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Full Length Research Paper

Development of a mathematical model for error propagation in engineering calculations involving many variables function

S. K. Fasogbon

Department of Mechanical Engineering, Obafemi Awolowo University, Ile-Ife, Nigeria. E-mail: kolasogbon@yahoo.com.

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In this research, a mathematical model for error propagations in many variables functions had been developed. The development employs both Taylor theorem and binomial coefficient of expansion. Verification of the models were done through computer simulation by comparing the results of the model with the actual difference between the result given by the function in question, when comp-uted with 'error_variable' and the same function when computed with 'error free variable'. The results given by the model proposed, gave good results for error propagation that were not signi-ficantly different from that obtained from the actual difference between the results of the error variable functions and 'error free variable' functions.

Key words: Mathematical model, error propagation, many variables function, engineering calculation.

INTRODUCTION

Mathematical functions denoted by g = f(x, y …z) are said to be many variables. Although, most time, engineers re-sort to representing real life situations using single varia-ble functions because many variables functions are either too difficult to model or too difficult to handle, yet, situa-tions do arise when engineers must call a spade , by modeling real life situations using many variables func-tions (Bajpai et al., 1973;Fasogbon, 2001). As stated above, many variables functions are characterized by many variables, but it is either the variables in question are related or not. Although in some cases, the variables are unrelated, notwithstanding, more often than not, some relationships between the variables must be satis-fied (Fasogbon, 2001; Kostelich and Armbruster, 1996; Yunfeng, 2008). Just as in the case of single variable functions, there are at least three sources of error in many variables functions, namely:

(i) Error in measuring initial conditions or initial error in the variable which characterizes any given mathematical function.
(ii) Error in the parameters (constants) in the function
(iii) An incorrect model of the underlying process.

It is always very difficult if not impossible to measure accurately the initial condition, this is because it is either the measuring equipments are malfunctioning, human mistake are in introduced or any other factors respon-sible. Most of the constants, parameters or the mathe-matical function depends upon are usually as a results of experimental analysis, and if the constants are not discrete, error is bound to set in, in the model. If the mathematical function describing an underlying process is not accurate, it shall be very difficult if not impossible to represent real life situation, even when errors from other sources are not present. Although error that propagates in mathematical functions serves as one of the factors militating against our ability to obtain exact results, it turns out that this factor plays a major role in our inability to predict far a head some engineering systems, even though the characteristic mathematical functions are highly determined. In fact, in large computations, the situation may go worse, such that our final result be-comes invalid, if at any point, the results in error happen to serve as input values in our subsequent calculations. Incidentally, more and more decisions in the development of science and technology are based on large scale computations and simulations (Kostelich and Armbruster, 1996; Hatim et al., 2004). Therefore, in order to gain a better understanding in to error propagation in many varia-bles functions, this study is aimed at developing mathe-matical models as well as simulates the responses of the models (mathematical functions) to initial error in the varia-bles which typify the functions.

Models developments

To start with, suppose we have two variables function f (x, y) which is continuous and partially differentiable (and the partial derivatives are continuous that is $f_{xy} = f_{yx}$) within the x and y range of interest, full Taylor expansion about the point($x=x_o$, $y=y_o$), is given by:

$$f(x, y) = \sum_{n=o}^{oo} \frac{1}{n!}\left[\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^n f(x, y)\right]_{xo, yo}$$

Expanding the function, we have:

$$f(x,y)=f(x_o, y_o) + \frac{\partial f}{dx}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2!}\left[\frac{\partial^{2f}}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^{2f}}{\partial y^2}(\Delta y)^2\right] + \ldots \quad (1)$$

Where $\Delta x = x - x_o$ and $\Delta y = y - y_o$, and all the derivatives are to be evaluated at (x_o, y_o).

First consideration

Suppose there are small errors 'δx' in the variable 'x' and 'δy' in the variable 'y' and we seek to investigate how these errors propagate in the function f(x, y), by analogy, it can be seen that:

$$f(x \pm \delta x, y \pm \delta y) = \sum_{n=0}^{oo} \frac{1}{n!}\left[\left(\left(\Delta x + \left(\pm \delta x\right)\right)\frac{\partial}{\partial x} + \left(\Delta y + \left(\pm \delta y\right)\right)\frac{\partial}{\partial y}\right)^n f(x, y)\right]_{xo, yo}$$

Expanding the function, neglecting the terms containing the terms error of second order, and rearranging the terms, we have:

$$f(x \pm \delta x, y \pm \delta y) = f(x_o, y_o) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2!}\left[\frac{\partial^{2f}}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^{2f}}{\partial y^2}(\Delta y)^2\right] + \frac{\partial f}{\partial x}(\pm \delta x) + \frac{\partial f}{\partial y}(\pm \delta y) + \frac{1}{2!}\left[2\frac{\partial^{2f}}{\partial x^2}\Delta x(\pm \delta x) + 2\frac{\partial^{2f}}{\partial x \partial y}\Delta x(\pm \delta y) + 2\frac{\partial^{2f}}{\partial x \partial y}\Delta y(\pm \delta x) + 2\frac{\partial^{2f}}{\partial y^2}\Delta y(\pm \delta y)\right] + \ldots \quad (2)$$

Close examination shows that expression (1) and (2) can be put in a compact notation forms with x_1=x and x_2=y, thus we have:

$$f(x,y)=f(x_{0}, y_{0}) + \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}\Delta x_i + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^{2f}}{\partial x_i \partial x_j} + \ldots \quad (3)$$

$$f(x \pm \delta x, y \pm \delta y) = f(x_o, y_o) + \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}(\Delta x_i + (\delta x_i)) + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^{2f}}{\partial x_i \partial x_j}\left(\Delta x_i \Delta x_j + 2\Delta x_i(\pm \delta x_j)\right) + \ldots \quad (4)$$

Expanding expression (4), we have:

$$f(x \pm \delta x, y \pm \delta y) = f(x_o, y_0) + \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}\Delta x_i + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}(\pm \delta x_i) + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^{2f}}{\partial x_i \partial x_j}.2\Delta x_i(\pm \delta x_j) + \ldots \quad (5)$$

If we define 'E(x, y)' as the propagated error in the function f(x, y) by small errors '$\pm \delta x$' in 'x', '$\pm \delta y$' in 'y', we have :

$$f(x \pm \delta x, y \pm \delta y) = f(x, y) + E(x, y) \quad (6)$$

Comparing expression (3), (5) and (6), we have:

$$E(x, y) = \sum_{i=1}^{2}\frac{\partial f}{\partial x_i}(\pm \delta x) + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{2}2\frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i(\pm \delta x_j) + \frac{1}{3!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3}^{2}3\frac{\partial^{3f}}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\Delta x_{i_1}\Delta x_{2}(\pm \delta x_{i_3}) + \ldots + \frac{1}{n!}\sum_{i_1=1}^{2}\ldots\sum_{i_n=1}^{2} n\left(\frac{\partial^n f}{\partial x_{i_1}\ldots\partial x_{i_n}}\right)\left(\Delta x_{i_1}\ldots\Delta x_{i_{n-1}}\right)\left(\pm \delta x_{i_n}\right)$$

Putting E(x, y) in a single compact notation, we have:

$$E(x, y) = \sum_{n=0}^{oo}\left[\frac{1}{n!}\sum_{i_1=1}^{2}\ldots\sum_{i_n=1}^{2} n\left(\frac{\partial^n f}{\partial x_{i_1}\ldots\partial x_{i_n}}\right)\left(\Delta x_{i_1}\ldots\Delta x_{i_{n-1}}\right)\left(\pm \delta x_{i_n}\right)\right] \quad (7)$$

It should be noted that for all Δx_i to $\Delta x_{i_{n-1}}$, the following exist:

If n= 0, $\Delta x_{i_{n-1}} = \Delta x_{i_{-1}} = 0$

If n=1, $\Delta x_{i_{n-1}} = \Delta x_{i_0} = 1$

If n $\geq 2, \Delta x_{i_{n-1}}$ is as defined

Second consideration

Suppose there is no error in one of the variables, or there is error $'\delta y'$ in 'y' but $''\delta x' = 0$.
Subtracting expression (i) from (ii), we have:

$$E(x, y) = \frac{\partial f}{\partial x}(\pm\delta x) + \frac{\partial f}{\partial y}(\pm\delta y) + \frac{1}{2!}\left[2\frac{\partial^{2f}}{\partial x^2}\Delta x(\pm\delta x) + 2\frac{\partial^{2f}}{\partial x\partial y}\Delta x(\pm\delta y) + 2\frac{\partial^{2f}}{\partial x\partial y}\Delta y(\pm\delta x) + 2\frac{\partial^{2f}}{\partial y^2}\Delta y(\pm\delta y)\right] + ...$$

Putting $'\delta x' = 0.$, we have:

$$E(x, y) = \frac{\partial f}{\partial y}(\pm\delta y) + \frac{1}{2!}[\ 2\frac{\partial^{2f}}{\partial x\partial y}\Delta x(\pm\delta y) + 2\frac{\partial^{2f}}{\partial y^2}\Delta y(\pm\delta y)\ ]$$

If we go further, we have:

$$E(x, y) = \sum_{i=1}^{1}\frac{\partial f}{\partial x_i}(\frac{+}{-}\delta x) + \frac{1}{2!}\sum_{i=1}^{2}\sum_{j=1}^{1}2\frac{\partial^2 f}{\partial x_i\partial x_j}\Delta x_i(\frac{+}{-}\delta x_{\ j}) + \frac{1}{3!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3}^{1}3\frac{\partial^{3f}}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\Delta x_{i_1}\Delta x_{\ 2}(\pm\delta x_{i_3}) + ...$$

$$+ \frac{1}{n!}\sum_{i_1=1}^{2}......\sum_{i_{n-1}=1}^{2}\sum_{i_n=1}^{1} n\left(\frac{\partial^n f}{\partial x_{i_1}.....\partial x_{i_n}}\right)\left(\Delta x_{i_1}.....\Delta x_{i_{n-1}}\right)\left(\frac{+}{-}\delta x_{i_n}\right)$$

Putting E(x, y) in a single compact notation, we have:

$$E(x, y) = \sum_{n=0}^{oo}[\ \frac{1}{n!}\sum_{i_1=1}^{2}......\sum_{i_{n-1}=1}^{2}\sum_{i_n=1}^{1} n\left(\frac{\partial^n f}{\partial x_{i_1}.....\partial x_{i_n}}\right)\left(\Delta x_{i_1}.....\Delta x_{i_{n-1}}\right)\left(\frac{+}{-}\delta x_{i_n}\right)\ ] \qquad (8)$$

It should be noted that for all i_1 to i_n, the following exist:
If n=0, the whole expression (8) equals zero
If n= 1, $i_1 = i \Rightarrow i = 1$
If n$\geq 2, i_1 = i_1 \Rightarrow i_1 = 1$ to 2, $i_n = 1$ to 1

Generalization

If we define 'L' as the number of variables which contain error, and 'M' as the number of variables a function depends upon, where L ≤ M, it is found that for many variables function f($x_1, x_2................x_m$) containing errors ($\delta x_1, \delta x_2...........\delta x_L$) in variables ($x_1, x_2...........x_m$) respectively, the propagated error E(**X**)

(Where **X =** $x_1, x_2............x_m$) is given by:

$$E(\mathbf{X}) = \sum_{n=0}^{oo}[\ \frac{1}{n!}\sum_{i_1=1}^{M}......\sum_{i_{n-1}=1}^{M}\sum_{i_n=1}^{L} n\left(\frac{\partial^n f}{\partial x_{i_1}.....\partial x_{i_n}}\right)\left(\Delta x_{i_1}.....\Delta x_{i_{n-1}}\right)\left(\frac{+}{-}\delta x_{i_n}\right)\ ] \qquad (9)$$

Performance evaluation of the model

We're now ready to test the performance of the model, but we need real numbers to use in the equations, presented here are one set of numbers (and the ones used to develop the results in the next section). You may choose any other set or relevant numbers you prefer.

Parameters used:

n=15;
Function, f = cos(x$* y - y^2 - x)$
x value = 3;
y value = 2;
b value = 0;
a value = 0;
δy = 0.000;
δx = 0.005;

RESULTS AND DISCUSSION

From Table 1, when n=0, the actual difference between $f(x \pm \delta x)$ and f(x) that is [$f(x \pm \delta x)$ -f(x)] = -2.2222e-5 while proposed model E(x) = 0.0 subsequently the difference between [$f(x \pm \delta x)$ and E(x) = -2.2222e-5. When n= 1, [$f(x \pm \delta x)$ -f(x)] = -8.7847e-6, E(x) = 2.2211e-5 and their difference = 1.3426e-5.When n= 2, [$f(x \pm \delta x)$ -f(x)] = -7.7847e-6, E(x) = -8.7719e-6 and their difference = 9.8712e-7. When n= 3, [$f(x \pm \delta x)$ - f(x)] = -7.9863e-6, E(x) = -7.7724e-6 and their difference

Table 1. Comparison of results given by the developed model and the actual differences between the functions.

n	f(x)	f(x +δx)	f(x+ δx) - f(x)	E(x)	{f(x+δx) - f(x)}-E(x)
0	-0.08957	-0.089592	-0.000022222	0	-0.000022222
1	-0.10291	-0.10292	-8.7847E-06	-0.000022211	0.000013426
2	-0.098879	-0.098887	-7.7847E-06	-8.7719E-06	9.8712E-07
3	-0.098679	-0.098687	-7.9863E-06	-7.7724E-06	-2.1392E-07
4	-0.098709	-0.098717	-7.9938E-06	-0.000007974	-1.9835E-08
5	-0.09871	-0.098718	-7.9929E-06	-7.9815E-06	-1.1432E-08
6	-0.09871	-0.098718	-7.9929E-06	-7.9806E-06	-1.2317E-08
7	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2341E-08
8	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
9	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
10	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
11	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
12	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
13	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
14	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
15	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
16	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
17	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
18	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
19	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08
20	-0.09871	-0.098718	-7.9929E-06	-7.9805E-06	-1.2339E-08

= -2.1392e-7. When n= 4, [$f(x \pm \delta x)$ -f(x)] = -7.9938e-6,- E(x) = -7.974e-6 and their difference = -1.9835e-8. When n= 5, [$f(x \pm \delta x)$ -f(x)] = -7.9929e-6, E(x) = -7.9805e-6 and their difference = -1.1432e-8. When n= 6, [$f(x \pm \delta x)$ -f(x)] = -7.9929e-6, E(x) = -7.9806e-6 and their difference = -1.2317e-8. This continues until when n= 8, where the values of [$f(x \pm \delta x)$ -f(x)], E(x) and their difference stabilize (up to n= 21) and equal to -7.9929e-6, -7.9805e-6 and -1.2339e-8 respectively. In a nut shell, we are unable to see any significant difference between the results given by the model and the actual difference between the functions. It must be stretched out that the choice of trigonometric function for performance evaluation is arbitrary and no special reason was attached for choosing it.

Finally, it can be noticed that out of the three sources of error possible in many variables function, only initial error in the variable(s) which characterize(s) the given mathematical function was investigated. In order words, the two other sources of error that is error in the parameter (constants) in the function and an incorrect model of the underlying process are treated as absent. In subsequent research, search light will be beamed unto them.

Conclusion

Considering the absolute values of the differences between [$f(x \pm \delta x)$ -f(x)] and E(x) which have the high-est value of 0.000022222 and lowest value of 0.000007993, thus, it can be seen that the results given by the developed model is in good agreement with the results given by the actual difference between the functions f(x) and $f(x \pm \delta x)$, and as such the model can be used to predict the behaviour of any error introduced into any such function.

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