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Full Length Research Paper

New generalizations of generalized geometric series distributions

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New generalizations of generalized geometric series distributions (NGOGGSD) and some of its subclasses are introduced. Recurrence relations and factorial moments are obtained. Several properties of NGOGGSD and its relations with other distributions are also discussed. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. Some sports sets of data have been fitted to the proposed model and comparison has been made with its subclasses.

Key words: NGOGGSD, GGSD, GSD, $ACEGGSD_0$, Quasi-uniform distribution, Bernoulli distribution.

INTRODUCTION

Mishra (1982) using the results of the lattice path analysis obtained a two parameter GGSD given by its probability function

$$p(x;\alpha,\beta)=\frac{\Gamma(\beta x+1)\alpha^{x}(1-\alpha)^{1+\beta x-x}}{x!\,\Gamma(\beta x-x+2)}\ ;x=0,1,2\ldots,0<\alpha<1,\ |\alpha\beta|<1 \tag{1}$$

It can be easily seen that at β=1 this distribution reduces to the geometric series distribution (GSD) and is a particular case of Jain and Consul's (1971) generalized negative binomial distribution in the same way as the geometric distribution is a particular case of the negative binomial distribution. The various aspects, interesting properties and fields of application of this distribution have been studied by Mishra (1982), Singh (1989), Mishra and Singh (1992) and Hassan (1995). Hassan et al. (2002; 2007; 2008) studied estimation of GGSD. They found this distribution to provide much closer fits to all those observed distributions where the geometric distribution and the various compound geometric distributions have been fitted earlier by many authors. A brief list of authors and their works can be seen in Johnson et al. (1992) and Consul and Famoye (2006).

In this paper we have made an attempt to define NGOGGSD which represents a wider class than GGSD. One naturally may be curious to know whether the properties of GGSD are also shared by NGOGGSD class. It has been found that this indeed is the case. Moreover, a general expression for moments of various distributions from NGOGGSD was obtained. Several properties of NGOGGSD and its relations with other distributions are also discussed. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. Some sports data sets have been fitted to the proposed model and comparison has been made with its subclasses.

NEW GENERALIZATIONS OF GENERALIZED GEOMETRIC SERIES DISTRIBUTIONS

For a discrete random variable X, NGOGGSD with two parameters α and β and a given integer j is defined by a probability function

$$p_j(x;\alpha,\beta)=\frac{\Gamma(1+\beta x)\alpha^{x}(1-\alpha)^{1+\beta x-x+j}}{x!\,M_j(\alpha,\beta)\,\Gamma(\beta x-x+j+2)};x=0,1,\ldots,0<\alpha<1,\ |\alpha\beta|<1 \tag{2}$$

$$=0,\ x\geq n \text{ if } 1+n\beta+j\prec 0$$

$$where M_j(\alpha,\beta x)=\sum_{x=0}^{\infty}\frac{\Gamma(1+\beta x)\alpha^x(1-\alpha)^{1+\beta x-x+j}}{x!\,\Gamma(\beta x-x+j+2)} \tag{3}$$

And n is maximum value of x.

Special cases: It can easily be seen that for different value of j and β in (2) which gives as;

i) $j=0$ it reduces to (1)

ii) $\beta=1$, it reduces to a class of extended geometric series distribution.

iii) $j=0$ and $\beta=1$, It reduces to simple geometric series distribution.

iv) $\beta=0$, It gives the probability function of a class of extended Bernoulli distributions.

v) $j=0$ and $\beta=0$, it reduces to simple Bernoulli distribution.

Recurrence relations for M_j (α,β)

Using $m^{(n)}=m(m-1)........(m-n+1)$, we can write (3) as:

$$M_j(\alpha,\beta x)=\sum_{x=0}^{\infty}\frac{(\beta x)!\ \alpha^x(1-\alpha)^{1+\beta x-x+j}}{x!(\beta x-x+j+1)!}$$

$$=\sum_{x=0}^{\infty}\beta x(\beta x-1)....(\beta x-x+j+2)\frac{\alpha^x}{x!}(1-\alpha)^{1+\beta x-x+j}$$

$$=\sum_{x=0}^{\infty}(\beta x)^{(x-j-1)}\frac{\alpha^x}{x!}(1-\alpha)^{1+\beta x-x+j}$$

$$=\frac{(j+2)}{(1-\alpha)}\sum_{x=0}^{\infty}(\beta x)^{(x-j-2)}\frac{\alpha^x}{x!}(1-\alpha)^{2+\beta x-x+j-1}$$

$$+\frac{\alpha(\beta-1)}{(1-\alpha)}\sum_{x=0}^{\infty}(1+\beta+\beta x-1)^{(x-j-1)}\frac{\alpha^x}{x!}(1-\alpha)^{1+\beta+\beta x-x+j}$$

$$=\frac{1}{(1-\alpha)}\left[(j+2)M_{j+1}(\alpha,\beta x)+\alpha(\beta-1)M_j(\alpha,\beta+\beta x)\right] \tag{3}$$

The repeated use of which may give $M_{j+1}(\alpha,\beta)$ is known. The equation (3) can also be written as

$$M_{j+1}(\alpha,\beta x)=\frac{1}{(j+2)}\left[(1-\alpha)M_j(\alpha,\beta x)-\alpha(\beta-1)M_j(\alpha,\beta+\beta x)\right] \tag{4}$$

Factorial moments

$$\mu'_{(r)}=\frac{1}{M_j(\alpha,\beta x)}\sum_{x=r}^{\infty}x^{(r)}\frac{\Gamma(1+\beta x)\alpha^x}{\Gamma(\beta x-x+j+2)x!}(1-\alpha)^{1+\beta x-x+j}$$

$$=\frac{1}{M_j(\alpha,\beta x)}\sum_{x=r}^{\infty}\frac{\Gamma(1+\beta x)\alpha^x}{\Gamma(\beta x-x+j+2)(x-r)!}(1-\alpha)^{1+\beta x-x+j}$$

$$=\frac{1}{M_j(\alpha,\beta x)}\sum_{x=0}^{\infty}\frac{\Gamma(1+\beta r+\beta x)\alpha^{x+r}(1-\alpha)^{1+\beta r+\beta x-x-r+j}}{\Gamma(\beta x-x+\beta r-r+j+2)x!}$$

$$=\alpha^r\frac{M_{j-r}(\alpha,\beta x+\beta r)}{M_j(\alpha,\beta x)};\ r=1,2,3,4,.... \tag{5}$$

Sub-classes of a class of extended GGSD

From Mishra's (1982) GGSD (1), it is clear $M_0(\alpha,\beta x)=1$ and we may call this ACEGGSD as GGSD or ACEGGSD$_0$. The expressions for some M_j's are summarized in Table 1 which provides some particular ACEGGSD'S.

SOME PROPERTIES

Mishra and Singh (1992) proved some properties of the GGSD. A question arises whether there exists a wider class of distributions which shares these properties. It turns out that NGOGGSD is the desired class. Below we list and prove the respective properties.

Property

Let X_k have the distribution $P_{j_k}(X_k,\alpha,\beta)$ (k =1, 2) and X_k be independent .The sum $X_1+X_2=z$ (z is a non – negative integer) has the distribution $P_{j_1+j_2}(z,\alpha,\beta)$

Table 1. Special cases of $\dfrac{1}{M_j(\alpha,\beta x)}$ in the NGOGGSD class.

$NGOGGSD_j$	j	$\dfrac{1}{M_j(\alpha,\beta x)}$
$NGOGGSD_0$	0	1
$NGOGGSD_1$	1	$\dfrac{2(1+\beta)}{(1+\beta-2\alpha\beta)}$
$NGOGGSD_2$	2	$\dfrac{6(1+\beta)(2+\beta)(1+2\beta)}{(1-\alpha)\{[(1+\beta)-2\alpha\beta](2+\beta)(1+2\beta)\}-2\alpha(\beta-1)[1+2\beta-\alpha\beta(2+\beta)]}$

Proof

The model (2) can be easily written as

$$p_j(x,\alpha,\beta)=\frac{(\beta x)!\,\alpha^x(1-\alpha)^{1+\beta x-x+j}}{(\beta x-x+j+1)!\quad x!\;M_j(\alpha,\beta x)}$$

$$=\binom{1+j+\beta x}{x}\frac{\alpha^x(1-\alpha)^{1+\beta x-x+j}}{M_j(\alpha,\beta x)(1+\beta x)^{[j+1]}} \tag{6}$$

Where
$(1+\beta x)^{[j+1]}=(1+\beta x)(2+\beta x)\ldots\ldots\ldots(1+\beta x+j)$

We have

$$P(X_1+X_2=z)=\sum_{x=0}^{z}p_{j_1}(x,\alpha,\beta)p_{j_2}(z-x,\alpha,\beta)$$

$$=\sum_{x=0}^{z}\left[\binom{1+j_1+\beta x}{x}\frac{\alpha^x(1-\alpha)^{1+\beta x-x+j_1}}{M_{j_1}(\alpha,\beta x)(1+\beta x)^{[j_1+1]}}\right]\left[\binom{1+j_2+\beta z-\beta x}{z-x}\frac{\alpha^{z-x}(1-\alpha)^{1+\beta z-\beta x-z+x+j_2}}{M_{j_2}(\alpha,\beta z-\beta x)(1+\beta z-\beta x)^{[j_2+1]}}\right]$$

$$=\frac{\alpha^z(1-\alpha)^{2+\sum_{k=1}^{2}j_k+\beta z-z}}{M_{j_1}(\alpha,\beta x)M_{j_2}(\alpha,\beta z-\beta x)}\;\frac{\sum_{x=0}^{z}\binom{1+j_1+\beta x}{x}\binom{1+j_2+\beta z-\beta x}{z-x}}{(1+\beta x)^{[j_1+1]}(1+\beta z-\beta x)^{[j_2+1]}} \tag{7}$$

Equating the coefficient of $\alpha^x(1-\alpha)^{\beta x-x}$ on both sides of the identity

$$(1-\alpha)^{-2-\sum_{k=1}^{2}j_k}=(1-\alpha)^{-1-j_1}(1-\alpha)^{-1-j_2},$$

(7) Can be written as

$$P(X_1+X_2=z)=\frac{\alpha^z(1-\alpha)^{2+\sum_{k=1}^{2}j_k+\beta z-z}\binom{2+\sum_{k=1}^{2}j_k+\beta z}{z}}{M_{j_1+j_2}(\alpha,\beta x)(2+\beta z)^{[j_1+j_2+1]}}. \tag{8}$$

Table 2. Runs scored by Gavaskar in 136 completed innings.

Runs (units of 30)	Observed frequency	Expected frequency		
		GSD	GGSD or NGOGGSD$_0$	NGOGGSD$_1$
0	64	58.84	56.1	59.2
1	27	35.62	35.4	34.7
2	20	18.58	20.3	19.4
3	13	10.27	12.3	10.6
4	5	5.67	5.8	5.7
5	3			
6	2	7.02	6.1	6.4
7	1			
8	1			
Total	136	136	136	136
Mean = 1.2279				
Variance = 2.5289				
	χ^2	3.45	3.39	2.74
	d.f.	4	3	3
Estimates	$\hat{\alpha}$	0.4473679	0.5871091	0.5646005
	$\hat{\beta}$		0.8888888	0.9567918

Property

Let X_k have the distribution $P_{j_k}(X_k, \alpha, \beta)$ (k=1, 2) and X_k be independent. The conditional distribution of $X_1 = x$ given $X_1 + X_2 = z$ has a class of extended quasi-uniform distribution.

Proof

Mishra and Singh (1992) defined a quasi-uniform distribution with probability mass function as

$$P\left(x, z-x, \beta\right) = \frac{\binom{1+\beta z}{x}\binom{1+\beta z-\beta x}{z-x}(2+\beta z)}{(1+\beta z)(1+\beta z-\beta x)\,2\binom{2+\beta z}{z}} \tag{9}$$

Taking the conditional probability of $X_1 = x$ given $X_1 + X_2 = z$ leads to a class of extended quasi-uniform distribution as

We have

$$P\left(X_1 = x \middle/ X_1 + X_2 = z\right) = \frac{P_{j_1}(x,\alpha,\beta)P_{j_2}(z-x,\alpha,\beta)}{\sum_{x=0}^{z} P_{j_1}(x,\alpha,\beta)P_{j_2}(z-x,\alpha,\beta)} \tag{10}$$

Using (6) and (9), the conditional distribution in (10) can be written as

$$P\left(X_1 = x \middle/ X_1 + X_2 = z\right)$$

$$= \frac{\binom{1+j_1+\beta x}{x}\binom{1+j_2+\beta z-\beta x}{z-x}}{M_{j_1}(\alpha,\beta x)M_{j_2}(\alpha,\beta z-\beta x)(1+\beta x)^{[j_1+1]}}$$

$$\frac{M_{j_1+j_2}(\alpha,\beta x)(2+\beta z)^{[j_1+j_2+1]}}{(1+\beta z-\beta x)^{[j_2+1]}\binom{2+\sum_{k=1}^{2} j_k+\beta z}{z}} \tag{11}$$

Table 3. Wickets taken by Sobers in 158 completed innings.

Runs (units of 30)	Observed frequency	Expected frequency		
		GSD	GGSD or $NGOGGSD_0$	$NGOGGSD_1$
0	49	57.0	54.7	51.1
1	41	36.4	41.6	43.0
2	31	23.3	27.1	29.2
3	23	14.9	17.2	17.4
4	28	9.5	9.1	9.4
5	5		6.2	6.3
6	1	9.9	2.1	1.6
Total	158	158	158	158
Mean = 1.487317				
Variance = 1.97135876				
Estimates	χ^2		3.89	2.69
	d.f		3	3
	$\hat{\alpha}$		0.6538156	0.674130
	$\hat{\beta}$		0.8571429	0.8060461

This is extended class of quasi uniform distribution.

Special cases

i) For $j_1 = j_2 = 0$ in (11), we get Mishra and Singh's (1992) quasi-uniform distribution (9)

ii) For $j_1 = j_2 = 0$, $\beta = 1$ in (11), we get classical uniform distribution.

SOME APPLICATIONS

In this section, some sports data sets have been taken to examine the fitting of proposed model NGOGGSD (2) and comparing that with GSD and GGSD models. Finally, a computer programmed in R-Software has been developed to ease the computations while estimating the parameters for data. The data sets presented are Tables 2 and 3.

Thus, it is evident from the tables that $NGOGGSD_1$ gives closer fits as compared to GSD, GGSD or $NGOGGSD_0$ model. Therefore it may be suggested that in all those situations where other distributions have been fitted by different authors, the $NGOGGSD_1$ can be fitted since the discrepancy between the observed and expected frequency is small.

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