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Triple Shehu transform and its properties with applications

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In the current paper, the concept of one-dimensional Shehu Transform have been generalized into three-dimensional Shehu Transform namely, Triple Shehu Transform (TRHT). Further, some main properties, several theorems and properties related to the TRHT have been established. Triple Shehu transform was used in solving fractional partial differential equations, with the fractional derivative described in Caputo sense. The proposed scheme finds the solution without any discretization, transformation or restrictive assumptions. Several examples are given to check the reliability and efficiency of the proposed technique.

Key words: Caputo fractional derivative, exponential order, Triple Shehu transforms, partial derivative, uniqueness.

INTRODUCTION

integral transforms is one of the most easy and effective methods for solving problems arising in Mathematical Physics, Applied Mathematics and Engineering Science which are defined by differential equations, difference equations and integral equations. The main idea in the application of the method is to transform the unknown function of some variable t to a different function of a complex variable. With this, the associated differential equation can be directly reduced to either a differential equation of lower dimension or an algebraic equation in the new variable. There are several forms of integral transforms such as Laplace transform (Papoulis, 1957, Debnath and Bhatta, 2014, Rehman et al., 2014 and Dhunde et al., 2013), Sumudu transform (Kılıcman and Gadain, 2010; Mahdy et al., 2015 and Mahdy et al., 2015a), Eltayeb and Kilicman, 2010, and Mechee and Naeemah, 2020. Aboodh transform (Aboodh, 2013), Elzaki transform (Elzaki, 2011), Variational homotopy perturbation method (Mahdy et al., 2015b), Alternative variational iteration method (Mtawal et al., 2020) and one form may be obtained from the other by a transformation of the coordinates and the functions. Recently, in 2019 Maitama and Zhao introduced a new type of integral transform as a generalization of both Laplace transform and Sumudu transform for solving differential equations in the time domain, and provided some theorems on this transform. The study was further reinforced by Issa and Mensah (2020), Alfaqeih and Misirli (2020), Aggarwal et al. (2019), Mahdy and Mtawal (2016), Mtawal and Alkaleeli (2020). In this paper, we extend and generalized

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some results of Thakur et al. (2018), Abdon (2013) and Alfaqeih (2019); in particular, we extend one-dimensional Shehu transform into three-dimensional Shehu transform, and provide some examples to show the effectiveness of our results.

PRELIMINARIES

Here, we recall the definitions of Shehu and Double Shehu transform.

Definition 1

The Shehu Transform ($\mathcal{H}$) (Maitama and Zhao, 2019) is defined over the set of the functions

$$B=\left\{ f(t):\exists M,\ \mu_1,\ \mu_2>0,\ |f(t)|<M e^{\left(\frac{|x|}{\mu_j}\right)},\ if\ t\in(-1)^j\times[0,\infty) \right\}$$

by the following formula

$$\begin{aligned} \mathrm{H}(f(t)) &= F(s,u)=\int_0^\infty e^{\left(\frac{-st}{u}\right)} f(x)\,dt \\ &= \lim_{\alpha\to\infty}\int_0^\alpha e^{\left(\frac{-st}{u}\right)} f(t)\,dt;\ s,u>0. \end{aligned} \tag{1}$$

And the inverse Shehu transform is defined by

$$\begin{aligned} \mathrm{H}^{-1}(F(s,u)) &= f(t), t\geq 0, \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{st}{u}\right)}F(s,u)\,ds\ . \end{aligned} \tag{2}$$

Definition 2

A real function $f(t), t>0,$ is considered to be in the space $C_m, m\in R,$ if there exists a real number $\sigma>m,$ so that $f(t)=t^{\sigma}g(t),$ where $g(t)\in[0,\infty),$ and it is said to be in the space $C_\sigma^m,$ if $f^m\in C_\sigma, m\in N$. (Podlubny, 1999; He, 2014).

Definition 3

The left-sided Riemann–Liouville fractional integral of order $\alpha\geq 0,$ of a function $f\in C_\sigma, \sigma\geq -1,$ (Podlubny, 1999; He, 2014) is defined as:

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ t,\alpha>0. \tag{3}$$

Here $\Gamma(.)$ is the gamma function.

Definition 4

If $f\in C_m^n, n\in N\cup\{0\}$. The left Caputo fractional derivative of f in the Caputo sense (Podlubny, 1999; He, 2014) is defined as follows:

$$D_t^\alpha f(t)=\begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau, & n\text{-}1<\alpha\leq n, \\ D_t^\alpha f(t), & \alpha=n. \end{cases} \tag{4}$$

Definition 5

The Mittag–Leffler function E_α, $E_{\alpha,\beta}$ (Kilbas et al., 2004) are defined as

$$E_\alpha(t)=\sum_{n=0}^\infty \frac{t^n}{\Gamma(n\alpha+1)},\ \ \alpha\in R,\ \mathrm{Re}(\alpha)>0, \tag{5}$$

$$E_{\alpha,\beta}(t)=\sum_{n=0}^\infty \frac{t^n}{\Gamma(n\alpha+\beta)},\ \ \alpha,\beta\in R,\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0. \tag{6}$$

These functions are generalization of the exponential function. Some special cases of the Mittag-Leffler function are as follows:

$$E_1(t)=e^t,\ E_{\alpha,1}(t)=E_\alpha(t).$$

Theorem 1

If $\alpha>0$, $a\in R$ and $|a|<\left(\frac{s}{u}\right)^\alpha$, (Khalouta and Kadem, 2019), then

$$\mathrm{H}^{-1}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^\alpha+a}\right]=t^{\beta-1}E_{\alpha,\beta}(-at^\alpha).$$

Definition 6

The single Shehu Transform ($\mathcal{H}$) of a function $f(x,y,z)$

with respect to the variables *x*, *y* and *z* respectively (Maitama and Zhao, 2019) are defined by:

$$\mathrm{H}_x\left(f\left(x,y,z\right)\right)=\int_0^\infty e^{-\left(\frac{sx}{u}\right)} f\left(x,y,z\right)dx, \tag{7}$$

$$\mathrm{H}_y\left(f\left(x,y,z\right)\right)=\int_0^\infty e^{-\left(\frac{qy}{v}\right)} f\left(x,y,z\right)dy, \tag{8}$$

$$\mathrm{H}_z\left(f\left(x,y,z\right)\right)=\int_0^\infty e^{-\left(\frac{rz}{k}\right)} f\left(x,y,z\right)dz\ . \tag{9}$$

Definition 7

The double Shehu Transform (H^2) of a function $f(x,y)$ (Alfaqeih and Misirli, 2020) over the set of the functions

$$A=\left\{f\left(x,y\right):\exists M,\ \mu_1,\ \mu_2>0,\left|f\left(x,y\right)\right|<M\,e^{\left(\frac{|x+y|}{\mu_j}\right)},\ if\ \left(x,y\right)\in \mathbb{R}_+^2,\ j=1,2\right\}$$

by the following formula

$$\mathrm{H}_{xy}^2\left(f\left(x,y\right)\right)=F\left[\left(s,q\right),\left(u,v\right)\right]=\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)} f\left(x,y\right)dx\,dy\ , \tag{10}$$

and the inverse double Shehu transform is defined by

$$\mathrm{H}_{xy}^{-2}(F[(s,q),(u,v)])=f\left(x,y\right) \tag{11}$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\mathrm{H}_2\left(f\left(x,y\right)\right)dq\right)ds.$$

Definition 8

The double Shehu Transform $\left(\mathrm{H}^2\right)$ of a function $f(x,y,z)$ with respect to *xy*, *xz* and *yz* respectively (Alfaqeih and Misirli, 2020), are defined by:

$$\begin{aligned}
\mathrm{H}_{xy}^2\left(f\left(x,y,z\right)\right)&=\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)} f\left(x,y,z\right)dx\,dy,\\
\mathrm{H}_{xz}^2\left(f\left(x,y,z\right)\right)&=\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{rz}{k}\right)} f\left(x,y,z\right)dxdz,\\
\mathrm{H}_{yz}^2\left(f\left(x,y,z\right)\right)&=\int_0^\infty\int_0^\infty e^{-\left(\frac{qy}{v}+\frac{rz}{k}\right)} f\left(x,y,z\right).
\end{aligned} \tag{12}$$

RESULTS

Here, we introduce the definition of Triple Shehu transform and Triple Shehu transform of partial and fractional derivatives which are used further in this paper; moreover, we apply Triple Shehu transform for some basic functions.

Definition 9

Let f be a continuous function of three variables; then, the Triple Shehu transform (TRST) of (, y, z) is defined by

$$\mathrm{H}_{xyz}^3\left(f\left(x,y,z\right)\right)=F\left[\left(s,q,r\right),\left(u,v,k\right)\right]=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} f\left(x,y,z\right)dx\,dy\,dz. \tag{13}$$

Where $x,y,z\geq 0$ and s,q,r,u,v and k, are Shehu variables, provided the integral exists.

Also, the inverse Triple Shehu transform is defined by

$$\mathrm{H}_{xyz}^{-3}(F[(s,q,r),(u,v,k)])=f\left(x,y,z\right)$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{k}e^{\left(\frac{rz}{v}\right)}\mathrm{H}_3\left(f\left(x,y,z\right)\right)dr\right)dq\right]ds. \tag{14}$$

Existence and uniqueness of TRST

Here, we debate the existence and uniqueness of the Triple Shehu transform and prove it.

Definition 10

A function $f\left(x,y,z\right)$ is said to be of exponential order $a>0,b>0,c>0,$ as $x,y,z\rightarrow\infty$ if there are positive constants M, X, Y and Z (Alfaqeih, 2019) such that

$$\left|f\left(x,y,z\right)\right|\leq M\,e^{(ax+by+cz)}\quad \text{for all } x>X,\ y>Y,\ z>Z,$$

and, we write

$f\left(x,y,z\right)=O\left(e^{ax+by+cz}\right)$ (as $x,y,z\rightarrow\infty$).

Or, equivalently,

$$\sup_{x,y,z>0}\left(\frac{\left|f\left(x,y,z\right)\right|}{e^{(ax+by+cz)}}\right)<\infty.$$

Theorem 2

Let $f\left(x,y,z\right)$ be a continuous function on the interval $(0,X),(0,Y),(0,Z)$ and of exponential order $e^{(ax+by+cz)}$. Then the Triple Shehu transform of $f(x,y,z)$ exists

$\forall s>au,\ q>bv,\ r>ck.$

Proof

Let $f(x,y,z)$ be of exponential order $e^{(ax+by+cz)}$ such that

$$\left|f\left(x,y,z\right)\right| \le M\,e^{(ax+by+cz)} \quad \text{for all } x>X,\ y>Y,\ z>Z.$$

Then, we have

$$\left|\mathrm{H}_{xyz}^{3}\left(f\left(x,y,z\right)\right)\right| = \left|\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} f\left(x,y,z\right)dx\,dy\,dz\right|$$

$$\le \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left|f\left(x,y,z\right)\right|dx\,dy\,dz$$

$$\le M\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}e^{ax+by+cz}\,dx\,dy\,dz$$

$$= M\int_0^\infty e^{-\left(\frac{r-kc}{k}\right)z}\left(\int_0^\infty e^{-\left(\frac{q-bv}{v}\right)y}\left(\int_0^\infty e^{-\left(\frac{s-au}{u}\right)x}dx\right)dy\right)dz$$

$$= \frac{Mukv}{(s-au)(q-bv)(r-kc)}.$$

Thus, the proof is complete.

In the next theorem, we show that $f(x,y,z)$ can be uniquely obtained from $F\left[(s,q,r),(u,v,k)\right]$.

Theorem 3

Let $F_1\left[(s,q,r),(u,v,k)\right]$ and $F_2\left[(s,q,r),(u,v,k)\right]$ be the Shehu transform of the continuous functions $f_1(x,y,z)$ and $f_2(x,y,z)$ defined for $x,y,z\ge 0$ respectively. If $F_1\left[(s,q,r),(u,v,k)\right]=F_2\left[(s,q,r),(u,v,k)\right]$, then $f_1(x,y,z)=f_2(x,y,z)$.

Proof

If we presume α,β,γ to be sufficiently large, then since

$$f(x,y,z)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{k}e^{\left(\frac{rz}{v}\right)}F\left[(s,q,r),(u,v,k)\right]dr\right)dq\right]ds$$

We deduce that

$$f_1(x,y,z)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{k}e^{\left(\frac{rz}{k}\right)}F_1\left[(s,q,r),(u,v,k)\right]dr\right)dq\right]ds$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\left(\frac{sx}{u}\right)}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{v}e^{\left(\frac{qy}{v}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{k}e^{\left(\frac{rz}{k}\right)}F_2\left[(s,q,r),(u,v,k)\right]dr\right)dq\right]ds$$

$$= f_2(x,y,z).$$

This proves the uniqueness of the TRHT.

TRHT of some elementary functions

(1) If $f(x,y,z)=A, x,y,z>0$. Then $\mathrm{H}_{xyz}^{3}\left(A\right)=A\left(\frac{uvk}{sqr}\right)$.

(2) If $f(x,y,z)=x\ y\ z$. Then $\mathrm{H}_{xyz}^{3}\left(x\ y\ z\right)=\left(\frac{uvk}{sqr}\right)^2$.

(3) If $f(x,y,z)=e^{ax+by+cz}$. Then

$$\mathrm{H}_{xyz}^{3}\left(e^{ax+by+cz}\right)=\frac{ukv}{(s-au)(q-bv)(r-kc)}.$$

(4) If $f(x,y,z)=e^{i(ax+by+cz)}$. Then

$$\mathrm{H}_{xyz}^{3}\left(e^{i(ax+by+cz)}\right)=\frac{ukv}{(s-iau)(q-ibv)(r-ikc)}$$

$$=\frac{ukv\left[(rsq-abuvr-sbkvc-auqkc)+i\,(sqkc-abuvkc+sbvr+auqr)\right]}{\left(s^2+a^2u^2\right)\left(q^2+b^2v^2\right)\left(r^2+k^2c^2\right)}.$$

Consequently,

$$\mathrm{H}_{xyz}^{3}\left(\cos(a\,x+b\,y+c\,z)\right)=\frac{ukv\,(rsq-abuvr-sbkvc-auqkc)}{\left(s^2+a^2u^2\right)\left(q^2+b^2v^2\right)\left(r^2+k^2c^2\right)},$$

and,

$$\mathrm{H}_{xyz}^{3}\left(\sin(a\,x+b\,y+c\,z)\right)=\frac{ukv\,(sqkc-abuvkc+sbvr+auqr)}{\left(s^2+a^2u^2\right)\left(q^2+b^2v^2\right)\left(r^2+k^2c^2\right)}, \tag{5}$$

If $f(x,y,z)=f_1(x)\,f_2(y)\,f_3(z)$. Then

$$\mathrm{H}^3_{xyz}\left(f\left(x,y,z\right)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left(f_1(x)f_2(y)\ f_3(z)\right)dx\,dy\,dz$$

$$=\int_0^\infty e^{-\left(\frac{sx}{u}\right)}f_1(x)\left(\int_0^\infty e^{-\left(\frac{qy}{v}\right)}f_2(y)\left(\int_0^\infty e^{-\left(\frac{rz}{k}\right)}f_3(z)dz\right)dy\right)dx$$

$$=\mathrm{H}_x\left(f_1(x)\right)\mathrm{H}_y\left(f_2(y)\right)\mathrm{H}_z\left(f_3(z)\right).$$

So that,

$$\mathrm{H}^3_{xyz}\left(\cos x\ \cos y\ \cos z\right)=\left(\frac{su}{s^2+u^2}\right)\left(\frac{qv}{q^2+v^2}\right)\left(\frac{rk}{r^2+k^2}\right);$$

$$\mathrm{H}^3_{xyz}\left(\sin x\ \sin y\ \sin z\right)=\left(\frac{u^2}{s^2+u^2}\right)\left(\frac{v^2}{q^2+v^2}\right)\left(\frac{k^2}{r^2+k^2}\right).$$

$$\mathrm{H}^3_{xyz}\left(x^n\ y^m\ z^p\right)=n!\left(\frac{u}{s}\right)^{n+1}m!\left(\frac{v}{q}\right)^{m+1}p!\left(\frac{k}{r}\right)^{p+1},\ n,m,p=\ 0,1,2,\ldots.$$

$$\mathrm{H}^3_{xyz}\left(\left(x\ y\ z\right)^n\right)=\left(n!\right)^3\left(\frac{uvk}{sqr}\right)^{n+1},\ n=\ 0,1,2,\ldots,$$

$$\mathrm{H}^3_{xyz}\left(x^\alpha\ y^\beta\ z^\gamma\right)=\Gamma(\alpha+1)\left(\frac{u}{s}\right)^{\alpha+1}\Gamma(\beta+1)\left(\frac{v}{q}\right)^{\beta+1}\Gamma(\gamma+1)\left(\frac{k}{r}\right)^{\gamma+1},\ \alpha,\beta,\gamma\geq-1.$$

Some main properties of TRHT

Linearity property

Let $f\left(x,y,z\right),g\left(x,y,z\right)$ be two functions such that

$$\mathrm{H}^3_{xyz}\left(f\left(x,y,z\right)\right)=F\left[\left(s,q,r\right),\left(u,v,k\right)\right],$$ and

$$\mathrm{H}^3_{xyz}\left(g\left(x,y,z\right)\right)=G\left[\left(s,q,r\right),\left(u,v,k\right)\right]$$

Then for any constants, α,β, we have

$$\mathrm{H}^3_{xyz}\left(\alpha f\left(x,y,z\right)+\beta g\left(x,y,z\right)\right)=\alpha\,\mathrm{H}^3_{xyz}\left(f\left(x,y,z\right)\right)+\beta\,\mathrm{H}^3_{xyz}\left(g\left(x,y,z\right)\right).$$

Proof

Using definition of TRST, we obtain

$$\mathrm{H}^3_{xyz}\left(\alpha f\left(x,y,z\right)+\beta g\left(x,y,z\right)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left(\alpha f\left(x,y,z\right)+\beta g\left(x,y,z\right)\right)dx\,dy\,dz$$

$$=\ \alpha\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}f\left(x,y,z\right)dx\,dy\,dz+\ \beta\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}g\left(x,y,z\right)dx\,dy\,dz$$

$$=\alpha\,\mathrm{H}^3_{xyz3}\left(f\left(x,y,z\right)\right)+\beta\,\mathrm{H}^3_{xyz}\left(g\left(x,y,z\right)\right).$$

Change of scale property

Let $f\left(x,y,z\right)$ be a functions such that

$$\mathrm{H}^3_{xyz}\left(f\left(x,y,z\right)\right)=F\left[\left(s,q,r\right),\left(u,v,k\right)\right]$$

Then for $a,b,c>0,$ we have

$$\mathrm{H}^3_{xyz}\left(f\left(a\,x,b\ y,c\ z\right)\right)=\ \frac{1}{a\,b\,c}F\left[\left(\frac{s}{a},\frac{q}{b},\frac{r}{c}\right),\left(u,v,k\right)\right].$$

Proof

We have

$$\mathrm{H}^3_{xyz}\left(f\left(a\,x,b\ y,c\ z\right)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}f\left(ax,by,cz\right)dx\,dy\,dz. \quad 5)$$

Let $t=a\,x,w=b\,y,\ l=c\,z.$

Then

$$\mathrm{H}^3_{xyz}\left(f\left(t,w,l\right)\right)=\frac{1}{a\,b\,c}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\left(\frac{s}{a}\right)t}{u}+\frac{\left(\frac{q}{b}\right)w}{v}+\frac{\left(\frac{r}{c}\right)l}{k}\right)}f\left(t,w,l\right)dt\,dw\,dl$$

$$=\ \frac{1}{a\,b\,c}F\left[\left(\frac{s}{a},\frac{q}{b},\frac{r}{c}\right),\left(u,v,k\right)\right].$$

First shifting property

Let $f\left(x,y,z\right)$ be a functions such that

$$\mathrm{H}^3_{xyz}\left(f\left(x,y,z\right)\right)=F\left[\left(s,q,r\right),\left(u,v,k\right)\right]$$

Then for real constants a,b,c, we have

$$\mathrm{H}^3_{xyz}\left(e^{(ax+by+cz)}f\left(x,y,z\right)\right)=\ F\left[\left(s-au,q-bv,r-c\,k\right),\left(u,v,k\right)\right]$$

Proof

We have, by definition

$$\mathrm{H}^3_{xyz}\left(e^{-(ax+by+cz)}f\left(x,y,z\right)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}e^{(ax+by+cz)}f\left(x,y,z\right)dx\,dy\,dz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{(s-au)x}{u}+\frac{(q-bv)y}{v}+\frac{(r-ck)z}{k}\right)}f\left(x,y,z\right)dx\,dy\,dz$$

$$=\ F\left[\left(s-au,q-bv,r-c\,k\right),\left(u,v,k\right)\right].$$

TRST of derivative of a function of three variables

1) The TRST of mixed derivative of a function of three variables is given by:

$$\mathrm{H}^3_{xyz}\left(\frac{\partial^3 f\,(x,y,z)}{\partial x\,\partial y\,\partial z}\right)=\left(\frac{sqr}{uvk}\right)F\left((s,q,r),(u,v,k)\right)-\left(\frac{sq}{uv}\right)\mathrm{H}^2_{xy}\left(f\,(x,y,0)\right)$$
$$-\left(\frac{qr}{vk}\right)\mathrm{H}^2_{yz}\left(f\,(0,y,z)\right)-\left(\frac{sr}{uk}\right)\mathrm{H}^2_{xz}\left(f\,(x,0,z)\right)$$
$$+\left(\frac{s}{u}\right)\mathrm{H}_x\left(f\,(x,0,0)\right)+\left(\frac{r}{k}\right)\mathrm{H}_z\left(f\,(0,0,z)\right)$$
$$+\left(\frac{q}{v}\right)\mathrm{H}_y\left(f\,(0,y,0)\right)-f\,(0,0,0)\,.$$

2) The TRST of nth partial derivative of a function of three variables is given by

$$\mathrm{H}^3_{xyz}\left(\frac{\partial^n f\,(x,y,z)}{\partial x^n}\right)=\left(\frac{s}{u}\right)^n F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{s}{u}\right)^{n-m-1}\mathrm{H}^2_{yz}\left(\frac{\partial^m f\,(0,y,z)}{\partial x^m}\right),$$
$$\mathrm{H}^3_{xyz}\left(\frac{\partial^n f\,(x,y,z)}{\partial y^n}\right)=\left(\frac{q}{v}\right)^n F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{q}{v}\right)^{n-m-1}\mathrm{H}^2_{xz}\left(\frac{\partial^m f\,(x,0,z)}{\partial y^m}\right),$$
$$\mathrm{H}^3_{xyz}\left(\frac{\partial^n f\,(x,y,z)}{\partial z^n}\right)=\left(\frac{r}{k}\right)^n F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{r}{k}\right)^{n-m-1}\mathrm{H}^2_{xy}\left(\frac{\partial^m f\,(x,y,0)}{\partial z^m}\right).$$

3) The TRST of the partial fractional Caputo derivatives of a function of three variables is given by:

$$\mathrm{H}^3_{xyz}\left(\frac{\partial^\alpha f\,(x,y,z)}{\partial x^\alpha}\right)=\left(\frac{s}{u}\right)^\alpha F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-m-1}\mathrm{H}^2_{yz}\left(\frac{\partial^m f\,(0,y,z)}{\partial x^m}\right),$$
$$\mathrm{H}^3_{xyz}\left(\frac{\partial^\alpha f\,(x,y,z)}{\partial y^\alpha}\right)=\left(\frac{q}{v}\right)^\alpha F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{q}{v}\right)^{\alpha-m-1}\mathrm{H}^2_{xz}\left(\frac{\partial^m f\,(x,0,z)}{\partial y^m}\right),$$
$$\mathrm{H}^3_{xyz}\left(\frac{\partial^\alpha f\,(x,y,z)}{\partial z^\alpha}\right)=\left(\frac{r}{k}\right)^\alpha F\left((s,q,r),(u,v,k)\right)-\sum_{m=0}^{n-1}\left(\frac{r}{k}\right)^{\alpha-m-1}\mathrm{H}^2_{xy}\left(\frac{\partial^m f\,(x,y,0)}{\partial z^m}\right).$$

Multiplying by $x^n\ y^m\ z^p$

Let $f\,(x,y,z)$ be a functions such that

$$\mathrm{H}^3_{xyz}\left(f\,(x,y,z)\right)=F\left[(s,q,r),(u,v,k)\right]$$

Then

$$\mathrm{H}^3_{xyz}\left(x^n y^m z^p f\,(x,y,z)\right)=(-1)^{n+m+p}u^n v^m k^p\frac{\partial^{n+m+p}}{\partial s^n\partial q^m\partial r^p}F\left[(s,q,r),(u,v,k)\right].$$

Proof

We have

$$\mathrm{H}^3_{xyz}\left(f\,(x,y,z)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}f\,(x,y,z)dx\,dy\,dz$$

Therefore,

$$(-1)^{n+m+p}\frac{\partial^{n+m+p}}{\partial s^n\partial q^m\partial r^p}\mathrm{H}^3_{xyz}\left(f\,(x,y,z)\right)=(-1)^{n+m+p}\frac{\partial^{n+m+p}}{\partial s^n\partial q^m\partial r^p}\int_0^\infty\int_0^\infty\int_0^\infty\left(e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}f\,(x,y,z)\right)dx\,dy\,dz$$

$$=(-1)^{n+m+p}\frac{\partial^{n+m}}{\partial s^n\partial q^m}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)}\left(\int_0^\infty\frac{\partial^p}{\partial r^p}e^{-\frac{rz}{k}}f\,(x,y,z)dz\right)dx\,dy$$
$$=(-1)^{n+m+p}\frac{\partial^{n+m}}{\partial s^n\partial q^m}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)}\left[\frac{(-1)^p}{k^p}\mathrm{H}^3_z\left(z^p f\,(x,y,z)\right)\right]dx\,dy$$
$$=\frac{(-1)^{n+m}}{k^p}\frac{\partial^{n+m}}{\partial s^n\partial q^m}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}\right)}\left[\mathrm{H}^3_z\left(z^p f\,(x,y,z)\right)\right]dx\,dy. \tag{16}$$

In the same way, integrating Equation (16) with respect to *y, x*, we can get the required result.
For $n=m=p=1$, we get

$$\mathrm{H}^3_{xyz}\left(x\ y\ z\ f\,(x,y,z)\right)=-(uvk)\frac{\partial^3}{\partial s\,\partial q\,\partial r}\mathrm{H}^3_{xyz}\left(f\,(x,y,z)\right).$$

Recall that the Heaviside unit step function $U\left(x-a,y-b,z-c\right)$ (Thakur et al., 2018) is defined by

$$U\left(x-a,y-b,z-c\right)=\begin{cases}1 & x>a,\ y>b,\ z>c,\\ 0 & \text{Otherwise.}\end{cases}$$

Theorem 4

Let $f\,(x,y,z)$ be a functions such that

$$\mathrm{H}^3_{xyz}\left(f\,(x,y,z)\right)=F\left[(s,q,r),(u,v,k)\right]$$

Then for a constants a,b,c we have

$$\mathrm{H}^3_{xyz}\left(f\,(x-a,y-b,z-c)U\left(x-a,y-b,z-c\right)\right)=e^{-\left(\frac{sa}{u}+\frac{qb}{v}+\frac{rc}{k}\right)}F\left[(s,q,r),(u,v,k)\right],$$

Where $U\,(x,y,z)$ is the Heaviside unit step.

Proof

Using definition of TRST, we get

$$\mathrm{H}^3_{xyz}\left(f\,(x-a,y-b,z-c)U\left(x-a,y-b,z-c\right)\right)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left[f\,(x-a,y-b,z-c)\right.$$
$$\left.U\left(x-a,y-b,z-c\right)\right]dx\,dy\,dz$$
$$=\int_c^\infty\int_b^\infty\int_a^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left[f\,(x-a,y-b,z-c)\right]dx\,dy\,dz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{s(t+a)}{u}+\frac{q(w+b)}{v}+\frac{r(l+c)}{k}\right)}\left[f\,(t,w,l)\right]dt\,dw\,dl$$

$$= e^{-\left(\frac{sa}{u}+\frac{qb}{v}+\frac{rc}{k}\right)} F\left[(s,q,r),(u,v,k)\right].$$

Convolution Theorem for the Triple Shehu Transform

The convolution of the functions $f(x,y,z), g(x,y,z)$ is denoted by $(f***g)(x,y,z)$ and defined by

$$(f***g)(x,y,z) = \int_0^x\int_0^y\int_0^z f(x-t_1,y-t_2,z-t_3)\, g(t_1,t_2,t_3)dt_1dt_2dt_3$$

$$= \int_0^x\int_0^y\int_0^z g(x-t_1,y-t_2,z-t_3)\, f(t_1,t_2,t_3)dt_1dt_2dt_3.$$

Theorem 5

Let $f(x,y,z), g(x,y,z)$ be of exponential order, such that

$$F\left[(s,q,r),(u,v,k)\right] = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} f(x,y,z)dx\,dy\,dz,$$

is converge, and in addition if

$$G\left[(s,q,r),(u,v,k)\right] = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} g(x,y,z)dx\,dy\,dz,$$

is absolutely converge, then

$$\mathrm{H}^3_{xyz}\left[(f***g)(x,y,z)\right] = \mathrm{H}^3_{xyz}\left[f(x,y,z)\right]\mathrm{H}^3_{xyz}\left[g(x,y,z)\right].$$

Proof

We have

$$\mathrm{H}^3_{xyz}\left[(f***g)(x,y,z)\right] = \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} (f***g)(x,y,z)dx\,dy\,dz$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left(\int_0^x\int_0^y\int_0^z f(x-t_1,y-t_2,z-t_3)\, g(t_1,t_2,t_3)dt_1dt_2dt_3\right)dx\,dy\,dz$$

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)}\left(\int_0^x\int_0^y\int_0^z f(x-t_1,y-t_2,z-t_3)\, g(t_1,t_2,t_3)dt_1dt_2dt_3\right)dx\,dy\,dz$$

Using the Heaviside unit step function, we obtained

$$= \int_0^\infty\int_0^\infty\int_0^\infty g(t_1,t_2,t_3)\left(\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{qy}{v}+\frac{rz}{k}\right)} f(x-t_1,y-t_2,z-t_3)\, U(x-t_1,y-t_2,z-t_3)dx\,dy\,dz\right)dt_1dt_2dt_3$$

By using Theorem 4,

$$= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{st_1}{u}+\frac{qt_2}{v}+\frac{rt_3}{k}\right)} F\left[(s,q,r),(u,v,k)\right] g(t_1,t_2,t_3)dt_1dt_2dt_3$$

$$= F\left[(s,q,r),(u,v,k)\right]\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{st_1}{u}+\frac{qt_2}{v}+\frac{rt_3}{k}\right)} g(t_1,t_2,t_3)dt_1dt_2dt_3$$

$$= F\left[(s,q,r),(u,v,k)\right]\, G\left[(s,q,r),(u,v,k)\right]$$

$$= \mathrm{H}^3_{xyz}\left[f(x,y,z)\right]\mathrm{H}^3_{xyz}\left[g(x,y,z)\right].$$

APPLICATIONS

Here, the Triple Shehu Transform is illustrated by studying the following examples.

Example 1

Consider the following fractional partial differential equation (Alfaqeih, 2019)

$$D_z^{\alpha}\psi(x,y,z) = \frac{\partial^2\psi(x,y,z)}{\partial x^2},\ 0<\alpha\le 1. \qquad (17)$$

With

$$\begin{cases}\psi_x(0,y,z) = \sin(y)\,E_\alpha(-z^{\alpha}),\\ \psi(0,y,z)=0,\\ \psi(x,y,0)=\sin(x)\sin(y).\end{cases} \qquad (18)$$

Applying Triple Shehu Transform to Equation (17), we get

$$\left(\frac{r}{k}\right)^{\alpha}\bar{\psi}(x,y,z) - \left(\frac{r}{k}\right)^{\alpha-1}\psi(x,y,z) = \left(\frac{s}{u}\right)^2\bar{\psi}(x,y,z) - \left(\frac{s}{u}\right)\mathrm{H}^2_{yz}\psi(0,y,z) - \mathrm{H}^2_{yz}\psi_x(0,y,z)$$

Using initial conditions (18), we obtain

$$\bar{\psi}\left((s,q,r),(u,v,k)\right) = \frac{u^2}{s^2+u^2}\,\frac{v^2}{q^2+v^2}\,\frac{\left(\frac{r}{k}\right)^{\alpha-1}}{\left(\frac{r}{k}\right)^{\alpha}+1}. \qquad (19)$$

Operating with the Triple Shehu inverse on both sides of Equation (19) gives

$$\psi\left(x, y, z\right) = \sin(x)\sin(y)E_{\alpha}\left(-z^{\alpha}\right).$$

Example 2

Consider the following fractional partial differential Equation (Alfaqeih, 2019):

$$D_z^{\alpha}\psi\left(x, y, z\right) = \frac{1}{5}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right], \quad 0 < \alpha \leq 1. \qquad (20)$$

With

$$\begin{cases} \psi(x, y, 0) = e^{x+2y}, \quad \psi(0, y, z) = e^{2y}E_{\alpha}\left(z^{\alpha}\right), \\ \psi(x, 0, z) = e^{x}E_{\alpha}\left(z^{\alpha}\right), \quad \psi(x, 0.5, z) = e^{x+1}E_{\alpha}\left(z^{\alpha}\right), \\ \psi(1, y, z) = e^{1+2y}E_{\alpha}\left(z^{\alpha}\right). \end{cases} \qquad (21)$$

Applying Triple Shehu Transform to Equation (20), we get,

$$\left(\frac{r}{k}\right)^{\alpha}\overline{\psi}(x, y, z) - \left(\frac{r}{k}\right)^{\alpha-1}\psi(x, y, z) = \frac{1}{5}\left[\left(\frac{s}{u}\right)^{2}\overline{\psi}(x, y, z) - \left(\frac{s}{u}\right)\mathrm{H}_{yz}^{2}\psi(0, y, z) - \mathrm{H}_{yz}^{2}\psi_x(0, y, z) + \left(\frac{q}{v}\right)^{2}\overline{\psi}(x, y, z) - \left(\frac{q}{v}\right)\mathrm{H}_{xz}^{2}\psi(x, 0, z) - \mathrm{H}_{xz}^{2}\psi_y(x, 0, z)\right]$$

Using the conditions (21), we obtain

$$\overline{\psi}\left((s, q, r), (u, v, k)\right) = \left(\frac{u}{s-u}\right)\left(\frac{v}{q-2v}\right)\left(\frac{\left(\frac{r}{k}\right)^{\alpha-1}}{\left(\frac{r}{k}\right)^{\alpha} - 1}\right). \qquad (22)$$

Operating with the Triple Shehu inverse on both sides of Equation 22 gives

$$\psi\left(x, y, z\right) = e^{x+2y}E_{\alpha}\left(z^{\alpha}\right).$$

Conclusion

In this paper, we introduced a new type of generalized integral transforms called a Triple Shehu transform, which is generalization of single Shehu transform. Furthermore, several properties, examples and theorems of this transform were presented. To see the efficiency of Triple Shehu transform, this transform was applied on some examples and the results show that the Triple Shehu transform method is an appropriate method for solving the fractional partial differential equations. As a new work, it will be interesting to extend known results on a Triple Laplace transform, Triple Aboodh transform, etc, to our results on a Triple Shehu transform. Finally, based on the mathematical formulations, simplicity and the findings of the proposed Triple Shehu transform, we conclude that it is highly efficient.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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