academicJournals

Vol. 6(5), pp. 118-120, May 2013
DOI: 10.5897/AJMCSR 09.046
ISSN 2006-9731 © 2013 Academic Journals
http://www.academicjournals.org/AJMCSR

African Journal of Mathematics and Computer Science Research

Short Communication

Cardinality and idempotency of partial one-one convex and contraction transformation semi group

Adeniji, A. O.

Department of Mathematics, Faculty of Science, University of Abuja, Nigeria.

Accepted 10 May, 2013

Let X_n be a set with finite number of elements following natural ordering of numbers. The formulae for the total number of elements in partial one – one convex and contraction transformation semigroup and its idempotents are obtained and discussed in this paper. The relationship between fix α and idempotency is obtained and stated; an element $_\alpha$ is an idempotent if |Imα |= |f (α) |. Also, idempotents commute and the product of two or more idempotents is an idempotent. An element y is said to be unique if x is an invertible element such that xy = yx = 1, x, y $\in$ S.

Key words: Convex, contraction, fix element, idempotent and partial one – one transformation semigroup.

INTRODUCTION

Let X_n = $\{1,2,3\ldots\}$. Then a (partial) transformation α is said to be one – one if α : Domα $\subseteq x_n \rightarrow$ *Imα* $\subseteq x_n$, such that each element in Domα has a unique image. The transformation is full if Domα = X_n. A transformation α for which | $x\alpha -$ yα | ≤ | x − y |, $\forall x$, *y* Є X_n is said to be a contraction (Howie, 2006). A set *C* $\subseteq X_n$ is said to be convex if *C* has the form [$i, i+t$], for some $i \in X_n$ and 0 ≤ $t \leq n-1$, (Catarino and Higgins, 1999).

The set *S* = $conv_n \wedge cont_n$ is the semigroup of all full convex and contraction transformations. *P* ($conv_n \wedge cont_n$) and *I* ($conv_n \wedge cont_n$) denote the partial and partial one- one convex and contraction transformation semigroups respectively. The semigroup of partial one – one convex and contraction transformation is of interest in this paper. The fix element is defined as f(α) = {x Є X_n: xα = x} and the set of idempotents, *E(S)* = {α Є *S:* α^2 = α}. Various special subsemigroups of full transformation semigroups have been studied by many authors like Ganyushkin and Mazorchuk (2003), Laradji and Umar (2004 and 2007) and Umar (2010).

Howie et al. (1988) showed the relationship between fix α and idempotency for α Є $Sing_n$ and the equivalence relation {(*x*, *y*) Є N^2 : $x\alpha$ = $y\alpha$} is denoted by *Ker*α. *Ker*α - classes are refered to as blocks, where *N* = {1, 2, 3…*n*}. $A_1, A_2 \ldots A_r$ are the blocks of α and $x\alpha$ = c_i, if and only $x \in A_i$. If c_i Є A_i then A_i is a stationary block. He also showed that the number of stationary blocks is equal to fix α and element α is idempotent if holds for partial one – one convex and contraction

E-mail: adeniji4love@yahoo.com. Tel: +2348035793204.

AMS Subject classification: 20M20.

Table 1. Values of elements in *S*.

n/h	0	1	2	3	4	5	6	7	Total
1	1	1							2
2	1	3	2						6
3	1	6	12	2					21
4	1	10	36	16	2				65
5	1	15	80	60	20	2			178
6	1	21	150	160	90	24	2		448
7	1	28	252	350	280	126	28	2	1067

transformation semigroup. The method employed in this work was by listing and studying the elements using combinatorial approach.

Cardinality and idempotency of partial one – one convex and contraction transformation semigroup

Let *S* = *I* ($conv_n \ \Lambda \ cont_n$). The number of elements in *S* is denoted by |S| and the following results were obtained.

Theorem 1

Let $\alpha \in S$ and $n \geq 2$. Then | *S* |= $\sum_{k=0}^{n-2} 2(k+1)\binom{n}{k} + \frac{n}{2}(n+1) + 1$.

Proof

k- images, $y_1, y_2, ...y_n$ of α can be chosen in $\binom{n}{k}$ ways and each image has a pre- image in a one – one fashion. In partial one – one transformation, there is always an empty map for each n. Combinatorially, the empty map can be written as $\binom{n}{0}$. Also it is known that the length of image and domain of α is the same in *I* ($conv_n \ \Lambda \ cont_n$) which is a unique property of partial one – one transformation and its subsets. Let h =|*Im*α| then h can be partitioned into three different simple parts as follows:

*h = 0 *implies an empty map.*

*$h = 1$ means the single element maps are $\frac{n(n+1)}{2}$ in number for the semi-group under consideration.
*The number of elements for which $n \geq 2$ is $\sum_{k=0}^{n-2} 2(k+1)\binom{n}{k}$.

Hence *S* = $\binom{n}{0}$ + $\frac{n(n+1)}{2}$ + $\sum_{k=0}^{n-2} 2(k+1)\binom{n}{k}$ as shown in the following table of values.

Lemma 1

Let α Є *S*, then α is an idempotent if | Imα |= |*f* (α) | (Table 1).

Proof

Let $X_n = \{a_1, a_2, ...a_n\}$ and $\alpha_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_3 & a_2 \end{pmatrix}$.

Here,| *Imα* | = 3 and |*f* (α) |= 2 but $\alpha_1^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \neq \alpha_1$.

Thus α_1 is not an idempotent. Also, let $\alpha_2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_2 & a_3 \end{pmatrix}$ where | *Imα* |= 2, |*f* (α) |= 2 and $\alpha_2^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_2 & a_3 \end{pmatrix} = \alpha_2$.

Thus α_2 is an idempotent.

Theorem 2

|*E*(*S*) |= $\frac{n(n+1)}{2} + 1$.

Proof

Let α be a transformation in *S*. From Lemma 1, α is an idempotent element in *S* if |*Imα* |= | *f* (*α*)|. This fact holds for an empty map. The number of elements atisfying the stated properly is $\frac{1}{2}(n^2 + n + 2) \equiv \frac{n(n+1)}{2} + 1$.

Lemma 2

Idempotents commute in *S.*

Proof

Let fg and *h*= $(fg)^{-1}$ be two idempotents in *S.*

Fg.h= $fg(fg)^{-1} = fg(g^{-1}f^{-1}) = f(gg^{-1})f^{-1} = 1$.
Also, *h.fg* $= (fg)^{-1}fg = (g^{-1}f^{-1})fg = g^{-1}(f^{-1}f)g = 1$.
Thus, idempotents in *S* commute.
For example, let $\alpha_{1,}\ \alpha_2 \in E(S)\ such\ that,$
$\alpha_{1,} = \binom{a\ b}{a\ b}, \alpha_2 = \binom{b\ c}{b\ c}, a, b, c \in X_n$.
$\alpha_1 . \alpha_2 = \binom{b}{b}$ and $\alpha_2 . \alpha_1 = \binom{b}{b}$.
Thus $\alpha_1 . \alpha_2 = \alpha_2 . \alpha_1$.
An element $x \in S$ is called invertible or a unit provided that there exists $y \in S$ such that $xy = yx = 1$.

Lemma 3

Let x be an invertible element in S such that $xy = yx = 1, y \in S$. Then y is unique.

Proof

Assume that y_1 and y_2 are two different elements and 1 is an identity element in S, $y_1 = y_1.1 = y_1(xy_2) = (y_1x)y_2$.
Since x is an invertible element, then $y_1x = 1$
$\Rightarrow y_1 (x)y_2 = 1.y_2 = y_2$.
$\Rightarrow y_1 = y_2$.

Lemma 4

Let $x = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}$. Then x is invertible.

Proof

Since $x.x = 1$, then x is a unit of itself. There is only one invertible element in *S* with the property $x.x = 1$.

Theorem 3

The product of two or more idempotents in *S* is an idempotent.

Proof

Let fg and $h = (fg)^{-1}$ be idempotents. Then $fg.h.fg = fg(fg)^{-1}fg = fg(g^{-1}f^{-1})fg = f(gg^{-1})(ff^{-1})g = fg$.
Also, $h.fg.h = (fg)^{-1}.fg.(fg)^{-1} = g^{-1}f^{-1}(fg)g^{-1}f^{-1}$
$= g^{-1}(f^{-1}f)(gg^{-1})f^{-1} = g^{-1}f^{-1} = (fg)^{-1}$.
Thus, $(fg)(fg)^{-1} = f(gg^{-1})f^{-1} = ff^{-1} = 1$ which is an idempotent.

Conclusively, for S = I(Conv$_n$ ΛCont$_n$),
$|S| = \sum_{k=0}^{n-2} 2(k+1)\binom{n}{k} + \frac{n}{2}(n+1) + 1, n \geq 2;$
$|E(S)| = \frac{n(n+1)}{2} + 1, \forall\, n.$

Idempotents commute in S; the product of two or more idempotents in S is an idempotent and y is unique if x is an invertible element in S such that $xy = yx = 1, y \in S$.

REFERENCES

Catarino PM, Higgins PM (1999). The monoid of orientation – preserving mappings on a chain, Semigroup Forum 58:190-206.
Ganyushkin O, Mazorchuk V (2003). On the structure of IO_n, Semigroup Forum 66:455-483.
Howie JM (2006). Semigroup of mappings, Technical Report series TR 357:1-30.
Howie JM, Robertson EF, Schein BM (1988). A combinatorial property of finite full transformation Semigroups. Proc. Royal Soc. Edinburgh 109A:319-328.
Laradji A, Umar A (2004). On the number of nilpotents in the partial symmertric semigroups. Commun. Algebra 32(8):3017-3032.
Laradji A, Umar A (2007). Combinatorial results for the symmetric inverse semigroup. Semigroup Forum 75:221-236.
Umar A (2010). Some combinatorial problems in the theory of symmetric inverse semigroups. Algebra Discrete Math. 9:115-126.