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# More accurate approximate analytical solution of pendulum with rotating support

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**Based on the energy balance method (EBM), a more accurate analytical solution of the pendulum equation with rotating support was presented. The results were compared with those obtained by the differential transformation method (DTM) and He's improved energy balance method. It was shown that the results are more accurate than the said methods.**

**Key words:** Energy balance method, approximate solutions, nonlinear oscillators, pendulum with rotating support.

## INTRODUCTION

Many scientific problems in natural sciences and engineering are inherently nonlinear, but it is difficult to determine their exact solutions. Many analytical methods are available to find their approximate solution. The perturbation methods (Nayfeh, 1973; He, 2006) were originally developed for handling weak nonlinear problems. Recently, some of them were modified (Cheung et al., 1991) to investigate strong nonlinear problems. Homotopy perturbation (Belendez, 2007; Ganji and Sadighi, 2006; Belendez et al., 2008; Ozis and Yildirim, 2007), iteration method (Haque et al., 2013; Jamshidi and Ganji, 2010; Lim et al., 2006; Baghani et al., 2012; Rafei et al., 2007) are useful for obtaining approximate periodic solution with large amplitude of oscillations; however, they are applicable only for odd nonlinearity problems. Harmonic balance method (Mickens, 1986; Lim et al., 2005; Belendez et al., 2006; Wu et al., 2006; Mickens, 2007; Alam et al., 2007; Belendez et al., 2009; Lai et al., 2009; Hosen et al., 2012) is a powerful method in which truncated Fourier series is used. Iterative homotopy harmonic balance (Guo and Leung, 2010), differential transformation (Ghafoori et al., 2011) and max-min (Yazdi et al., 2012) methods have been developed for solving strongly nonlinear oscillators. Energy balance method (He, 2002; Khan and Mirzabeigy, 2014; Alam et al., 2016; Mehdipour et al., 2010; Ebru et al., 2016; Zhang et al., 2009) is another widely used technique for solving strongly nonlinear oscillators. Though, all these analytical methods have been developed for handling nonlinear oscillator, they provide almost similar results for a particular approximation. Recently, EBM has been modified by truncating some higher order terms of the algebraic equations of related variables to the solution (Alam et al., 2016) and it measures more correct result than the usual method. Moreover, the modification on EBM used in Alam et al. (2016) is valid for some nonlinear oscillators, especially when $f(-x) = -f(x)$.

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In this article, the EBM (Alam et al., 2016) was utilized to determine the approximate solution of pendulum equation with rotating support. This type of oscillator was analyzed by Ghafoori et al. (2011) applying differential transformation method (DTM), Belendez et al. (2006) using harmonic balance method and Yazdi et al. (2012) using max-min approach. He (2002) first introduced energy balance method and Khan and Mirzabeigy (2014) was used to improve accuracy of He's energy balance method to obtain the solution of pendulum equation with rotating support. The present method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

## THE BASIC IDEA OF HE'S ENERGY BALANCE METHODS

A general form of nonlinear oscillator is

$$\ddot{x} + f(x) = 0, \quad x(0) = A, \dot{x}(0) = 0, \tag{1}$$

where over dot denotes the derivative with respect to time $t$, $f(x)$ is a nonlinear function such that $f(-x) = -f(x)$.
According to the variational principle, Equation (1) can be written as:

$$J(x) = \int_0^{T/4} [-\frac{1}{2}\dot{x}^2 + F(x)]dt, \tag{2}$$

where $T = 2\pi/\omega$ is a period of the oscillation, $\omega$ is the frequency of the oscillator (to be determined) and $F(x) = \int f(x)dx$. The Hamiltonian of Equation (2) is presented by the following equation:

$$H(x) = -\frac{1}{2}\dot{x}^2 + F(x) = F(A) \tag{3}$$

which provides the following residual

$$R(x) = -\frac{1}{2}\dot{x}^2 + F(x) - F(A) = 0. \tag{4}$$

The first-order approximate solution of Equation (1) is assumed in the following form:

$$x(t) = A\cos\omega t. \tag{5}$$

Substituting Equation (5) into Equation (4), we obtain:

$$R(t) = -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0. \tag{6}$$

When $\omega t = \pi/4$ (collocation principle), it becomes:

$$\omega = \frac{2}{A}\sqrt{F(A) - F\left(\frac{\sqrt{2}}{2}A\right)}. \tag{7}$$

### Improved energy balance method

Khan and Mirzabeigy (2014) considered the solution:

$$x(t) = b\cos\omega t + b_1 \cos 3\omega t. \tag{8}$$

According to initial conditions, it becomes

$$A = b + b_1 \tag{9}$$

Eliminating $b_1$ from Equations (8)-(9), the solution takes the form:

$$x(t) = b\cos\omega t + (A - b)\cos 3\omega t. \tag{10}$$

By substitution of Equation (10) into Equation (4), the residual is obtained, which contain two unknown parameters, $\omega$ and $b$. For determining these parameters, two equations are essential; the first equation obtained by collocation method as follows:

$$\lim_{\omega t \to \frac{\pi}{4}} R(t) = 0. \tag{11}$$

Then the second equation is obtained using Galerkin-Petrov method as follows:

$$\int_0^{T/4} R(t)\cos\omega t dt = 0. \tag{12}$$

By solving Equations (11) and (12) simultaneously, $\omega$ and $b$ are determined.

### More accurate solution

Let us consider $b = A(1-u)$, $b_1 = Au$ (where $u$ is an unknown constant) and then Equation (8) becomes:

$$x(t) = A((1-u)\cos\omega t + u\cos 3\omega t). \tag{13}$$

Substituting Equation (13) into Equation (4) residual is obtained as:

$$R(t) = -\frac{1}{2}(A\omega((1-u)\sin\omega t + 3u\sin 3\omega t))^2 + F(A((1-u)\cos\omega t + u\cos 3\omega t)) - F(A) = 0. \tag{14}$$

This residual contain two unknown constants $\omega$ and $u$. In order to determine these constants, we need two equations which are obtained from:

$$\int_0^{T/4} \frac{R(t)\cos(2n-2)\omega t dt}{\sin^2 \omega t} = 0, n = 1, 2. \tag{15}$$

Solving these two equations simultaneously, we obtain $\omega$ and $u$.

### Application

Mathematical model of a pendulum attached to a rotating support

**Table 1**. Comparison of the present solution with Ghafoori et al. (2011), Khan and Mirzabeigy (2014) and corresponding numerical solution for x(t) and $A=1$.

| t | Nuerical solution | DTM (Ghafoori et al., 2011) | (% error) | Improved EBM (Khan and Mirzabeigy, 2014) | (% error) | Present method | (% error) |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0.000000 | 1 | 0.000000 | 1 | 0.000000 |
| 1 | 0.820933 | 0.825443 | 0.549375 | 0.824193 | 0.397109 | 0.820853 | 0.009745 |
| 2 | 0.412192 | 0.422288 | 2.449340 | 0.416311 | 0.999292 | 0.410502 | 0.410003 |
| 3 | -0.04611 | 0.032546 | 29.416612 | -0.045082 | 2.229451 | -0.046006 | 0.225548 |
| 4 | -0.50203 | -0.485740 | 3.244826 | -0.503957 | 0.383842 | -0.501056 | 0.194012 |
| 5 | -0.88241 | -0.869266 | 1.489557 | -0.883181 | 0.087374 | -0.882731 | 0.036377 |
| 6 | -0.99224 | -0.996193 | 0.398392 | -0.992855 | 0.061981 | -0.992178 | 0.006248 |
| 7 | -0.75008 | -0.776877 | 3.572552 | -0.756222 | 0.818846 | -0.749394 | 0.091456 |
| 8 | -0.32098 | -0.358097 | 11.563648 | -0.327212 | 1.941553 | -0.318759 | 0.691943 |
| 9 | 0.138314 | 0.097636 | 29.409893 | 0.135246 | 2.218141 | 0.138176 | 0.099772 |
| 10 | 0.589446 | 0.548088 | 7.016418 | 0.589124 | 0.054627 | 0.589274 | 0.029179 |

Where (% error) denotes the absolute percentage error.

(Ghafoori et al., 2011; Khan and Mirzabeigy, 2014) is expressed as

$$\ddot{x}+\sin(x)(1-\Lambda\cos(x))=0,\ x(0)=A,\ \dot{x}(0)=0, \tag{16}$$

where, $\Lambda=\dfrac{\Omega^2 r}{g}$, $x=\omega t.$

Without loss of generality, we set $\Lambda=1.$ Equation (16) can be expanded up to seventh order as follows:

$$\ddot{x}+\frac{x^3}{2}-\frac{x^5}{8}+\frac{x^7}{80}=0,\ x(0)=A,\ \dot{x}(0)=0. \tag{17}$$

According to the variational principle, Equation (17) can be written as:

$$J(x)=\int_0^t\left[\frac{\dot{x}^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}+\frac{x^8}{640}\right]dt. \tag{18}$$

Its Hamiltonian becomes

$$H=\frac{\dot{x}^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}+\frac{x^8}{640}=\frac{A^4}{8}-\frac{A^6}{48}+\frac{A^8}{640}. \tag{19}$$

Also, residual of Equation (19) is:

$$R(t)=\dot{x}^2/2+(x^4-A^4)/8-(x^6-A^6)/48+(x^8-A^8)/640=0. \tag{20}$$

Substituting Equation (13) into Equation (20) and using Equation (15), we obtain respectively for $n=1,2$

$$\begin{aligned}&-1920A^2+400A^4-35A^6+5120\omega^2+10(-768A^2+160A^4-14A^6+2048\omega^2)u\\&+(7680A^2-2000A^4+196A^6+112640\omega^2)u^2+(-7680A^2+3200A^4-392A^6)u^3\\&+10(384A^2-400A^4+70A^6)u^4+10(320A^4-98A^6)u^5=0\end{aligned} \tag{21}$$

$$\begin{aligned}&-3840A^2+1280A^4-141A^6+(-61440A^2+14400A^4-1344A^6+368640\omega^2)u\\&+(23040A^2-9600A^4+1176A^6+737280\omega^2)u^2+(6400A^4-1344A^6)u^3\\&+(-7680A^2+1050A^6)u^4-4800A^4u^5=0\end{aligned} \tag{22}$$

Eliminating $\omega^2$ between Equations (21) and (22) and ignoring more than third order terms of $u$, we obtain:

$$\begin{aligned}&1-(24(3200-600A^2+49A^4)u+24(12480-2800A^2+259A^4)u^2\\&-16(224640-49000A^2+4431A^4)u^3)/(3840-1280A^2+141A^4)=0\end{aligned} \tag{23}$$

From Equation (23), the value of $u$ is obtained. But it is a cubical equation. It is noted that the coefficient of $u^3$ is small. Ignoring this term, its solution is obtained as $u_0$. Then, $u^3$ can be written as $u^2u_0$. Thus, Equation (23) again becomes a quadratic equation, whose smallest solution is the required value of $u$. Substituting the value of $u$ into Equation (21), $\omega$ is obtained.

## RESULTS AND DISCUSSION

A more accurate solution of Equation (16) was determined. The solution was compared with those presented by Ghafoori et al. (2011) and Khan and Mirzabeigy (2014). All the results together with numerical solution (obtained by fourth-order Runge–Kutta formula) are presented in Table 1. From the results shown in the table, it is clear that the percentage error of the present solution did not exceed 0.69%. On the contrary, the maximum percentage errors of DTM (Ghafoori et al., 2011) and improved EBM (Khan and Mirzabeigy, 2014) are respectively 29.41 and 2.22%. Thus, the present method provides more accurate solution.

## Conclusion

Based on EBM, an analytical approximate solution was presented for solving pendulum equation with rotating support. The solution is nicely close to the exact results and are much better than those obtained by differential transformation method (DTM) (Ghafoori et al., 2011) and improved accuracy of He's energy balance method (Khan and Mirzabeigy, 2014). The relative error of the present method is lower than those obtained by others (Ghafoori et al., 2011; Khan and Mirzabeigy, 2014).

## CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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