re c-a)||\Gamma(\Re c-b)|}[|(a)_2||(b)_2|+3|ab|\{(\Re c-2\Re a-2)(\Re c-2\Re b-2)\}^{\frac{1}{2}}$$
$$+\{(\Re c-2\Re a-2)_2(\Re c-2\Re b-2)_2\}^{\frac{1}{2}}]\le 1+\frac{(A-B)|\tau|}{1+|B|},$$

is satisfied, then $I_c^{a,b}$ maps the class S (or ST) into $\mathcal{R}^{\tau}(A,B)$.

Taking $b=\bar{a}$ in Corollary 1 and after simplification, we get the following.

Corollary 2

Let $a\in\mathbb{C}\setminus\{0\}$ and $c\in\mathbb{C}$ satisfy (Mishra and Panigrahi, 2011, Corollary 1, p. 57)

$$\Re c > \max\{0, 2\Re a + 2\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\Gamma(\Re c - 2\Re a - 2)}{|\Gamma(\Re c - a)|^2}[|(a)_2|^2 + 3|a|^2(\Re c - 2\Re a - 2) + (\Re c - 2\Re a - 2)_2] \le 1 + \frac{(A-B)|\tau|}{1+|B|},$$

is satisfied, then $\mathrm{I}_c^{a,\bar{a}}$ maps the class S or ST into $\mathrm{R}^{\tau}(A,B)$.

Letting $b = 1$ in Theorem 1 gives:

Corollary 3

Let $a \in \mathrm{C}\setminus\{0\}$ and $c \in \mathrm{C}$ satisfy

$$\Re c > \max\{7, 2\Re a + 5\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a - 5)\Gamma(\Re c - 7)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - 1)|}[120\gamma\mu\delta|(a)_5| + 24(\mu\delta + \mu\gamma - \gamma\delta + 13\gamma\mu\delta)|(a)_4|$$

$$\{(\Re c - 2\Re a - 5)(\Re c - 7)\}^{\frac{1}{2}} + 6(\mu - \delta + \gamma + 9\mu\delta + 8\gamma\mu - 8\gamma\delta + 46\gamma\mu\delta)|(a)_3|$$

$$\{(\Re c - 2\Re a - 5)_2(\Re c - 7)_2\}^{\frac{1}{2}} + 2(1 + 5\mu - 5\delta + 5\gamma + 19\mu\delta + 14\gamma\mu - 14\gamma\delta + 46\gamma\mu\delta)|(a)_2|$$

$$\{(\Re c - 2\Re a - 5)_3(\Re c - 7)_3\}^{\frac{1}{2}} + (3 + 4\mu - 4\delta + 4\gamma + 8\mu\delta + 4\gamma\mu - 4\gamma\delta + 8\gamma\mu\delta)|a|$$

$$\{(\Re c - 2\Re a - 5)_4(\Re c - 7)_4\}^{\frac{1}{2}} + \{(\Re c - 2\Re a - 5)_5(\Re c - 7)_5\}^{\frac{1}{2}}] \le 1 + \frac{(A-B)|\tau|}{1+|B|}$$

is satisfied, then $\mathrm{L}(a,c)$ maps the class S (or ST) into the class $\mathrm{R}_{\gamma}^{\tau}(A,B)$.

Remark 1

Taking $\mu = \delta = \gamma = 0$ in Corollary 3, we get the result of (Mishra and Panigrahi (2011), Corollary 2, p. 57)

Theorem 2

Let $a, b \in \mathrm{C}\setminus\{0\}$, $p_1 = p_1(k)$ be defined by Equation 13 and $c \in \mathrm{C}$ satisfy

$$\Re c > \max\{0, 2\Re a + p_1 + 3, 2\Re b + p_1 + 3\}. \qquad (24)$$

If the hypergeometric inequality

$$\{{}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\}^{\frac{1}{2}}\{{}_3F_2(b, \bar{b}, p_1; \Re c, 1; 1)\}^{\frac{1}{2}} + (1 + \mu - \delta + \mu\delta + \gamma)\frac{|ab|p_1}{\Re c}$$

$$\{{}_3F_2(a+1, \bar{a}+1, p_1+1; \Re c+1, 2; 1)\}^{\frac{1}{2}}\{{}_3F_2(b+1, \bar{b}+1, p_1+1; \Re c+1, 2; 1)\}^{\frac{1}{2}}$$

$$+ (\mu - \delta + 3\mu\delta + \gamma + 2\mu\gamma - 2\delta\gamma + 4\gamma\mu\delta)\frac{|ab|p_1}{\Re c}\{{}_3F_2(a+1, \bar{a}+1, p_1+1; \Re c+1, 1; 1)\}^{\frac{1}{2}}$$

$$\{{}_3F_2(b+1, \bar{b}+1, p_1+1; \Re c+1, 1; 1)\}^{\frac{1}{2}} + (\mu\delta + \gamma\mu - \gamma\delta + 4\gamma\mu\delta)\frac{|(a)_2||(b)_2|(p_1)_2}{(\Re c)_2}$$

$$\{{}_3F_2(a+2, \bar{a}+2, p_1+2; \Re c+2, 2; 1)\}^{\frac{1}{2}}\{{}_3F_2(b+2, \bar{b}+2, p_1+2; \Re c+2, 2; 1)\}^{\frac{1}{2}} + \gamma\mu\delta\frac{|(a)_2||(b)_2|(p_1)_2}{(\Re c)_2}$$

$$\{{}_3F_2(a+2, \bar{a}+2, p_1+2; \Re c+2, 1; 1)\}^{\frac{1}{2}}\{{}_3F_2(b+2, \bar{b}+2, p_1+2; \Re c+2, 1; 1)\}^{\frac{1}{2}}$$

$$\le 1 + \frac{(A-B)|\tau|}{1+|B|} \qquad (25)$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $k-\mathrm{ST}$ into $\mathrm{R}_{\gamma}^{\tau}(A,B)$.

Proof

Let the function $f \in \mathrm{A}$ given by Equation 1 be in class $k-\mathrm{ST}$. In view of Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n[1 + \gamma(n-1)][1 + (n-1)(\mu - \delta + n\mu\delta)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|$$

$$\le \frac{(A-B)|\tau|}{1+|B|}.$$

Using the coefficient estimate of Equation 14 and the elementary inequality $|(c)_p| > (\Re c)_p$ $(p \in \mathrm{N})$, it is again sufficient to show that

$$S_2 = \sum_{n=2}^{\infty} n[1 + \gamma(n-1)][1 + (n-1)(\mu - \delta + n\mu\delta)]\frac{|(a)_{n-1}(b)_{n-1}|(p_1)_{n-1}}{(\Re c)_{n-1}(1)_{n-1}(1)_{n-1}}$$

$$\le \frac{(A-B)|\tau|}{1+|B|}. \qquad (26)$$

Now,

$$S_2 = \sum_{n=1}^{\infty}(n+1)(1+n\gamma)[1 + n(\mu - \delta) + n(n+1)\mu\delta]\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}$$

$$= \sum_{n=1}^{\infty}[1+n(1+\mu-\delta+\mu\delta+\gamma)+n^2(\mu-\delta+2\mu\delta+\gamma+\mu\gamma-\delta\gamma+\gamma\mu\delta)$$
$$+n^3(\mu\delta+\mu\gamma-\delta\gamma+2\gamma\mu\delta)+\gamma\mu\delta n^4]\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n} = \left(\sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}-1\right)$$
$$+(1+\mu-\delta+\mu\delta+\gamma)\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_{n+1}(1)_n}+(\mu-\delta+2\mu\delta+\gamma+\mu\gamma-\delta\gamma+\gamma\mu\delta)$$
$$\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_n(1)_n}+(\mu\delta+\mu\gamma-\delta\gamma+2\gamma\mu\delta)\sum_{n=0}^{\infty}\frac{|(a)_{n+2}(b)_{n+2}|(p_1)_{n+2}}{(\Re c)_{n+2}(1)_{n+1}(1)_n}$$
$$+(\mu\delta+\mu\gamma-\delta\gamma+2\gamma\mu\delta)\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_n(1)_n}+\gamma\mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+2}(b)_{n+2}|(p_1)_{n+2}}{(\Re c)_{n+2}(1)_n(1)_n}+2\gamma\mu\delta$$
$$\sum_{n=0}^{\infty}\frac{|(a)_{n+2}(b)_{n+2}|(p_1)_{n+2}}{(\Re c)_{n+2}(1)_{n+1}(1)_n}+\gamma\mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_n(1)_n} = \sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}+(1+\mu-\delta$$
$$+\mu\delta+\gamma)\frac{|ab|\,p_1}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}+(\mu-\delta+3\mu\delta+\gamma+2\mu\gamma-2\delta\gamma+4\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}$$
$$\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}+(\mu\delta+\gamma\mu-\gamma\delta+4\gamma\mu\delta)\frac{|(a)_2(b)_2|(p_1)_2}{(\Re c)_2}\sum_{n=0}^{\infty}\frac{|(a+2)_n(b+2)_n|(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}$$
$$+\gamma\mu\delta\frac{|(a)_2(b)_2|(p_1)_2}{(\Re c)_2}\sum_{n=0}^{\infty}\frac{|(a+2)_n(b+2)_n|(p_1+2)_n}{(\Re c+2)_n(1)_n(1)_n}-1. \tag{27}$$

Applications of Cauchy-Schwarz inequality to individual sum in Equation 27 give

$$S_2 \le \left\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}+(1+\mu-\delta+\mu\delta+\gamma)\frac{|ab|\,p_1}{\Re c}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}+(\mu-\delta+3\mu\delta+\gamma$$
$$+2\mu\gamma-2\delta\gamma+4\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$+(\mu\delta+\gamma\mu-\gamma\delta+4\gamma\mu\delta)\frac{|(a)_2\|(b)_2|(p_1)_2}{(\Re c)_2}\left\{\sum_{n=0}^{\infty}\frac{(a+2)_n(\bar{a}+2)_n(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+2)_n(\bar{b}+2)_n(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$+\gamma\mu\delta\frac{|(a)_2\|(b)_2|(p_1)_2}{(\Re c)_2}\left\{\sum_{n=0}^{\infty}\frac{(a+2)_n(\bar{a}+2)_n(p_1+2)_n}{(\Re c+2)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+2)_n(\bar{b}+2)_n(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}-1.$$

Since the condition in Equation 24 holds, the aforementioned summation can be written as evaluation of generalized hypergeometric functions and we get

$$S_2 \le \{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}+(1+\mu-\delta+\mu\delta+\gamma)$$
$$\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$

$$+(\mu-\delta+3\mu\delta+\gamma+2\mu\gamma-2\delta\gamma+4\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,\,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}$$

$$\{{}_3F_2(b+1,\bar{b}+1,\,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}+(\mu\delta+\gamma\mu-\gamma\delta+4\gamma\mu\delta)\frac{|(a)_2||(b)_2|(p_1)_2}{(\Re c)_2}$$

$$\{{}_3F_2(a+2,\bar{a}+2,\,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,\,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}+\gamma\mu\delta$$

$$\frac{|(a)_2||(b)_2|(p_1)_2}{(\Re c)_2}\{{}_3F_2(a+2,\bar{a}+2,\,p_1+2;\Re c+2,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,\,p_1+2;\Re c+2,1;1)\}^{\frac{1}{2}}-1.$$

Therefore, in view of Equation 26, if the hypergeometric inequality (Equation 25) is satisfied, then $H^{a,b,c}_{\mu,\delta}(f)\in \mathrm{R}^{\tau}_{\gamma}(A,B)$. The proof of Theorem 2 is complete.

Putting $\mu=\delta=\gamma=0$ in Theorem 3 after simplification, we get the following result.

Corollary 4

Let $a,b\in \mathbb{C}\setminus\{0\}$, $p_1=p_1(k)$ be defined by Equation 13 and $c\in\mathbb{C}$ satisfy (Mishra and Panigrahi, 2011), Theorem 2(i), p. 57)

$$\Re c>\max\{0,2\Re a+p_1,2\Re b+p_1\}.$$

If the hypergeometric inequality

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}+\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,\,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$\{{}_3F_2(b+1,\bar{b}+1,\,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\le 1+\frac{(A-B)|\tau|}{1+|B|},$$

is satisfied, then $\mathrm{I}^{a,b}_{c}$ maps the class $k-\mathrm{ST}$ into the class $\mathrm{R}^{\tau}_{\gamma}(A,B)$.

Taking $b=\bar{a}$ in Theorem 2, we have the following.

Corollary 5

Let $a\in \mathbb{C}\setminus\{0\}$, $p_1=p_1(k)$ be defined by Equation 13 and $c\in\mathbb{C}$ satisfy

$$\Re c>\max\{0,2\Re a+p_1+3\}.$$

If the hypergeometric inequality

$${}_3F_2(a,\bar{a},p_1;\Re c,1;1)+(1+\mu-\delta+\mu\delta+\gamma)\frac{|a|^2\,p_1}{\Re c}{}_3F_2(a+1,\bar{a}+1,\,p_1+1;\Re c+1,2;1)$$
$$+(\mu-\delta+3\mu\delta+\gamma+2\mu\gamma-2\delta\gamma+4\gamma\mu\delta)\frac{|a|^2\,p_1}{\Re c}{}_3F_2(a+1,\bar{a}+1,\,p_1+1;\Re c+1,1;1)+(\mu\delta$$
$$+\gamma\mu-\gamma\delta+4\gamma\mu\delta)\frac{|(a)_2|^2\,(p_1)_2}{(\Re c)_2}{}_3F_2(a+2,\bar{a}+2,\,p_1+2;\Re c+2,2;1)+\gamma\mu\delta\frac{|(a)_2|^2\,(p_1)_2}{(\Re c)_2}$$
$${}_3F_2(a+2,\bar{a}+2,\,p_1+2;\Re c+2,1;1)\le 1+\frac{(A-B)|\tau|}{1+|B|}$$

is satisfied, then $H^{a,\bar{a},c}_{\mu,\delta}$ maps the class $k-\mathrm{ST}$ into the class $\mathrm{R}^{\tau}_{\gamma}(A,B)$.

Letting $\mu=\delta=\gamma=0$ in Corollary 5 and after simplification we get the following result due to Mishra and Panigrahi (2011):

Corollary 6

Let $a\in \mathbb{C}\setminus\{0\}$, $p_1=p_1(k)$ be defined by Equation 13 and $c\in\mathbb{C}$ satisfy (Mishra and Panigrahi, 2011, Corollary 3, p. 59)

$$\Re c>\max\{0,2\Re a+p_1\}.$$

If the hypergeometric inequality

$${}_3F_2(a,\bar{a},p_1;\Re c,1;1)+\frac{|a|^2\,p_1}{\Re c}{}_3F_2(a+1,\bar{a}+1,\,p_1+1;\Re c+1,2;1)\le 1+\frac{(A-B)|\tau|}{1+|B|},$$

is satisfied, then $\mathrm{I}^{a,\bar{a}}_{c}$ maps the class $k-\mathrm{ST}$ into the class $\mathrm{R}^{\tau}(A,B)$.

Corollary 7

Let $a\in \mathbb{C}\setminus\{0\}$, $p_1=p_1(k)$ be defined by Equation

13 and $c \in \mathbb{C}$ satisfy the inequality (Mishra and Panigrahi, 2011, Corollary 4, p. 59)

$$\Re c > \max\{2 + p_1, 2\Re a + p_1\}.$$

If the hypergeometric inequality

$$\left[\frac{(\Re c - 1)}{(\Re c - p_1 - 1)}\right]^{\frac{1}{2}}\left[\{{}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\}^{\frac{1}{2}} + \frac{|a|\, p_1}{\{\Re c(\Re c - p_1 - 2)\}^{\frac{1}{2}}}\{{}_3F_2(a+1, \bar{a}+1, p_1+1; \Re c+1, 2; 1)\}^{\frac{1}{2}}\right] \le 1 + \frac{(A-B)|\tau|}{1+|B|}$$

is satisfied, then $L(a,c)$ maps the class $k-ST$ into $R^{\tau}(A,B)$.

Proof

Take $b = 1$ in Corollary 4. Using summation formula (Equation 7), we have ${}_3F_2(1,1,p_1;\Re c,1;1) = {}_2F_1(1,p_1;\Re c;1) = \frac{\Gamma(\Re c)\Gamma(\Re c - p_1 - 1)}{\Gamma(\Re c - 1)\Gamma(\Re c - p_1)}$

$$= \frac{\Re c - 1}{\Re c - p_1 - 1},$$

$$\begin{aligned} {}_3F_2(2,2,p_1+1;\Re c+1,2;1) &= {}_2F_1(2,p_1+1;\Re c+1;1) \\ &= \frac{\Gamma(\Re c+1)\Gamma(\Re c - p_1 - 2)}{\Gamma(\Re c - 1)\Gamma(\Re c - p_1)} \\ &= \frac{(\Re c)(\Re c - 1)}{(\Re c - p_1 - 1)(\Re c - p_1 - 2)}. \end{aligned}$$

Hence, the result follows.

Theorem 4

Let $a, b \in \mathbb{C}\setminus\{0\}$, $p_1 = p_1(k)$ be defined by Equation 13 and $c \in \mathbb{C}$ satisfy

$$\Re c > \max\{0, 2\Re a + p_1 + 2, 2\Re b + p_1 + 2\}. \quad (28)$$

If the hypergeometric inequality

$$\begin{aligned} &\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} + (1+\mu-\delta+4\mu\delta)\frac{|ab|\,p_1}{\Re c} \\ &\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}} \\ &+(\mu-\delta)\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}} \\ &+\mu\delta\frac{|(a)_2(b)_2|(p_1)_2}{(\Re c)_2}\{{}_3F_2(a+2,\bar{a}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}} \\ &+3\mu\delta\frac{|(a)_2(b)_2|(p_1)_2}{2(\Re c)_2}\{{}_3F_2(a+2,\bar{a}+2,p_1+2;\Re c+2,3;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,p_1+2;\Re c+2,3;1)\}^{\frac{1}{2}} \le 2 \end{aligned} \quad (29)$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $k-ST$ into ST.

Proof

The proof follows the same line to that of Theorem 2. In this case we use Lemma 4 instead of Lemma 2. The proof of Theorem 4 is complete.

Takiing $\mu = \delta = 0$ in Theorem 4 we get the following.

Corollary 8

Let $a, b \in \mathbb{C}\setminus\{0\}$, $p_1 = p_1(k)$ be defined by Equation 13 and $c \in \mathbb{C}$ satisfy (Mishra and Panigrahi, 2011, Theorem 2(ii), p. 58)

$$\Re c > \max\{0, 2\Re a + p_1, 2\Re b + p_1\}.$$

If the hypergeometric inequality

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} + \frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}} \le 2$$

is satisfied, then $I_c^{a,b}$ maps the class $k-ST$ into ST.

Theorem 5

Let $a, b \in \mathbb{C}\setminus\{0\}$, $p_1 = p_1(k)$ be defined by Equation 13 and

$c \in \mathbb{C}$ satisfy

$$\Re c > \max\{0, 2\Re a + p_1 + 2, 2\Re b + p_1 + 2\}. \qquad (30)$$

If the hypergeometric inequality

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} + (\mu-\delta+\mu\delta+\gamma)\frac{|ab|\,p_1}{\Re c}$$
$$\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$+(\mu\delta+\gamma\mu-\gamma\delta+2\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}$$
$$\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}} + \gamma\mu\delta\frac{|(a)_2||(b)_2|(p_1)_2}{(\Re c)_2}$$
$$\{{}_3F_2(a+2,\bar{a}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}$$

$$\leq 1 + \frac{(A-B)|\tau|}{1+|B|} \qquad (31)$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class $k-\mathrm{UCV}$ into $\mathrm{R}_{\gamma}^{\tau}(A,B)$.

Proof

Let the function f given by Equation 1 be a member of $k-\mathrm{UCV}$. The proof follows the same line to that of Theorem 1. Making use of Lemma 2, the coefficient estimate (Equation 15) for a_n and the elementary inequality $|(c)_p| > (\Re c)_p$, it is sufficient to show that:

$$S_3 = \sum_{n=2}^{\infty} n[1+\gamma(n-1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p_1)_{n-1}}{(1)_n}\frac{|(a)_{n-1}(b)_{n-1}|}{(\Re c)_{n-1}(1)_{n-1}}$$
$$\leq \frac{(A-B)|\tau|}{1+|B|}. \qquad (32)$$

The term S_3 can be equivalently written as

$$S_3 = \sum_{n=1}^{\infty}[1+n(\mu-\delta)+n(n+1)\mu\delta+n\gamma+n^2\gamma(\mu-\delta)+n^2(n+1)\gamma\mu\delta]\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}$$
$$= \sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n} + (\mu-\delta+\mu\delta+\gamma)\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_{n+1}(1)_n} + (\mu\delta+\gamma\mu-\gamma\delta+2\gamma\mu\delta)$$
$$\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_n(1)_n} + \gamma\mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+2}(b)_{n+2}|(p_1)_{n+2}}{(\Re c)_{n+2}(1)_{n+1}(1)_n} - 1.$$

An applications of Cauchy-Schwarz inequality and the relation $\overline{(d)_n} = (\bar{d})_n \quad (n \in \mathbb{N}_0)$ for any complex number d to the individual sum give

$$S_3 \leq \left\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}} + (\mu-\delta+\mu\delta+\gamma)\frac{|ab|\,p_1}{\Re c}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}} + (\mu\delta+\gamma\mu-\gamma\delta+2\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}} + \gamma\mu\delta\frac{|(a)_2(b)_2|(p_1)_2}{(\Re c)_2}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(a+2)_n(\bar{a}+2)_n(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b+2)_n(\bar{b}+2)_n(p_1+2)_n}{(\Re c+2)_n(2)_n(1)_n}\right\}^{\frac{1}{2}} - 1. \qquad (33)$$

Since the condition (Equation 13) holds which ensure that sum in the r.h.s of Equation 33 are convergent hypergeometric series. Therefore,

$$S_3 \leq \{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} + (\mu-\delta+\mu\delta+\gamma)\frac{|ab|\,p_1}{\Re c}$$
$$\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$+(\mu\delta+\gamma\mu-\gamma\delta+2\gamma\mu\delta)\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}$$
$$+\gamma\mu\delta\frac{|(a)_2(b)_2|(p_1)_2}{(\Re c)_2}\{{}_3F_2(a+2,\bar{a}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+2,\bar{b}+2,p_1+2;\Re c+2,2;1)\}^{\frac{1}{2}} - 1.$$

Hence, in view of Equation 32, if the hypergeometric inequality (Equation 31) is satisfied, then $H_{\mu,\delta}^{a,b,c}(f) \in \mathrm{R}_{\gamma}^{\tau}(A,B)$ as asserted. This complete the proof of Theorem 5.

Putting $\mu = \delta = \gamma = 0$ in Theorem 5, then we have:

Corollary 9

Let $a,b \in \mathbb{C}\setminus\{0\}$, $p_1 = p_1(k)$ be defined by (2.3) and $c \in \mathbb{C}$ satisfy (Mishra and Panigrahi, 2011, Theorem 3(i), p.60)

$$\Re c > \max\{0, 2\Re a + p_1 - 1, 2\Re b + p_1 - 1\}.$$

If the hypergeometric inequality

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} \leq 1 + \frac{(A-B)|\tau|}{1+|B|}$$

is satisfied, then $\mathrm{I}_c^{a,b}$ maps the class $k-\mathrm{UCV}$ into the class $\mathrm{R}^{\tau}(A,B)$.

Corollary 10

Let the complex numbers a,b and c be as in Theorem 5 and further satisfy

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}+(\mu-\delta+\mu\delta)\frac{|ab|\,p_1}{\Re c}$$
$$\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$+\mu\delta\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}$$
$$\leq 1+(1-\beta)cos\eta.$$

Then the operator $H_{\mu,\delta}^{a,b,c}$ maps the class $k-\mathrm{UCV}$ into the class $\mathrm{R}_\eta(\beta)$.

Proof

Taking $A=1-2\beta\ (0\leq\beta<1),\ B=-1,\ \gamma=0,\ \tau=e^{-i\eta}cos\eta$ in Theorem 5 we get the desire result.

Remark 2

Putting $\mu=\delta=0$ in Corollary 10, we get the result of (Mishra and Panigrahi (2011), Corollary 5, p. 61).

Theorem 6

Let $a,b\in\mathrm{C}\setminus\{0\}$, $p_1=p_1(k)$ be defined by Equation 13 and $c\in\mathrm{C}$ satisfy

$$\Re c>\max\{0,2\Re a+p_1+1,2\Re b+p_1+1\}. \qquad (34)$$

If the hypergeometric inequality

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}+(\mu-\delta+\mu\delta)\frac{|ab|\,p_1}{\Re c}$$
$$\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$+\mu\delta\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}$$
$$\leq 2, \qquad (35)$$

is satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps the class of $k-\mathrm{UCV}$ into ST.

Proof

Let the function f given by (1.1) be in the class $k-\mathrm{UCV}$. In view of Lemma 4, it is sufficient to show that $\sum_{n=2}^{\infty}n[1+(n-1)(\mu-\delta+n\mu\delta)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 1.$

By making use of Lemma 3 and elementary inequality $|(c)_p|>(\Re c)_p\ (p\in\mathrm{N})$, it is again sufficient to show that:

$$S_4=\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{|(a)_{n-1}(b)_{n-1}|(p_1)_{n-1}}{(\Re c)_{n-1}(1)_{n-1}(1)_{n-1}}\leq 1. \qquad (36)$$

Now

$$S_4=\sum_{n=1}^{\infty}[1+n(\mu-\delta)+n(n+1)\mu\delta]\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}=\sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}$$
$$+(\mu-\delta+\mu\delta)\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_{n+1}(1)_n}+\mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|(p_1)_{n+1}}{(\Re c)_{n+1}(1)_n(1)_n}$$
$$=\sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}+(\mu-\delta+\mu\delta)\frac{|ab|\,p_1}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}$$
$$+\mu\delta\frac{|ab|\,p_1}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}-1.$$

Applications of Cauchy-Schwarz inequality give

$$S_4\leq\left\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}+(\mu-\delta+\mu\delta)\frac{|ab|\,p_1}{\Re c}\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\right\}^{\frac{1}{2}}+\mu\delta\frac{|ab|\,p_1}{\Re c}\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(1)_n(1)_n}\right\}^{\frac{1}{2}}-1.$$

The conditions $\Re c>2\Re a+p_1+1$ and $\Re c>2\Re b+p_1+1$ given in Equation 34 ensure that the sum in the r.h.s of Equation 37 are convergent hypergeometric series so that

$$S_4\leq\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}+(\mu-\delta+\mu\delta)\frac{|ab|\,p_1}{\Re c}$$
$$\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}$$
$$+\mu\delta\frac{|ab|\,p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,1;1)\}^{\frac{1}{2}}-1$$

Therefore, in view of Equation 36 if the inequality (Equation 35) is satisfied, then $H_{\mu,\delta}^{a,b,c}(f)\in\mathrm{ST}$. This complete the proof of Theorem 6.

Remark 3

Putting $\delta=\mu=0$ in Theorem 6, we get the result due to (Mishra and Panigrahi (2011), Theorem 3(ii), p. 60).

Theorem 7

Let $a,b \in \mathbb{C}\setminus\{0\}$ and $c \in \mathbb{C}$ satisfy

$$\Re c > \max\{0, 2\Re a + 2, 2\Re b + 2\}. \tag{38}$$

If the hypergeometric inequality

$$\{{}_2F_1(a,\bar{a};\Re c;1)\}^{\frac{1}{2}}\{{}_2F_1(b,\bar{b};\Re c;1)\}^{\frac{1}{2}} + (\mu-\delta+2\mu\delta)\frac{|ab|}{\Re c}\{{}_2F_1(a+1,\bar{a}+1;\Re c+1;1)\}^{\frac{1}{2}}$$
$$\{{}_2F_1(b+1,\bar{b}+1;\Re c+1;1)\}^{\frac{1}{2}} + \mu\delta\frac{|(a)_2(b)_2|}{(\Re c)_2}\{{}_2F_1(a+2,\bar{a}+2;\Re c+2;1)\}^{\frac{1}{2}}$$
$$\{{}_2F_1(b+2,\bar{b}+2;\Re c+2;1)\}^{\frac{1}{2}} \le 1 + \frac{1}{1+|B|} \tag{39}$$

satisfied, then $H_{\mu,\delta}^{a,b,c}$ maps class of $\mathrm{R}_\gamma^\tau(A,B)$ into $\mathrm{R}_\gamma^\tau(A,B)$.

Proof

Let the function f given by Equation 1 be a member of $\mathrm{R}_\gamma^\tau(A,B)$. By virtue of Lemma 2 and coefficient estimate (Equation 17), it is sufficient to show that

$$(1+|B|)S_5 \le 1, \tag{40}$$

where

$$S_5 = \sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{|(a)_{n-1}(b)_{n-1}|}{(\Re c)_{n-1}(1)_{n-1}}.$$

The term S_5 can equivalently written as

$$S_5 = \sum_{n=1}^{\infty}[1+n(\mu-\delta)+n(n+1)\mu\delta]\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} = \sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$
$$+(\mu-\delta+2\mu\delta)\sum_{n=0}^{\infty}\frac{|(a)_{n+1}(b)_{n+1}|}{(\Re c)_{n+1}(1)_n} + \mu\delta\sum_{n=0}^{\infty}\frac{|(a)_{n+2}(b)_{n+2}|}{(\Re c+2)_{n+1}(1)_n} - 1$$
$$= \sum_{n=0}^{\infty}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} + (\mu-\delta+2\mu\delta)\frac{|ab|}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|}{(\Re c+1)_n(1)_n}$$
$$+\mu\delta\frac{|(a)_2||(b)_2|}{(\Re c)_2}\sum_{n=0}^{\infty}\frac{|(a+2)_n(b+2)_n|}{(\Re c+2)_n(1)_n} - 1.$$

Applications of Cauchy-Schwarz inequality gives

$$S_5 \le \left\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n}{(\Re c)_n(1)_n}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n}{(\Re c)_n(1)_n}\right\}^{\frac{1}{2}} + (\mu-\delta+2\mu\delta)\frac{|ab|}{\Re c}\left\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n}{(\Re c+1)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n}{(\Re c+1)_n(1)_n}\right\}^{\frac{1}{2}} + \mu\delta\frac{|(a)_2||(b)_2|}{(\Re c)_2}\left\{\sum_{n=0}^{\infty}\frac{(a+2)_n(\bar{a}+2)_n}{(\Re c+2)_n(1)_n}\right\}^{\frac{1}{2}}$$
$$\left\{\sum_{n=0}^{\infty}\frac{(b+2)_n(\bar{b}+2)_n}{(\Re c+2)_n(1)_n}\right\}^{\frac{1}{2}} - 1$$

Since the conditions $\Re c > 2\Re a + 2$ and $\Re c > 2\Re b + 2$ given by Equation 38 ensure that the sum in the r.h.s of Equation 19 are convergent hypergeometric series so that

$$S_5 \le \{{}_2F_1(a,\bar{a};\Re c;1)\}^{\frac{1}{2}}\{{}_2F_1(b,\bar{b};\Re c;1)\}^{\frac{1}{2}} + (\mu-\delta+2\mu\delta)\frac{|ab|}{\Re c}\{{}_2F_1(a+1,\bar{a}+1;\Re c+1;1)\}^{\frac{1}{2}}$$
$$\{{}_2F_1(b+1,\bar{b}+1;\Re c+1;1)\}^{\frac{1}{2}} + \mu\delta\frac{|(a)_2||(b)_2|}{(\Re c)_2}\{{}_2F_1(a+2,\bar{a}+2;\Re c+2;1)\}^{\frac{1}{2}}$$
$$\{{}_2F_1(b+2,\bar{b}+2;\Re c+2;1)\}^{\frac{1}{2}} - 1.$$

Thus, in view of Equation 40 if the inequalities (Equation 39) is satisfied, then $H_{\mu,\delta}^{a,b,c}(f) \in \mathrm{R}_\gamma^\tau(A,B)$ as asserted. This ends the proof of Theorem 7.

Conclusion

By making use of Cauchy-Schwarz inequalities, the authors obtain sufficient conditions for a linear operator define by means of normalized hypergeometric function to be certain close to convex class. In this direction, researchers (Bansal, 2013; Mostafa, 2009; Sharma et al., 2013; Sivasubramanian et al., 2011; Swaminathan, 2010; Sudharsan et al., 2014; Sivasubramanian et al., 2013) have already obtained sufficient conditions for vaious class without making use of Cauchy-Schwarz inequlaities.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

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