Vol. 11(7), pp. 96-102, October 2018
DOI: 10.5897/AJMCSR2018.0735
Article Number: 76FBE6A58983
ISSN: 2006-9731

http://www.academicjournals.org/AJMCSR

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African Journal of Mathematics and Computer Science Research

Full Length Research Paper

Derivation of a new block fifth order method for numerical solution of first order initial value problems

Salman H. Abbas[1*] and Noor A. Jalil Alshakhoori[2]

[1]Department of Mathematical Sciences, College of Arts and Science, Ahlia University, P. O. Box 10887, Kingdom of Bahrain.
[2]Department of Mathematics, College of Science, University of Bahrain, P. O. Box 32038, Kingdom of Bahrain.

Received 27 February, 2018; Accepted 28 August, 2018

A new block method of order five for the numerical solution of initial value problems is derived. The coefficients of the matrix of the method are chosen such that low power of the blocksize appears in the principal local truncation error. The stability polynomial is shown to be a perturbation of $(p+1)^{th}$ order explicit Runge-Kutta method.

Key words: Predictor-corrector methods, ordinary differential equations, block methods.

INTRODUCTION

Numerical methods for the solution of the initial value problems (IVP's) are well established techniques in literature. In the last two decades, a number of papers have appeared on this topic (Abbas, 1995, 1997; Abbas and Delves, 1989; Voss and Abbas, 1997; Birta and Abou-Rabia, 1987; Chu and Hamilton, 1987; Enright and Pryce, 1987; Franklin, 1978; Henrici, 1962; Houwen and Sommeijer, 1989; Katz et al., 1977; Lambert, 1973; Luther and Konen, 1965; Miranker, 1971; Miranker and Liniger, 1967; Moody and Hans, 1987; Moulton, 1926; Mouney et al., 1991; Rosser, 1967; Shampine and Watts, 1969; Watts and Shampine, 1972; Worland, 1976; Adesanya et al., 2013; Dhaigude and Devkate, 2017; Adesanya et al., 2013). The earliest research on block methods was proposed by Shampine and Watts (1969) and Watts and Shampine (1972) with block implicit one step methods, Chu and Hamilton (1987) with multi-block methods, and Voss and Abbas (1997) with predictor-corrector schemes. Other block methods are discussed by several researches such as Houwen and Sommeijer (1989) with block Runge-Kutta methods. One of such techniques is the block method which by means of a single application of a calculation unit yields a sequence of new estimates for y in the differential equation:

$$y' = f(x,y), \qquad y(x_0) = y_0 \tag{1}$$

where $y, f \in \mathbb{R}^s$. If the block size k ≥ 1, then in simple cases the values of x for which solutions are computed will be evenly separated. In other words, each basic cycle of calculation has the potential to advance the solution by k new points in the x direction. The block method of the present study has the properties of Runge-Kutta method for being self-starting and does not

*Corresponding author. Email: dr_salman121@hotmail.com.

refuse the development of separate predictors or starting values. Block method was found to be cost effective and to give better approximation.

DERIVATION OF A FIFTH ORDER METHOD

The requirement that the method be fifth order for any block length k complicates the search for good methods. The aim was to produce a fifth order method which might be very efficient to implement. As far as the order of the accuracy and the stability are concerned, if we require a method of order s to be achieved then the local truncation error will normally be of order $k^{s+1}h^{s+1}$ which is a very high dependence upon the block length k. We seek some methods where the local truncation errors have a lower power of k, as example, $k^{s}h^{s+1}$, but then over experience with order 2 methods suggest that the stability problem will be possibly serious.
Therefore, we consider the fifth order method:

$$y_{n+r,s+1} = y_n + h\,(\alpha_{r,0}\,y'_n + \alpha_{r,1}\,y'_{n+1,s} + \alpha_{r,2}\,y'_{n+2,s} \quad (2)$$

$$+ \alpha_{r,3}\,y'_{n+3,s} + \alpha_{r,4}\,y'_{n+4,s});$$
$$r = 1,2,\dots,k; \quad s = 0,1,2,3,4.$$

Note that the method will be fifth order if the coefficients of the method satisfy the five characteristic polynomials, which are given as:

$$\sum_{j=0}^{k} \alpha_{r,j} = r \;, \quad \sum_{j=0}^{k} j\,\alpha_{r,j} = \frac{r^2}{2} \;, \quad \sum_{j=0}^{k} j^2\,\alpha_{r,j} = \frac{r^3}{3}$$

$$\sum_{j=0}^{k} j^3\,\alpha_{r,j} = \frac{r^4}{4} \;, \quad \sum_{j=0}^{k} j^4\,\alpha_{r,j} = \frac{r^5}{5} \;; \quad r = 1,2,\dots k. \qquad (3)$$

Therefore, the coefficients of the method is given by

$$\alpha_{r,0} = \frac{1}{120}r^5 - \frac{5}{48}r^4 + \frac{35}{72}r^3 - \frac{25}{24}r^2 + r$$
$$\alpha_{r,1} = -\frac{1}{30}r^5 + \frac{3}{8}r^4 - \frac{13}{9}r^3 + 2r^2$$
$$\alpha_{r,2} = \frac{1}{20}r^5 - \frac{1}{2}r^4 + \frac{19}{12}r^3 - \frac{3}{2}r^2$$
$$\alpha_{r,3} = -\frac{1}{30}r^5 + \frac{7}{24}r^4 - \frac{7}{9}r^3 + \frac{2}{3}r^2$$
$$\alpha_{r,4} = \frac{1}{120}r^5 - \frac{1}{16}r^4 + \frac{11}{72}r^3 - \frac{1}{8}r^2; \qquad r = 1,2,\dots,k.$$

Hence,

$$y_{n+r,s+1} = y_n + h\,(\alpha_{r,0}\,y'_{n,s} + \alpha_{r,1}\,y'_{n+1,s} + \alpha_{r,2}\,y'_{n+2,s} + \alpha_{r,3}\,y'_{n+3,s}$$

$$+ \alpha_{r,4}\,y'_{n+4,s}); \quad r = 1,2,\dots,10; \quad s = 0,1,2,3,4.$$

is equivalent to

$$\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6}\\ y_{n+7}\\ y_{n+8}\\ y_{n+9}\\ y_{n+10} \end{pmatrix} = \begin{pmatrix} y_n\\ y_n\\ y_n\\ y_n\\ y_n\\ y_n\\ y_n\\ y_n\\ y_n\\ y_n \end{pmatrix} + h \begin{pmatrix} 251/720 & 323/360 & -11/30 & 53/360 & -19/720\\ 29/90 & 62/45 & 4/15 & 2/45 & -1/90\\ 27/80 & 51/40 & 9/10 & 21/40 & -3/80\\ 14/45 & 64/45 & 8/15 & 64/45 & 14/45\\ 95/144 & -25/72 & 25/6 & -175/72 & 425/144\\ 33/10 & -66/5 & 144/5 & -126/5 & 123/10\\ 9107/720 & -20629/360 & 3283/30 & -33859/360 & 26117/720\\ 1648/45 & -7552/45 & 4576/15 & -11392/45 & 3928/45\\ 7011/80 & -15957/40 & 7047/10 & -22707/40 & 14661/80\\ 3305/18 & -7450/9 & 4300/3 & -10150/9 & 6275/18 \end{pmatrix} \begin{pmatrix} y'_n\\ y'_{n+1}\\ y'_{n+2}\\ y'_{n+3}\\ y'_{n+4} \end{pmatrix}$$

Now, we would give the formulae of the method when k = 5 and 10.

(1) When k = 5, the formulae are

$$y_{n+1} = y_n + h\left[\frac{251}{720}y'_n + \frac{323}{360}y'_{n+1} - \frac{11}{30}y'_{n+2} + \frac{53}{360}y'_{n+3} - \frac{19}{720}y'_{n+4}\right]$$
$$y_{n+2} = y_n + h\left[\frac{29}{90}y'_n + \frac{62}{45}y'_{n+1} + \frac{4}{15}y'_{n+2} + \frac{2}{45}y'_{n+3} - \frac{1}{90}y'_{n+4}\right]$$
$$y_{n+3} = y_n + h\left[\frac{27}{80}y'_n + \frac{51}{40}y'_{n+1} + \frac{9}{10}y'_{n+2} + \frac{21}{40}y'_{n+3} - \frac{3}{80}y'_{n+4}\right] \qquad (4)$$

$$y_{n+4} = y_n + h\left[\frac{14}{45}y'_n + \frac{64}{45}y'_{n+1} + \frac{8}{15}y'_{n+2} + \frac{64}{45}y'_{n+3} + \frac{14}{45}y'_{n+4}\right]$$
$$y_{n+5} = y_n + h\left[\frac{95}{144}y'_n - \frac{25}{72}y'_{n+1} + \frac{25}{6}y'_{n+2} - \frac{175}{72}y'_{n+3} + \frac{425}{144}y'_{n+4}\right]$$

(2) When k = 10, the formulae are

$$\begin{aligned}
y_{n+1} &= y_n + h\left[\frac{251}{720}y'_n + \frac{323}{360}y'_{n+1} - \frac{11}{30}y'_{n+2} + \frac{53}{360}y'_{n+3} - \frac{19}{720}y'_{n+4}\right]\\
y_{n+2} &= y_n + h\left[\frac{29}{90}y'_n + \frac{62}{45}y'_{n+1} + \frac{4}{15}y'_{n+2} + \frac{2}{45}y'_{n+3} - \frac{1}{90}y'_{n+4}\right]\\
y_{n+3} &= y_n + h\left[\frac{27}{80}y'_n + \frac{51}{40}y'_{n+1} + \frac{9}{10}y'_{n+2} + \frac{21}{40}y'_{n+3} - \frac{3}{80}y'_{n+4}\right]\\
y_{n+4} &= y_n + h\left[\frac{14}{45}y'_n + \frac{64}{45}y'_{n+1} + \frac{8}{15}y'_{n+2} + \frac{64}{45}y'_{n+3} + \frac{14}{45}y'_{n+4}\right]\\
y_{n+5} &= y_n + h\left[\frac{95}{144}y'_n - \frac{25}{72}y'_{n+1} + \frac{25}{6}y'_{n+2} - \frac{175}{72}y'_{n+3} + \frac{425}{144}y'_{n+4}\right]\\
y_{n+6} &= y_n + h\left[\frac{33}{10}y'_n - \frac{66}{5}y'_{n+1} + \frac{144}{5}y'_{n+2} - \frac{126}{5}y'_{n+3} + \frac{123}{10}y'_{n+4}\right]\\
y_{n+7} &= y_n + h\left[\frac{9107}{720}y'_n - \frac{20629}{360}y'_{n+1} + \frac{3283}{30}y'_{n+2} - \frac{33859}{360}y'_{n+3} + \frac{26117}{720}y'_{n+4}\right]\\
y_{n+8} &= y_n + h\left[\frac{1648}{45}y'_n - \frac{7552}{45}y'_{n+1} + \frac{4576}{15}y'_{n+2} - \frac{11392}{45}y'_{n+3} + \frac{3928}{45}y'_{n+4}\right]\\
y_{n+9} &= y_n + h\left[\frac{7011}{80}y'_n - \frac{15957}{40}y'_{n+1} + \frac{7047}{10}y'_{n+2} - \frac{22707}{40}y'_{n+3} + \frac{14661}{80}y'_{n+4}\right]\\
y_{n+10} &= y_n + h\left[\frac{3305}{18}y'_n - \frac{7450}{9}y'_{n+1} + \frac{4300}{3}y'_{n+2} - \frac{10150}{9}y'_{n+3} + \frac{6275}{18}y'_{n+4}\right]
\end{aligned} \quad (5)$$

Principal local truncation error

The r^{th} component of local truncation error of Equation 1 is given by

$$\varepsilon_r = y_{n+r,s+1} - y_n - h(\alpha_{r,0}y'_n + \alpha_{r,1}y'_{n+1,s} + \alpha_{r,2}y'_{n+2,s} \quad (6)$$

$$+ \alpha_{r,3}y'_{n+3,s} - \alpha_{r,4}y'_{n+4,s})$$

where $y_{n+r} = y(x_{n+r})$ denotes the exact value of the solution at x_{n+r}. ε_r can be expressed as a power series in h, and the remaining term of Taylor's series expansion used to derive ε_r.

Therefore, the k^{th} component of local truncation error of Equation 1 is given by:

$$\varepsilon_k = y_{n+k,s+1} - y_n - h(\alpha_{k,0}y'_n + \alpha_{k,1}y'_{n+1,s} + \alpha_{k,2}y'_{n+2,s}$$
$$+ \alpha_{k,3}y'_{n+3,s} + \alpha_{k,4}y'_{n+4,s})$$

$$\therefore \varepsilon_k = k^2\left(\tfrac{1}{720}k^4 - \tfrac{1}{60}k^3 + \tfrac{7}{96}k^2 - \tfrac{5}{36}k + \tfrac{1}{10}\right)h^6y_n^6 = O(k^6h^6) \quad (7)$$

Then we can express $\underline{\varepsilon}$ as:

$$\left[\frac{3}{160}h^6y_n^6, \frac{1}{90}h^6y_n^6, \frac{3}{160}h^6y_n^6, \ldots, \frac{645}{2}h^6y_n^6, \ldots, k^2\left(\frac{1}{720}k^4 - \frac{1}{60}k^3 + \frac{7}{96}k^2 - \frac{5}{36}k + \frac{1}{10}\right)h^6y_n^6\right]^T$$

for $r = 1,2,3, \ldots, k$. So, the local error of our method (Equation 1) estimated depend on k^6h^6. The global error of this method is of order k^5h^5.

Stability analysis

The following analysis is carried out according to the stability of a fixed step size studied by Watts and Shampine (1972). To discuss the stability of this block method "fifth order method", we will check the case when k = 5.

$$y_{n+r,s+1} = y_n + h(\alpha_{r,0}y'_{n,s} + \alpha_{r,1}y'_{n+1,s} + \alpha_{r,2}y'_{n+2,s} + \alpha_{r,3}y'_{n+3,s}$$
$$+ \alpha_{r,4}y'_{n+4,s}); \quad r = 1,2,\ldots,5; \quad s = 0,1,2,3,4.$$

Now, applying the method to the test equation $y' = \lambda y$, and let $H = \lambda h$, we get:

$$y_{n+r,s+1} = y_n + H(\alpha_{r,0} y_{n,s} + \alpha_{r,1} y_{n+1,s} + \alpha_{r,2} y_{n+2,s} + \alpha_{r,3} y_{n+3,s} + \alpha_{r,4} y_{n+4});$$

$$r = 1,2,\dots,5.$$

Then the general form when k = 5 of this method in this representation is given by:

$$\begin{pmatrix} 1-\frac{323}{360}H & \frac{11}{30}H & -\frac{53}{360}H & \frac{19}{720}H & 0 \\ -\frac{62}{45}H & 1-\frac{4}{15}H & -\frac{2}{45}H & \frac{1}{90}H & 0 \\ -\frac{51}{40}H & -\frac{9}{10}H & 1-\frac{21}{40}H & \frac{3}{80}H & 0 \\ -\frac{64}{45}H & -\frac{8}{15}H & -\frac{64}{45}H & 1-\frac{14}{45}H & 0 \\ \frac{25}{72}H & -\frac{25}{6}H & \frac{175}{72}H & -\frac{425}{144}H & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 1+\frac{251}{720}H \\ 1+\frac{29}{90}H \\ 1+\frac{27}{80}H \\ 1+\frac{14}{45}H \\ 1+\frac{95}{144}H \end{pmatrix} \quad (8)$$

Let A be the coefficient matrix in Equation 7 and B the vector in the right hand side. Now, let us define q(H), $q_r(H)$ by

$$q(H) = \det(A) \quad \text{and} \quad q_r(H) = \det(A_r)$$

where A_r is the matrix A with the r^{th} column replaced by the vector B. Using the aforementioned definitions and Cramer's rule in Equation 3, we find that

$$y_{n+r} = \frac{q_r(H)}{q(H)} y_n, \qquad r = 1,2,\dots,k;$$

where q(H) and $q_r(H)$ are polynomials with degree k and it is necessary that A to be invertible for all H.

For $n = mk, m = 0,1,2, \dots,$ we have

$$y_{n+r} = \left[\frac{q_r(H)}{q(H)}\right]\left[\frac{q_k(H)}{q(H)}\right]^m, \quad r = 1,2,\dots,k.$$

Hence, the block method is said to be A-stable if and only if

$$\left|\frac{q_k(H)}{q(H)}\right| < 1 \text{ for all } H, \mathrm{Re}(H) < 0.$$

Therefore, in the case when k = 5, q(H) and $q_5(H)$ for the aforementioned scheme (Equation 8) are given by:

$$q(H) = 1 - 2H + \frac{7}{4}H^2 - \frac{5}{6}H^3 + \frac{1}{5}H^4$$

$$q_5(H) = 1 + 3H + \frac{17}{4}H^2 + \frac{15}{4}H^3 + \frac{137}{60}H^4 + H^5$$

Hence, the amplification factor is $\left|\frac{q_5(H)}{q(H)}\right| = \left|\frac{1+3H+\frac{17}{4}H^2+\frac{15}{4}H^3+\frac{137}{60}H^4+H^5}{1-2H+\frac{7}{4}H^2-\frac{5}{6}H^3+\frac{1}{5}H^4}\right|$ which is less than one for all $H \in (-2.32688,0)$. So, the method is A-stable for all $H \in (-2.32688,0)$. Then the method is A-stable when k = 5.

Test problems

Here, we present four select test problems given by:
Problem (i):

$$y' = y, \qquad y(0) = 1, \qquad 0 \le x \le 1$$

The analytic solution is $y = e^x$. Problem (ii), we use the following problem by Enright and Pryce (1987),

$$y' = -y^3/2, \qquad y(0) = 1, \qquad 0 \le x \le 1$$

The analytic solution is $y = \frac{1}{\sqrt{x+1}}$.

Problem (iii):

$$y' = xy, \qquad y(0) = 1, \qquad 0 \le x \le 1$$

The analytic solution is $y = e^{x^2/2}$.

Problem (iv):

$$y' = x + y, \qquad y(0) = 1, \qquad 0 \le x \le 1$$

The analytic solution is $y = -x - 1 + 2e^x$

COMPUTATIONAL RESULTS

Tables 1 to 4 show the experimental results for the aforementioned test problems and the accuracy of the solution which is measured by computing the differences between the approximation solution and the exact solution at the end of the steps.

For all the problems tested, the numerical results show that new block fifth order method gives good accuracy with the reduction of total steps.

COMPARISON WITH A RUNGE-KUTTA METHOD OF THE SAME ORDER

Here, we compare the experimental results which was obtained by this study block method (with block length = 5) and the following fifth order Runge-Kutta method given by Luther and Konen (1965). From Tables 5 to 8, we notice that the accuracy is satisfactory and much better

Table 1. Block method results using test problem (i).

Step-size h	Computed solution	Accuracy
0.01	2.7182818284339207	2.51×10^{-11}
0.001	2.7182818284590464	1.30×10^{-15}
0.0001	2.7182818284590429	2.20×10^{-15}

The exact solution is $y = e^x$; The exact solution at $x = 1$; $y(1) = e = 2.7182818284590451$.

Table 2. Block method results using test problem (ii).

Step-size h	Computed solution	Accuracy
0.01	0.70710678109597458	9.1×10^{-11}
0.001	0.70710678118654646	1.0×10^{-15}
0.0001	0.70710678118654713	3.3×10^{-15}

The exact solution is $y = 1/\sqrt{x+1}$. The exact solution at $x = 1$; $y(1) = 1/\sqrt{2} = 0.70710678118654746$.

Table 3. Block method results using test problem (iii).

Step-size h	Computed solution	Accuracy
0.01	1.6487212703448912	3.55×10^{-10}
0.001	1.6487212707001244	3.80×10^{-15}
0.0001	1.6487212707001289	7.00×10^{-16}

The exact solution is $y = e^{x^2/2}$. The exact solution at $x = 1$; $y(1) = e^{1/2} = 1.6487212707001282$.

Table 4. Block method results using test problem (iv).

Step-size h	Computed solution	Accuracy
0.01	3.4365636568678393	5.0×10^{-11}
0.001	3.4365636569180902	4.0×10^{-15}
0.0001	3.4365636569180955	4.9×10^{-16}

The exact solution is $y = -x - 1 + 2e^x$. The exact solution at $x = 1$; $y(1) = -1 - 1 + 2e = 3.4365636569180906$.

Table 5. Comparison between Block method and Runge-Kutta method using test problem (i).

Step-size h	Accuracy	
	Fifth order block method	Runge-Kutta method
0.01	3.18×10^{-11}	1.18×10^{-3}
0.001	1.30×1010^{-15}	1.18×1010^{-4}
0.0001	2.20×10^{-15}	1.20×1010^{-5}

than the fifth order Runge-Kutta method. Note that the accuracy of the solution is measured by evaluating (l_∞) norm of the differences between the approximation solution and the exact solution.

Table 6. Comparison between Block method and Runge-Kutta method using test problem (ii).

Step-size h	Accuracy	
	Fifth order block method	**Runge-Kutta method**
0.01	8.9×10^{-11}	8×10^{-5}
0.001	1.0×10^{-15}	8×10^{-6}
0.0001	3.3×10^{-16}	1×10^{-6}

Table 7. Comparison between Block method and Runge-Kutta method using test problem (iii).

Step-size h	Accuracy	
	Fifth order block method	**Runge-Kutta method**
0.01	3.56×10^{-10}	9.54×10^{-4}
0.001	3.80×10^{-15}	9.50×10^{-5}
0.0001	7.00×10^{-16}	9.00×10^{-6}

Table 8. Comparison between Block method and Runge-Kutta method using test problem (iv).

Step-size h	Accuracy	
	Fifth order block method	**Runge-Kutta method**
0.01	6.4×10^{-11}	2.36×10^{-3}
0.001	4.0×10^{-15}	2.36×10^{-4}
0.0001	4.9×10^{-16}	2.40×10^{-5}

Conclusion

This paper provides a new block method of higher order which can be implemented as a corrected formula of any block size together with Euler's method as a predictor formula. However, this research is concerned with solving initial value problems using block predictor-corrector method of fifth order. The study highlights the importance of solving initial value problems and their application in the real life problems. Furthermore, the result indicates that the block method which was derived in the study could be considered as a good method on the basis of the accuracy and the comparison with a Runge-Kutta method of the same order. Also, we noticed that the accuracy is satisfactory and much better than the 5th order Runge-Kutta method. Thus, for small h we can say that the algorithm may give a better accuracy. Therefore, the results are acceptable.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

ACKNOWLEDGEMENT

The authors are grateful to the referees for the very constructive comments and for suggesting various improvements to the manuscript.

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