African Journal of Mathematics and Computer Science Research Vol. 4(9), pp. 294-299, 9 September, 2011
Available online at http://www.academicjournals.org/AJMCSR
ISSN 2006-9731 ©2011 Academic Journals

Full Length Research Paper

Projective semimodules

Huda Mohammed J. Al-Thani

Department of Mathematics and Physics, Faculty of Arts and Sciences, University of Qatar, Doha Qatar.
E-mail: h.m.althani@qu.edu.qa.

Accepted 16 May, 2011

In this paper, a natural dual of an injective semimodule which was obtained by reversing the arrows in the definition of injective semimodule and replacing epimorphsims with monomorphisms was considered. These new objects are called "projective semimodules".

Key words: Projective semimodule, epimorphisim, left regular exact sequence, proper exact sequence.

INTRODUCTION

In any category χ , an object $P \in \chi$ is said to be 'projective' if every arrow $h : P \longrightarrow N$ in χ factors is through any epimorphism $f : M \longrightarrow N$ in χ : that is, there exists an arrow $g : P \longrightarrow M$ in χ with $h = f\ g$. Projective objects are dual to 'injective' objects, which are defined by reversing the arrows in the foregoing definition and replacing epimorphism with monomorphism. In contrast to the category of modules for rings, an epimorphism of semimodules must be surjective. This has led to some confusion between surjective morphisms and epimorphisms in the category of semimodules (Takahashi and Wang, 1982). This confusion has in turn led to an unnatural definition of projective semimodules, which are defined in many papers by replacing the epimorphism in the definition with a surjective morphism (Huda, 1995; Golan, 1992; Michihiro, 1983). This confusion was clarified in Takahashi and Wang (1982), where it was proved that an epimorphism of semimodules $f : M \longrightarrow N$ is surjective if and only if it is *i*-regular, that is, $f(M) = \mathrm{Im} f :=$ $\{n \in N : n + f(m) = f(m')$ for some $m, m' \in M\}$.

The aim of this paper is to introduce a more natural dual to injective semimodules. We call these new objects "projective semimodules". In this context, classical projective semimodules will be seen to be weakly projective. Subsequently, after preliminaries, the epimorphism *f* of cancelable semimodules was characterized in terms of Im *f* in epimorphisms over cancelable semimodules, after which the concept of projective semimodules was defined and its structure was studied. Also, an example of a weakly projective semimodule that is not a projective semimodule was given. Finally, projective semimodules were characterized over the class of cancelable semimodules.

PRELIMINARIES

Throughout this paper, *R* denotes a semiring with identity 1 and all semimodules *M* are unitary left *R*-semimodules: that is, $1.m = m \forall m \in M$. We recall here (Huda, 1995, 2002, 2003; Golan, 1992; Michichiro, 1982), the following facts and definitions:

1. Let $f : M \longrightarrow N$ be a homomorphism of semimodules. The subsemimodule $\mathrm{Im} f$ of *N* is defined as Im $f = \{n \in N : n + f(m') = f(m)$ for some $m, m' \in M\}$. Then f is *i-regular* if $f(M) =$ Im f ; f is *k-regular* if for any $m, m' \in M$, $f(m) = f(m')$ implies $m + k = m' + k'$ for some $k, k' \in$ Ker f ; f is *regular* if it is both *i*-regular and *k*-regular; *f* is a monomorphism if and only if whenever *g* and *g'* are distinct *R*-homomorphisms *M'* $\longrightarrow$ *M* for some left *R*-semimodules *M,* then *f g* ≠ *f g*'; and the dual *f* is *epic* if and only if whenever *g* and *g'* are distinct *R*-homomorphisms *N* $\longrightarrow$ *N'* for left *R*-semimodules *N',*

then $g f \neq g' f$.

2. P is a 'weak projective semimodule' if and only if for each surjective R- homomorphism

$f: M \longrightarrow N$ the induced homomorphism $\bar{f}$: Hom (P, M) $\longrightarrow$ Hom (P, N) is surjective.

3. The sequence $K \xrightarrow{h} M \xrightarrow{f} N$ is an 'exact sequence' if Ker f = Im h and 'properly exact' if Ker $f = h(K)$.

4. A short sequence

$0 \longrightarrow K \xrightarrow{h} M \xrightarrow{f} N \longrightarrow 0$ is 'left regular' if h is regular.

5. A semimodule M is cancelable if for all $m, m', m'' \in M$, $m + m' = m + m'' \Rightarrow m' = m''$.

Remark 1

For potential applications, it is noted that semimodules are important tools over semirings for studying the properties of semirings, and the latter arise in diverse areas of applied mathematics, including optimization theory, automata theory, mathematical modeling and parallel computation systems (Golan, 1992).

EPIMORPHISMS OVER CANCELABLE SEMIMODULES

The following result characterizes an epimorphism f of cancelable semimodules in terms of Im f.

Proposition 1

If $f: M \longrightarrow N$ is an R-homomorphism between cancelable R-semimodules, then f is an epimorphism if and only if Im $f = N$.

Proof

Let f be an epimorphism. Set N' = Im f. Then we have R–homomorphisms from N to N / N' defined by $g : n \mapsto 0 / N'$ and $g' : n \mapsto n / N'$. Moreover, $g f = g' f$. Since f is epic, $g = g'$. Hence, $n / N' = 0 / N'$ for every $n / N' \in N / N'$. Therefore, for every $n \in N$, there exist $n'_1, n'_2 \in N'$ such that $n + n'_1 = n'_2$ where $n'_1, n'_2 \in N'$. Thus, Im $f = N$. Conversely, it can be assumed that Im $f = N$ and $g, g' : N \longrightarrow N'$ are distinct R-homomorphisms to some left semimodule N' such that $g f = g' f$. Since Im $f = N$, it follows that for every $n \in N$, there exist m, m' such that $n + f(m) = f(m')$. Hence, $g(n) + gf(m) = gf(m')$ and $g'(n) + g'f(m) = g'f(m')$. Since N' is cancelable, $g'(n) = g(n)$. Therefore, $g = g'$ shows that f is epic.

The following example shows that epimorphisms of the class of cancelable semimodules need not to be surjective.

Example 1

The set N of non-negative integers and the set Z of integers are considered with the usual operations of addition and multiplication of integers. Clearly, both N and Z are cancelable semimodules over N. Let $i: N \longrightarrow Z$ be the inclusion homomorphism. It is easy to show that Im $i = Z$, and by using the previous proposition, i is epic. Consequently, i is an epimorphism which is not surjective.

PROJECTIVE SEMIMODULES

Here, the notion of projective semimodules is introduced. Also, some theorems about injective semimodules are dualized. In example 5, it was shown that a weakly projective semimodule does not need to be a projective semimodule. The notion of projective semimodules is concluded by showing that the class of all projective semimodules is closed under direct sum.

Definition 1

A semimodule P is 'projective', relative *to M* (or P is M projective) if for every epimorphism $f: M \longrightarrow N$ and for every homomorphism $g : P \longrightarrow N$ there is a homomorphism $\bar{g} : P \longrightarrow M$ such that the following diagram commutes (Figure 1a). In other words, P is M projective if and only if Hom (P, f): $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is a surjective N-homomorphism for every epimorphism $f : M \longrightarrow N$.

Definition 2

A semimodule *P* is *projective* if it is projective relative to every semimodule.

Example 2

Let *Z* be the set of all integers with usual addition and multiplication. Then *Z* is projective relative to itself over all *N*-cancelable semimodules, where *N* is the set of all non-negative integers.

In module theory, projective and weak projective are equivalent concepts. However, this is not true in semimodules, as shown in example 3.

Proposition 2

Every projective semimodule is a weekly projective semimodule.

Proof

This follows immediately from the fact that every surjective homomorphism is an epimorphism.

Recall that in the case of modules, every free module is projective. The analogue of this result for semimodules does not hold, as shown by the next example. Since every free semimodule is weakly projective, the example also shows that a weak projective semimodule does not need to be projective.

Example 3

Let *N* be the set of all non-negative integers. Clearly *N N* is a free semimodule, so it is a weakly projective *N*-semimodule.

Consider the homomorphism *h*: $N \times N \longrightarrow N \times N$ defined by the rule

$$h(n,m) = (2n+m,n).$$

It can be shown that $(1,1) \notin h(N \times N)$. Thus, *h* is not surjective. Now for every $(n, m) \in N \times N$, there exist $(n', m') \in N \times N$ such that

$$(n, m) + h(n' - m, m' + 2m - n) = h(n', m')$$

where $n' > m, m' > n$. Since $N \times N$ is a cancelable semimodule, *h* is an epimorphism. If $N \times N$ were to be projective, then there would be a homomorphism g: $N \times N \longrightarrow N \times N$ such that $hg = I_{N \times N}$ and *h* would be surjective. Thus, this would have been a contradiction, so $N \times N$ is not projective.

Since the projective is the dual of the injective, the theorems about the injective semimodule in Golan (1992) can be dualized. The proofs are easily established by dualizing the proofs of the analogous results for the injective case.

Proposition 3

Let $(P_\alpha)_{\alpha \in A}$ be an indexed set of left *R*-semimodules. Then, $\oplus P_\alpha$ is projective if and only if each P_α is projective.

Proof

Consider the following diagram (Figure 1b), where f is an epimorphism. Since $\oplus P_\alpha$ is projective, we have the following commutative diagram (Figure 1c): where π_α is the canonical projection: that is, $g\pi_\alpha = fh$. Therefore, we have the next commutative diagram (Figure 1d): where i_α is a canonical injection. Thus, P_α is projective.

Conversely, we consider the diagram (Figure 1e) where f is an epimorphism. For the fact that P_α is projective, we have, for each $\alpha \in A$, the commutative diagram (Figure 1f) and $gi_\alpha = fh_\alpha$. Thus, the diagram in Figure 1g commutes, where

$$h\left(\sum_{\alpha=1}^{n} P_\alpha\right) = \sum_{\alpha=1}^{n} h_\alpha(P_\alpha).$$

It follows that $\bigoplus_{\alpha \in A} P_\alpha$ is projective.

CHARACTERIZATION OF PROJECTIVE SEMIMODULES OVER THE CLASS OF CANCELABLE SEMIMODULES

At this point, the projective semimodules are considered

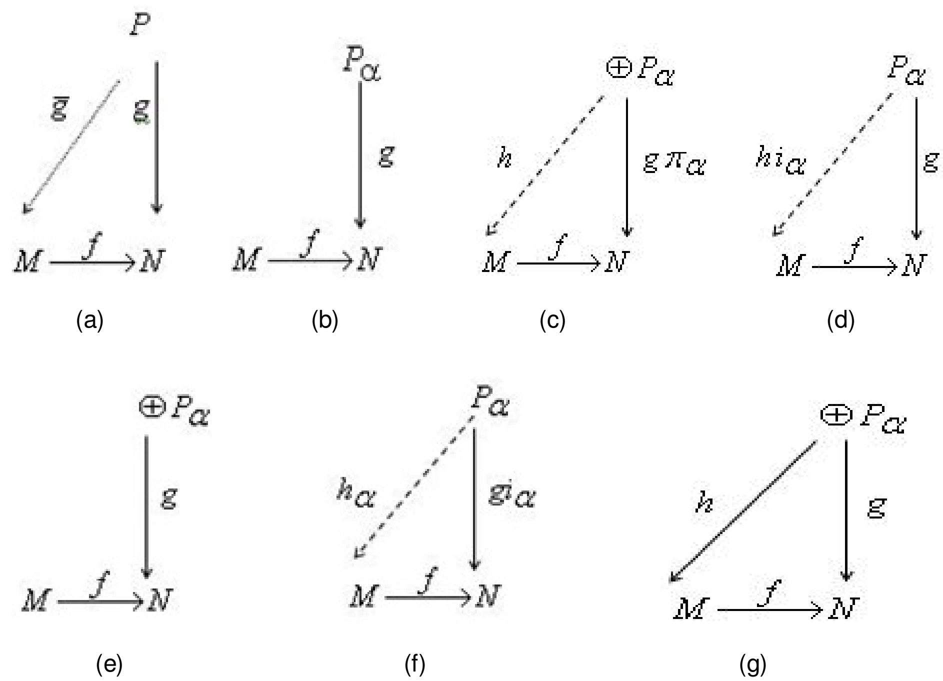


Figure 1. (a) Commutive homomorphism diagram (b) Homomorphism diagram (c) Commutive homomorphism diagram (d) Commutive homomorphism diagram (e) Homomorphism diagram (f) Commutive homomorphism diagram and (g) Commutive homomorphism diagram.

over the class of all cancelable semimodules μ . Theorem 1 asserts that P is projective if and only if Hom (P, -) preserves the exactness of all left regular exact sequences over cancelable semimodules.

Theorem 1

The following conditions on an R-semimodule $P \in \mu$ are equivalent:

(i) P is projective.

(ii) If $0 \longrightarrow K \xrightarrow{h} M \xrightarrow{f} N \longrightarrow 0$ is any left regular exact sequence of R- semimodules, where N, M, $K \in \mu$, then

$0 \longrightarrow \mathrm{Hom}_R(P, K) \xrightarrow{\bar{h}} \mathrm{Hom}_R(P, M) \xrightarrow{\bar{f}} \mathrm{Hom}(P, N) \longrightarrow 0$ is properly exact.

Proof

(i) $\Rightarrow$ (ii): By Theorem 2.6 in Michichiro (1982), the sequence $0 \longrightarrow \mathrm{Hom}_R(P, K) \xrightarrow{\bar{h}} \mathrm{Hom}_R(P, M) \xrightarrow{\bar{f}} \mathrm{Hom}(P, N)$ is exact and $\bar{h}$ is regular, so the sequence is properly exact. Since Imf = N, f is an epimorphism, and by (i) the sequence $0 \longrightarrow \mathrm{Hom}_R(P, K) \xrightarrow{\bar{h}} \mathrm{Hom}_R(P, M) \xrightarrow{\bar{f}} \mathrm{Hom}(P, N) \longrightarrow 0$ is properly exact.

ii) $\Rightarrow$ (i): If $f: M \longrightarrow N$ is an epimorphism, then clearly Imf = N. If we let K = Ker$f$ and apply (ii) to the exact sequence $0 \longrightarrow K \xrightarrow{\subset} M \xrightarrow{f} N \longrightarrow 0$, the result follows.

The next theorem shows that for the class of all cancelable semimodules, P is M projective if and only if Hom (P, -) preserves the exactness of all exact sequences $M'' \xrightarrow{g} M \xrightarrow{f} M'$, where f is k-regular.

Theorem 2

The following statements for cancelable semimodule P

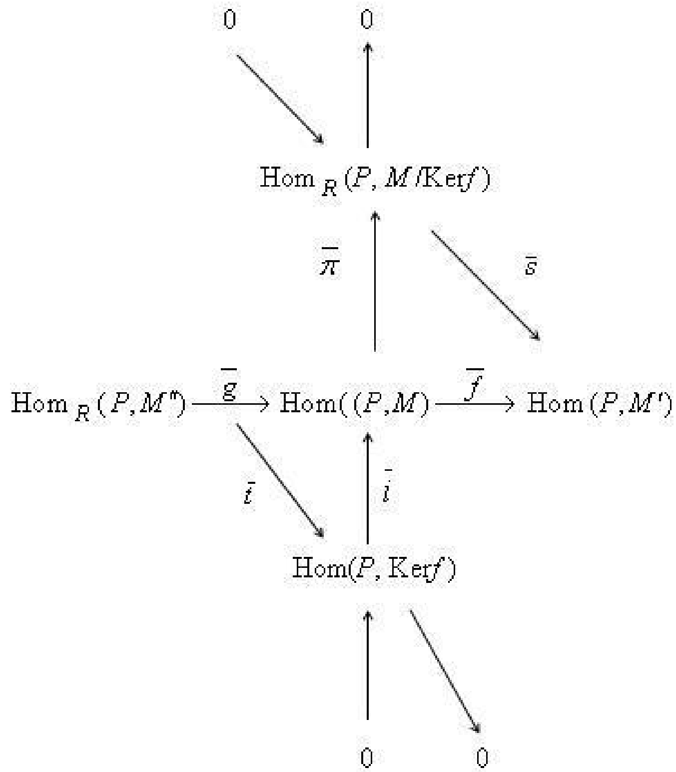


Figure 2. Commutive homomorphism diagram.

are equivalent:

(i) P is projective.
(ii) For every exact sequence of left R-semimodules.

$$M'' \xrightarrow{g} M \xrightarrow{f} M'$$

where f is k-regular and $M'', M, M' \in \mu$, the sequence Hom $(P, M'') \xrightarrow{\bar{g}}$ Hom $(P, M) \xrightarrow{\bar{f}}$ Hom(P, M') is properly exact.

Proof

(ii) $\Rightarrow$ (i): Let f: $M \longrightarrow N$ be an epimorphism. Since $M \xrightarrow{g} N \longrightarrow 0$ is an exact sequence with $N \longrightarrow 0$ k-regular, it follows by (ii) that Hom $(P, M) \longrightarrow$ Hom $(P, N) \longrightarrow 0$ is properly exact and therefore P is projective.

(i) $\Rightarrow$ (ii): Let P be a projective semimodule. Suppose that $M'' \xrightarrow{g} M \xrightarrow{f} M'$ is an exact sequence with f k-regular, the sequence $0 \longrightarrow$ Ker$f$ $\xrightarrow{\bar{i}}$ $M \xrightarrow{\pi} M$ / Ker$f \longrightarrow 0$ is left regular exact. By Theorem 1, the sequence $0 \longrightarrow$ Hom$_R$ $(P$, Ker$f) \xrightarrow{\bar{i}}$ Hom $(P, M) \xrightarrow{\bar{\pi}}$ Hom $(P$, M / Ker f $) \longrightarrow 0$ is properly exact.

Now let t: $M'' \longrightarrow$ Ker$f$ and s: M / Ker$f \longrightarrow M'$ be defined by $t(m'') = g(m'')$ and $s(m$ / Ker$f) = f(m)$. The following diagram commutes (Figure 2): where $\bar{t}(h) = t\,h$ and $s(q) = s\,q, h \in$ Hom$_R$ (P, M'') and $q \in$ Hom$_R$ $(P$, M / Ker$f)$. Now choose $h, q \in$ Hom $(P$, M / Ker$f)$ such that $s(h) = s(q)$. Since η is injective, $h = q$. Clearly, t is an epimorphism. Next, choose $h \in$ Hom$_R$ $(P$, Ker$f)$.

Since P is projective, there exists $k : P \longrightarrow M''$ such that the following diagram commutes (Figure 3a).

Thus, $\bar{t}$ is surjective. Therefore, Figure 3a is a commutative diagram in which the non-horizontal sequences are all properly exact. By corollary 3.2 in Huda (2002), the sequence

Hom $(P, M'') \longrightarrow$ Hom $(P, M) \longrightarrow$ Hom(P, M')

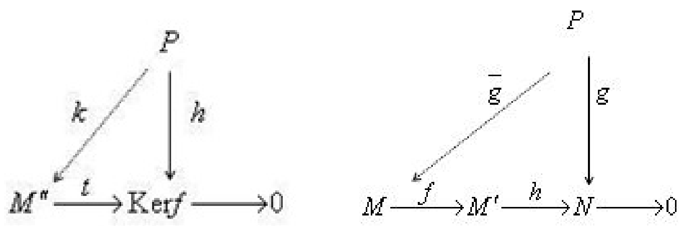
is properly exact. The next result shows that class $\Omega(P)$ of all semimodules $M \in \mu$, where P is M projective, is closed under homomorphic images.

Proposition 4

Let $P \in \mu$. If $M \xrightarrow{f} M' \longrightarrow 0$ is an exact sequence with M, $M' \in \mu$ and P is M projective, then P is projective relative to M'.

Proof

Let h: $M' \longrightarrow N$ be an epimorphism and $g: P \longrightarrow N$ a homomorphism. For the fact that f and h are epimorphisms, $h\,f$ is epic. Also, since P is M projective, there exists a homomorphism $\bar{g}: P \longrightarrow M$ such that



(a) (b)

Figure 3. (a) Commutive homomorphism diagram (b) Commutive homomorphism diagram.

the following diagram commutes (Figure 3b). Therefore, P is M' projective.

REFERENCES

Golan JS (1992). The theory of semirings with applications in mathematics and theoretical computer science, (Pitman, Monographs and surveys in pure and applied mathematics), Longman Scientific & Technical, London.

Huda MA (1995). A note on projective semimodules. Kobe J. Math., 12(2): 89-94.

Huda MA (2002). Characterizations of projective and *k*-projective semimodules, Int. J. Math. Math. Sci., 32(7): 439-448.

Huda MA (2003). Injective Semimodules. J. Inst. Math. Comput. Sci., 16(3): 143-152

Michichiro T (1982). On the Bordism Categories III, Functors Hom and for semimodules. Math. Sem. Notes (Kobe), 10: 211-236.

Michihiro T (1983). Extensions of semimodules II. Math. Sem. Notes, 11: 83-118.

Takahashi M, Wang HM (1982). On Epimorphisms of semimodules. Kobe J. Math., 6: 297-298.