New generalizations of generalized geometric series distributions

Anwar Hassan

Department of Statistics, University of Kashmir, Srinagar, (J and K), India. E-mail: anwar.hassan2007@gmail.com, anwar_hassan2007@hotmail.com.

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New generalizations of generalized geometric series distributions (NGOGGSD) and some of its subclasses are introduced. Recurrence relations and factorial moments are obtained. Several properties of NGOGGSD and its relations with other distributions are also discussed. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. Some sports sets of data have been fitted to the proposed model and comparison has been made with its subclasses.

Keywords: NGOGGSD, GGSD, GSD, ACEGGSD, Quasi-uniform distribution, Bernoulli distribution.

INTRODUCTION

Mishra (1982) using the results of the lattice path analysis obtained a two parameter GGSD given by its probability function

\[ p(x; \alpha, \beta) = \frac{I(\beta+1)\alpha^x(1-\alpha)^{\beta-x}}{xI(\beta-x+2)} \text{; } x=0,1,2,..., \alpha<1, \beta<1 \]

(1)

It can be easily seen that at \( \beta=1 \) this distribution reduces to the geometric series distribution (GSD) and is a particular case of Jain and Consul's (1971) generalized negative binomial distribution in the same way as the geometric distribution is a particular case of the negative binomial distribution. The various aspects, interesting properties and fields of application of this distribution have been studied by Mishra (1982), Singh (1989), Mishra and Singh (1992) and Hassan (1995). Hassan et al. (2002; 2007; 2008) studied estimation of GGSD. They found this distribution to provide much closer fits to all those observed distributions where the geometric distribution and the various compound geometric distributions have been fitted earlier by many authors. A brief list of authors and their works can be seen in Johnson et al. (1992) and Consul and Famoye (2006).

In this paper we have made an attempt to define NGOGGSD which represents a wider class than GGSD. One naturally may be curious to know whether the properties of GGSD are also shared by NGOGGSD class. It has been found that this indeed is the case. Moreover, a general expression for moments of various distributions from NGOGGSD was obtained. Several properties of NGOGGSD and its relations with other distributions are also discussed. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. Some sports data sets have been fitted to the proposed model and comparison has been made with its subclasses.

NEW GENERALIZATIONS OF GENERALIZED GEOMETRIC SERIES DISTRIBUTIONS

For a discrete random variable \( X \), NGOGGSD with two parameters \( \alpha \) and \( \beta \) and a given integer \( j \) is defined by a probability function

\[ p_j(x; \alpha, \beta) = \frac{I(\beta+1)\alpha^x\alpha^{j+x}\beta^{-x-j}}{xI(\alpha\beta)(\beta+1)^{x+2}} \text{; } x=0,1,0<\alpha<1, \alpha\beta<1 \]

(2)

\[ = 0, \text{ if } x \geq n \text{ if } 1 + n\beta + j < 0 \]
where \( M_j(\alpha, \beta) = \sum_{x=0}^{\infty} \frac{\Gamma(1+\beta x)(1-\alpha)^{1+\beta x+j}}{x! \Gamma(\beta x-x+j+2)} \)

(3)

And \( n \) is maximum value of \( x \).

**Special cases:** It can easily be seen that for different value of \( j \) and \( \beta \) in (2) which gives as;

i) \( j = 0 \) it reduces to (1)

ii) \( \beta = 1 \), it reduces to a class of extended geometric series distribution.

iii) \( j = 0 \) and \( \beta = 1 \), It reduces to simple geometric series distribution.

iv) \( \beta = 0 \), It gives the probability function of a class of extended Bernoulli distributions.

v) \( j = 0 \) and \( \beta = 0 \), it reduces to simple Bernoulli distribution.

Recurrence relations for \( M_j(\alpha, \beta) \)

Using \( m^{(n)} = m(m-1)....(m-n+1) \), we can write (3) as:

\[
M_j(\alpha, \beta x) = \sum_{x=0}^{\infty} \frac{\beta x! \alpha^x(1-\alpha)^{1+\beta x-x+j}}{x!(\beta x-x+j+1)!} \\
= \sum_{x=0}^{\infty} \beta x(\beta x-1)....(\beta x-x+j+2) \frac{\alpha^x}{x!} (1-\alpha)^{1+\beta x-x+j} \\
= \sum_{x=0}^{\infty} (\beta x)^{x-j-1} \frac{\alpha^x}{x!} (1-\alpha)^{1+\beta x-x+j} \\
= \frac{(j+2)}{(1-\alpha)} \sum_{x=0}^{\infty} (\beta x)^{(x-j-2)} \frac{\alpha^x}{x!} (1-\alpha)^{1+\beta x-x+j+1} \\
+ \frac{\alpha(\beta-1)}{(1-\alpha)} \sum_{x=0}^{\infty} (1+\beta x-1)^{(x-j-1)} \frac{\alpha^x}{x!} (1-\alpha)^{1+\beta x-x+j} \\
= \frac{1}{(1-\alpha)} \left[ (j+2)M_{j+1}(\alpha, \beta) + \alpha(\beta-1)M_j(\alpha, \beta + \beta) \right] \\
= \frac{1}{(j+2)} \left[ (j+2)M_{j+1}(\alpha, \beta) + \alpha(\beta-1)M_j(\alpha, \beta + \beta) \right] \\
(3)

The repeated use of which may give \( M_{j+1}(\alpha, \beta) \) is known. The equation (3) can also be written as

\[
M_{j+1}(\alpha, \beta) = \frac{1}{(j+2)} \left[ (j+2)M_{j+1}(\alpha, \beta) + \alpha(\beta-1)M_j(\alpha, \beta + \beta) \right] \\
= \frac{1}{(j+2)} \left[ (j+2)M_{j+1}(\alpha, \beta) + \alpha(\beta-1)M_j(\alpha, \beta + \beta) \right] \\
(4)

**Factorial moments**

\[
\mu_r = \frac{1}{M_j(\alpha, \beta)} \sum_{x=r}^{\infty} \frac{\Gamma(1+\beta x)^{r}}{\Gamma(\beta x-x+j+2)x!} (1-\alpha)^{1+\beta x-x+j} \\
= \frac{1}{M_j(\alpha, \beta)} \sum_{x=r}^{\infty} \frac{\Gamma(1+\beta x)^{r}}{\Gamma(\beta x-x+j+2)x!} (1-\alpha)^{1+\beta x-x+j} \\
= \frac{1}{M_j(\alpha, \beta)} \sum_{x=r}^{\infty} \frac{\Gamma(1+\beta x)^{r}}{\Gamma(\beta x-x+j+2)x!} (1-\alpha)^{1+\beta x-x+j} \\
= \alpha^r \frac{M_{j-1}(\alpha, \beta x)}{M_j(\alpha, \beta x)}; r = 1, 2, 3, 4, ....
(5)

**Sub-classes of a class of extended GGSD**

From Mishra’s (1982) GGSD (1), it is clear \( M_0(\alpha, \beta x) = 1 \) and we may call this ACEGGSD as

GGSD or ACEGGSD. The expressions for some \( M_j \)’s are summarized in Table 1 which provides some particular ACEGGSD’s.

**SOME PROPERTIES**

Mishra and Singh (1992) proved some properties of the GGSD. A question arises whether there exists a wider class of distributions which shares these properties. It turns out that NGOGGSD is the desired class. Below we list and prove the respective properties.

**Property**

Let \( X_k \) have the distribution \( P_k(X_k, \alpha, \beta) \) \((k = 1, 2)\) and \( X_k \) be independent .The sum \( X_1 + X_2 = z \) \((z \) is a non-negative integer) has the distribution \( P_{j+1}(z, \alpha, \beta) \)
Table 1. Special cases of $\frac{1}{M_j(\alpha, \beta x)}$ in the NGOGSD class.

<table>
<thead>
<tr>
<th>NGOGSD$^j$</th>
<th>$j$</th>
<th>$\frac{1}{M_j(\alpha, \beta x)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGOGSD$^0$</td>
<td>0</td>
<td>$\frac{1}{2(1 + \beta)}$</td>
</tr>
<tr>
<td>NGOGSD$^1$</td>
<td>1</td>
<td>$\frac{1}{(1 + \beta - 2\alpha \beta)}$</td>
</tr>
<tr>
<td>NGOGSD$^2$</td>
<td>2</td>
<td>$\frac{6(1 + \beta)(2 + \beta)(1 + 2\beta)}{(1 - \alpha)[(1 + \beta - 2\alpha \beta)(2 + \beta)(1 + 2\beta)] - 2\alpha(\beta - 1)[1 + 2\beta - \alpha \beta(2 + \beta)]}$</td>
</tr>
</tbody>
</table>

**Proof**

The model (2) can be easily written as

$$p_j(x, \alpha, \beta) = \frac{(\beta x)! x^j (1 - \alpha)^{1+\beta x - x + j}}{(\beta x - x + j + 1)!} \cdot x! \cdot M_j(\alpha, \beta x)$$

$$= \left(1 + \frac{j + \beta x}{x}\right) \frac{\alpha^x (1 - \alpha)^{1+\beta x - x + j}}{M_j(\alpha, \beta x)(1 + \beta x)^{[j+1]}}$$

Where

$$(1 + \beta x)^{[j+1]} = (1 + \beta x)(2 + \beta x)\ldots (1 + \beta x + j)$$

We have

$$P(X_1 + X_2 = z) = \sum_{x=0}^{z} p_j(x, \alpha, \beta)p_j(z-x, \alpha, \beta)$$

$$= \sum_{x=0}^{z} \left[ \frac{(1+j_1+j_2+\beta x)}{x} \frac{\alpha^x (1-\alpha)^{1+\beta x - x + j_1}}{M_{j_1}(\alpha, \beta x)(1+\beta x)^{[j_1+1]}} \right] \left[ \frac{(1+j_2+\beta x - \beta x)}{z-x} \frac{\alpha^{z-x} (1-\alpha)^{1+\beta x - z + j_2}}{M_{j_2}(\alpha, \beta x)(1+\beta x)^{[j_2+1]}} \right]$$

Equating the coefficient of $\alpha^x (1 - \alpha)^{\beta x - x}$ on both sides of the identity

$$(1 - \alpha)^{-2 - \sum_{i=1}^{2} j_i} = (1 - \alpha)^{-j_1} (1 - \alpha)^{-j_2} ,$$

(7) Can be written as

$$P(X_1 + X_2 = z) = \frac{\alpha^{-2 + \sum_{k=1}^{2} j_k + \beta z - z}}{M_{j_1+j_2}(\alpha, \beta x)(2 + \beta x)^{[j_1+j_2+1]}}.$$
Table 2. Runs scored by Gavaskar in 136 completed innings.

<table>
<thead>
<tr>
<th>Runs (units of 30)</th>
<th>Observed frequency</th>
<th>Expected frequency</th>
<th>GSD</th>
<th>GGSD or NGOGGSD</th>
<th>NGOGGSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>64</td>
<td>58.84</td>
<td>56.1</td>
<td>59.2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>27</td>
<td>35.62</td>
<td>35.4</td>
<td>34.7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>18.58</td>
<td>20.3</td>
<td>19.4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>10.27</td>
<td>12.3</td>
<td>10.6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5.67</td>
<td>5.8</td>
<td>5.7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>7.02</td>
<td>6.1</td>
<td>6.4</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>136</td>
<td>136</td>
<td>136</td>
<td>136</td>
<td></td>
</tr>
</tbody>
</table>

Mean = 1.2279
Variance = 2.5289

\[ \chi^2 = 3.45 \quad 3.39 \quad 2.74 \]

d.f. = 4 \quad 3 \quad 3

Estimates
\[ \hat{\alpha} = 0.4473679 \quad 0.5871091 \quad 0.5646005 \]
\[ \hat{\beta} = 0.8888888 \quad 0.9567918 \]

Property

Let \( X_k \) have the distribution \( P_{j_k}(X_k, \alpha, \beta) \) \((k = 1, 2)\) and \( X_k \) be independent. The conditional distribution of \( X_1 = x \) given \( X_1 + X_2 = z \) has a class of extended quasi-uniform distribution.

Proof

Mishra and Singh (1992) defined a quasi-uniform distribution with probability mass function as

\[
P(x, z-x, \beta) = \frac{(1 + \beta z)\left(1 + \beta z - \beta x\right)(2 + \beta z)}{(1 + \beta z)(1 + \beta z - \beta x)2\left(2 + \beta z\right)}
\]

Taking the conditional probability of \( X_1 = x \) given \( X_1 + X_2 = z \) leads to a class of extended quasi-uniform distribution as

\[
P\left(X_1 = \frac{x}{X_1 + X_2 = z}\right) = \frac{P_j(x, \alpha, \beta)P_j(z-x, \alpha, \beta)}{\sum_{x=0}^{z} P_j(x, \alpha, \beta)P_j(z-x, \alpha, \beta)}
\]

Using (6) and (9), the conditional distribution in (10) can be written as

\[
P\left(X_1 = \frac{x}{X_1 + X_2 = z}\right) = \frac{(1 + j_1 + \beta x)\left(1 + j_2 + \beta z - \beta x\right)}{x\left(z-x\right)}
\]

\[= M_{j_1}(\alpha, \beta x)M_{j_2}(\alpha, \beta z - \beta x)(1 + \beta x)^{(j_1 + j_2 + 1)}
\]

\[
M_{j_1+j_2}(\alpha, \beta x)(2 + \beta z)^{(j_1 + j_2 + 1)} \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)}
\]

\[
\left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)} \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)}
\]

\[= \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)} \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)}
\]

\[= \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)} \left(1 + \beta z - \beta x\right)^{(j_1 + j_2 + 1)}
\]
Table 3. Wickets taken by Sobers in 158 completed innings.

<table>
<thead>
<tr>
<th>Runs (units of 30)</th>
<th>Observed frequency</th>
<th>Expected frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>GSD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GGSD or NGOGGSD₀</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NGOGGSD₁</td>
</tr>
<tr>
<td>0</td>
<td>49</td>
<td>57.0</td>
</tr>
<tr>
<td>1</td>
<td>41</td>
<td>36.4</td>
</tr>
<tr>
<td>2</td>
<td>31</td>
<td>23.3</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
<td>14.9</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>9.5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>6.2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>9.9</td>
</tr>
<tr>
<td>Total</td>
<td>158</td>
<td>158</td>
</tr>
</tbody>
</table>

Mean = 1.487317
Variance = 1.97135876

\[ \hat{\alpha} = 0.6538156 \]
\[ \hat{\beta} = 0.8571429 \]

This is extended class of quasi uniform distribution.

**Special cases**

i) For \( j₁ = j₂ = 0 \) in (11), we get Mishra and Singh's (1992) quasi-uniform distribution (9)

ii) For \( j₁ = j₂ = 0, \beta = 1 \) in (11), we get classical uniform distribution.

**SOME APPLICATIONS**

In this section, some sports data sets have been taken to examine the fitting of proposed model NGOGGSD (2) and comparing that with GSD and GGSD models. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. The data sets presented are Tables 2 and 3.

Thus, it is evident from the tables that NGOGGSD₁ gives closer fits as compared to GSD, GGSD or NGOGGSD₀ model. Therefore it may be suggested that in all those situations where other distributions have been fitted by different authors, the NGOGGSD₁ can be fitted since the discrepancy between the observed and expected frequency is small.

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**REFERENCES**