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Full Length Research Paper

Triangular matrix summability of a series

S. K. Paikray[1], U. K. Misra[2]* and N. C. Sahoo[3]

[1]Deptartment of Mathematics, Dhaneshwar Rath institute of engineering and management studies, Tangi Cuttack-754022 Odisha, India.
[2]Deptartment of Mathematics, Berhampur University Berhampur-760007 Ganjam, Odisha, India.
[3]Deptartment of Mathematics, Sailabala Women's College (Auto.) Cuttack, Odisha, India.

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In this paper, in extending a result of Savas and Rhodes on indexed matrix summability, an analogue theorem has been established by using Euler-Totient function.

Key words: Summability, triangular matrix summability, Euler-Totient function.

INTRODUCTION

Let $A = (a_{mn})$ be a lower-triangular matrix and $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$ such that

$$A_n = \sum_{v=0}^{n} a_{nv} \; s_v \; .$$

Then, the series $\sum a_n$ is said to be summable $\left|A\right|_k$ $k \geq 1,$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|A_n - A_{n-1}\right|^k < \infty \; .$$

Associate with A we define two lower triangular matrices $\overline{A}$ and $\hat{A}$ as follows:

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr} , \; n = 0,1,2,... \; and \; v = 0,1,2,...$$

$$\hat{a}_{nv} = \overline{a}_{n,v} - \overline{a}_{n-1}, \; n = 1,2,3,... \; and \; v = 0,1,2,...$$

Euler-Totient function

For any positive integer $k \geq 1$ we define Euler-Totient function $\phi(k)$ as the number of positive integers not exceeding k and relatively prime to k .

KNOWN THEOREM

Savas and Rhodes (2007) proved the following theorem on $\left|A\right|_k$ -summability.

Theorem-A

Let A be a lower triangular matrix with non-negative entries satisfying

(i) $\overline{a}_{n0} = 1, \; n = 0,1,...$

(ii) $a_{n-1,v} \geq a_{nv} \; for \; n \geq v+1$

(iii) $n \; a_{nn} = O(1)$

If the sequence $\{s_n\}$is bounded and the sequence $\{\lambda_n\}$ is such that

(iv) $\sum_{n=1}^{m} \left|\Delta \lambda_n\right| = O(1)$,

*Corresponding author. E-mail: umukanta_misra@yahoo.com.

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(v) $$\sum_{n=1}^{m} a_{nn}|\lambda_n|^k = O(1)$$.

Then the series $\sum a_n \lambda_n$ is summable $|A|_k,\ k \geq 1$.

MAIN RESULT

In this chapter we take $\lambda_n = \sum_{k=1}^{n} \phi(k)$ and prove the following theorem.

Theorem

Let A be a lower-triangular matrix with non-negative entries satisfying

(i) $\overline{a}_{n,0} = 1,\ n = 0,1,2,...$

(ii) $a_{n-1,v} \geq a_{n,v}\ \ for\ n \geq v+1$

(iii) $na_{nn} = O(1)$

and for

$$t_n = \frac{1}{n+1}\sum_{k=1}^{n} k\ a_k, \qquad w_n = \frac{1}{n+1}\sum_{k=1}^{n} a_k \log k$$

(iv) $$O\left(|t_v|^k\ |a_{vv}|\right) = \frac{1}{(v+1)^{2k+2}}, \quad O\left(|w_v|^k\ |a_{vv}|\right) = \frac{1}{(v+1)^{2k+2}}$$

(v) $$\sum_{n=v+1}^{m+1}\left|\Delta_v\ \hat{a}_{n,v}\right| = O|a_{vv}|$$

(vi) $$n^k\left|\sum_{v=1}^{n-1} t_v\ \hat{a}_{n,v}\right|^{k-1} = O\left(\frac{1}{(v+1)^{k-1}a_{vv}}\right)$$

(vii) $$\sum_{v=1}^{\infty}\frac{t_v}{a_{vv}} = O(1)\ ;\ \sum_{v=1}^{\infty}\frac{w_v}{a_{vv}} = O(1)$$

(viii) $$\frac{1}{n}|t_n|^k = O\left(\frac{1}{(n+1)^{2k+2}}\right);\ \frac{1}{n}|w_n|^k = O\left(\frac{1}{(n+1)^{2k+2}}\right)$$

Then $\sum a_n \lambda_n$ is summable $|A|_k,\ k \geq 1$.

Lemmas

We need the following lemmas for the proof of our theorem.

Lemma-1

$$\sum_{v=0}^{n-1}\left|\Delta_v\ \hat{a}_{nv}\right| = a_{n,n}$$

Proof. We have,

$$\hat{a}_{n,v+1} = \overline{a}_{n,v+1} - \overline{a}_{n-1,v+1}$$

$$= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i}$$

$$= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{v} a_{n-1,i}$$

$$= \sum_{i=0}^{v}\left(a_{n-1,i} - a_{n,i}\right) \geq 0$$.

Now

$$\Delta_v \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1}$$

$$= \overline{a}_{nv} - \overline{a}_{n-1,v} - \overline{a}_{n,v+1} + \overline{a}_{n-1,v+1}$$

$$= \sum_{r=v}^{n} a_{nr} - \sum_{r=v}^{n-1} a_{n-1,r} - \sum_{r=v+1}^{n} a_{n,r} + \sum_{r=v+1}^{n-1} a_{n-1,r}$$

$$= 1 - \sum_{r=0}^{v-1} a_{n,r} - 1 + \sum_{r=0}^{r-1} a_{n-1,r} - 1 + \sum_{r=0}^{v} a_{n,r} + 1 - \sum_{r=0}^{v} a_{n-1,r}$$

$$= \left(\sum_{r=0}^{v} a_{n,r} - \sum_{r=0}^{v-1} a_{n,r}\right) - \left(\sum_{r=0}^{v} a_{n-1,r} - \sum_{r=0}^{v-1} a_{n-1,r}\right)$$

$$= a_{n,r} - a_{n-1,v} \leq 0$$

$$\sum_{v=0}^{n-1}\left|\Delta_v\ \hat{a}_{nv}\right| = \sum_{v=0}^{n-1}\left(a_{n-1,v} - a_{n,v}\right)$$

$$= \sum_{v=0}^{n-1} a_{n-1,v} - \sum_{v=0}^{n-1} a_{n,v}$$

$$= \sum_{v=0}^{n-1} a_{n-1,v} - \sum_{v=0}^{n} a_{n,v} + a_{n,n}$$

$$= 1 - 1 + a_{nn} = a_{nn} .$$

Lemma-2 (Apostol* 1989*)

For $n > 1,\ \lambda_n = \sum_{k \le n} \phi(k) = \frac{3}{\pi^2} n^2 + O(n \ \log \ n)$

Lemma-3

$$\sum_{n=v+1}^{m} \hat{a}_{n,v} = 1 .$$

Proof

$$\hat{a}_{n,v} = \overline{a}_{n,v} - \overline{a}_{n-1,v}$$

$$= \sum_{r=v}^{n} a_{n,r} - \sum_{r=v}^{n-1} a_{n-1,v}$$

$$= 1 - \sum_{r=0}^{v-1} a_{n,r} - 1 + \sum_{r=0}^{v-1} a_{n-1,r}$$

$$= \sum_{r=0}^{v-1} a_{n-1,r} - \sum_{r=0}^{v-1} a_{n,r}$$

$$= \sum_{r=0}^{v+1} \left(a_{n-1,r} - a_{n,r}\right)$$

$$\sum_{n=v+1}^{m} \hat{a}_{n,v} = \sum_{n=v+1}^{m} \left\{ \sum_{r=0}^{v-1} \left(a_{n-1,r} - a_{n,r}\right) \right\}$$

$$= \sum_{r=0}^{v-1} \left\{ \sum_{n=v+1}^{m} \left(a_{n-1,r} - a_{n,r}\right) \right\}$$

$$= \sum_{r=0}^{v-1} \left(a_{v,r} - a_{m+1,r}\right)$$

$$\le \sum_{r=0}^{v-1} a_{v,r}$$

$$\le \sum_{r=0}^{v} a_{v,r} = 1$$

3.2 **Proof of the theorem**

We have

$$T_n = \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \left(\sum_{v=0}^{i} \lambda_v a_v \right)$$

$$= \sum_{v=0}^{n} \lambda_v a_v \sum_{i=v}^{n} a_{ni}$$

$$= \sum_{v=0}^{n} \lambda_v a_v \ \overline{a}_{nv} = \sum_{v=1}^{n} \lambda_v a_v \overline{a}_{nv}$$

Then

$$T_n - T_{n-1} = \sum_{v=1}^{n} \lambda_r a_v \ \overline{a}_{nv} - \sum_{v=1}^{n-1} \lambda_v a_v \ \overline{a}_{n-1,v}$$

$$= \sum_{v=1}^{n} \lambda_v a_v \overline{a}_{nv} - \sum_{v=1}^{n} \lambda_v a_v \overline{a}_{n-1,v}$$

$$= \sum_{v=1}^{n} \left(\overline{a}_{nv} - \overline{a}_{n-1,v}\right) \lambda_v a_v = \sum_{v=1}^{n} \hat{a}_{nv} \ \lambda_v \ a_v$$

$$\le \sum_{v=1}^{n} \hat{a}_{n,v} a_v \left(v^2 + v \log v\right)$$

Using Lemma-2

$$= \sum_{v=1}^{n} \hat{a}_{nv} \ a_v v^2 + \sum_{v=1}^{n} \hat{a}_{nv} \ a_v \ v \log v$$

$$= \sum_{v=1}^{n} \left(v \, \hat{a}_{nv}\right)\left(v \, a_v\right) + \sum_{v=1}^{n} \left(v \, \hat{a}_{nv}\right)\left(a_v \ \log \ v\right)$$

$$= \sum_{v=1}^{n-1} \Delta_v \left(v \, \hat{a}_{nv}\right) \cdot \sum_{r=1}^{v} \left(r \, a_r\right) + \left(n \ \hat{a}_{nn}\right) \left(\sum_{v=1}^{n} v a_v \right)$$

$$+\sum_{v=1}^{n-1}\Delta v\left(v\,\hat{a}_{nv}\right)\cdot\sum_{v=1}^{v}a_r\ \log\ r+\left(n\ \hat{a}_{nn}\right)\sum_{v=1}^{n}\left(a_v\ \log\ v\right)$$

Using Abel's Lemma.

$$=\sum_{v=1}^{n-1}\Delta_v\left(v\hat{a}_{n,v}\right)(v+1)\cdot t_v+\left(n\,\hat{a}_{nn}\right)(n+1)\,t_n$$

$$+\sum_{v=1}^{n-1}\Delta_v\left(v\,\hat{a}_{n,v}\right)(v+1)\,w_v+\left(n\ \hat{a}_{nn}\right)(n+1)\,w_n$$

$$=\sum_{v=1}^{n-1}\left\{(v+1)\cdot\Delta_v\left(\hat{a}_{n,v}\right)+\left(\hat{a}_{n,v}\right)(-1)\right\}(v+1)\,t_v+\left(n\hat{a}_{nn}\right)(n+1)\,t_n$$

$$+\sum_{v=1}^{n-1}\left\{(v+1)\cdot\Delta_v\left(\hat{a}_{n,v}\right)+\left(\hat{a}_{n,v}\right)(-1)\right\}(v+1)\cdot w_v+\left(n\hat{a}_{n0}\right)(n+1)\,w_n$$

$$=\sum_{v=1}^{n-1}(v+1)^2\cdot t_v\cdot\Delta_v\left(\hat{a}_{n,v}\right)-\sum_{v=1}^{n-1}\left(\hat{a}_{n,v}\right)(v+1)\,t_v+\left(n\ \hat{a}_{nn}\right)(n+1)\,t_n$$

$$+\sum_{v=1}^{n-1}(v+1)^2\cdot w_v\cdot\Delta_v\left(\hat{a}_{n,v}\right)-\sum_{v=1}^{n-1}\left(\hat{a}_{n,v}\right)(v+1)\,w_v+\left(n\ \hat{a}_{nn}\right)(n+1)\,w_n$$

$$=T_1-T_2+T_3+T_4-T_5+T_6.$$

In order to establish the theorem it is sufficient to prove that

$$\sum_{n=1}^{\infty}n^{k-1}\left|T_{n,r}\right|^k<\infty,r=1,2,3,4,5,6.$$

Now

$$\sum_{n=1}^{m+1}n^{k-1}\left|T_{n,1}\right|_k=\sum_{n=1}^{m+1}n^{k-1}\left|\sum_{v=1}^{n-1}(v+1)^2\ t_v\cdot\Delta_v\left(\hat{a}_{n,v}\right)\right|^k$$

$$\leq\sum_{n=1}^{m+1}n^{k-1}\left|\left(\sum_{v=1}^{n-1}\left(\left((v+1)^2\cdot t_v\cdot\left(\Delta_v\ \hat{a}_{n,v}\right)^{\frac{1}{k}}\right)^k\right)^{\frac{1}{k}}\left(\sum_{v=1}^{n-1}\left(\left(\Delta_v\ \hat{a}_{n,v}\right)^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}\right)\right|^k$$

$$=\sum_{n=1}^{m+1}n^{k-1}\left|\left(\sum_{v=1}^{n-1}(v+1)^{2k}\ \left|t_v\right|^k\ \left|\Delta_v\,\hat{a}_{n,v}\right|\right)\right|\left|\left(\sum_{v=1}^{n-1}\Delta_v\ \hat{a}_{nv}\right)\right|^{k-1}$$

$$\leq\sum_{n=1}^{m+1}n^{k-1}\left|\sum_{v=1}^{n-1}(v+1)^{2k}\left|t_v\right|^k\Delta_v\ \hat{a}_{n,v}\right|\cdot\left|\ a_{nn}\right|^{k-1}\Bigg|$$

$$=\sum_{n=1}^{m+1}\left|\sum_{v=1}^{n-1}(v+1)^{2k}\left|t_v\right|^k\left|\Delta_v\,\hat{a}_{n,v}\right|\right|$$

$$=\sum_{v=1}^{m}(v+1)^{2k}\ \left|t_v\right|^k\ \sum_{n=v+1}^{m+1}\left|\Delta_v\ \hat{a}_{n,v}\right|$$

$$\leq\sum_{v=1}^{m}(v+1)^{2k}\left|t_v\right|^k\cdot\left|a_{vv}\right|$$

using condition 3 (v)

$$=\sum_{v=1}^{m}(v+1)^{2k}\cdot\frac{1}{(v+1)^{2k+2}}=\sum_{v=1}^{m}\frac{1}{(v+1)^2}=O(1),\ as\ m\to\infty.$$

Next

$$\sum_{n=1}^{m+1}n^{k+1}\left|T_{n,2}\right|=\sum_{n=1}^{m+1}n^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v}\ (v+1)\ t_v\right|^k$$

$$=\sum_{n=1}^{m+1}n^{k-1}\left|\sum_{v=1}^{n-1}(v+1)\cdot\left(t_v\ \hat{a}_{n,v}\right)^{\frac{1}{k}}\left(t_v\ a_{n,v}\right)^{\frac{k-1}{k}}\right|^k$$

$$=\sum_{n=1}^{m+1}n^{k-1}\left|\left(\sum_{v=1}^{n-1}\left((v+1)\left(t_v\cdot\hat{a}_{n,v}\right)^{\frac{1}{k}}\right)^k\right)^{\frac{1}{k}}\left(\sum_{v=1}^{n-1}\left(\left(t_v\ \hat{a}_{n,v}\right)^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}\right|^k$$

$$=\sum_{n=1}^{m+1}n^{k-1}\left|\sum_{v=1}^{n-1}(v+1)^k\ \left(t_v\cdot\hat{a}_{n,v}\right)\right|\left|\sum_{v=1}^{n-1}t_v\ \hat{a}_{n,v}\right|^{k-1}$$

$$=\sum_{n=1}^{m+1}n^{-1}\left|\sum_{v=1}^{n-1}(v+1)^k\ \left(t_v\cdot\hat{a}_{n,v}\right)\right|n^k\left|\sum_{v=1}^{n-1}t_v\ \hat{a}_{n,v}\right|^{k-1}$$

$$=\sum_{n=1}^{m+1}\frac{1}{n}\left|\sum_{v=1}^{n-1}(v+1)^k\left(t_v\ \hat{a}_{n,v}\right)\right|\cdot O\left(\frac{1}{(r+1)^{k-1}a_{vv}}\right)$$

using 3 (vi)

$$\leq\sum_{n=1}^{m+1}\frac{1}{n}\sum_{v=1}^{n-1}(v+1)\,t_v\ \hat{a}_{n,v}\cdot\frac{1}{a_{vv}}$$

$$=\sum_{v=1}^{m}(v+1)\frac{t_v}{a_{vv}}\cdot\sum_{n=v+1}^{m+1}\frac{1}{n}\,\hat{a}_{n,v}$$

$$\le \sum_{v=1}^{m} (v+1) \frac{t_v}{a_{vv}} \cdot \frac{1}{v+1} \cdot \sum_{n=v+1}^{m+1} \hat{a}_{n,v}$$

$$= \sum_{v=1}^{m} \frac{t_v}{a_{vv}} \cdot 1$$

using Lemma-3

$$= O(1) \ as \ m \to \infty$$

using condition 3 (vii)

Next

$$\sum_{n=1}^{m+1} n^{k-1} \left| T_{n,3} \right|^k = \sum_{n=1}^{m+1} n^{k-1} \left| n \ \hat{a}_{nn} \ (n+1) \ t_n \right|^k$$

$$\le \sum_{n=1}^{m+1} n^{k-1} (n+1)^k \ \left| t_n \right|^k$$

$$= \sum_{n=1}^{m+1} n^k (n+1)^k \ \frac{1}{n} \left| t_n \right|^k$$

$$\le \sum_{n=1}^{m+1} (n+1)^{2k} \ \frac{1}{n} \left| t_n \right|^k$$

$$= \sum_{n=1}^{m+1} (n+1)^{2k} \ O\left(\frac{1}{(n+1)^{2k+2}} \right)$$

using condition 3 (viii)

$$= \sum_{n=1}^{m+1} \frac{1}{(n+1)^2} = O(1) \ as \ m \to \infty$$

Further

$$\sum_{n=1}^{m+1} n^{k-1} \left| T_{n,4} \right|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} (v+1)^2 w_v \ \Delta_v \left(\hat{a}_{n,v} \right) \right|^k$$

$$\le \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} (v+1)^2 \ w_v \left(\Delta_v \left(\hat{a}_{n,v} \right) \right)^{\frac{1}{k}} \left(\Delta_v \left(\hat{a}_{n,v} \right) \right)^{\frac{k-1}{k}} \right|^k$$

$$= \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} (v+1)^{2k} \ \left| w_v \right|^k \ \left| \Delta_v \ \hat{a}_{n,v} \right| \sum_{v=1}^{n-1} \Delta_v \ \hat{a}_{n,v} \right|^{k-1}$$

$$\le \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{n} (v+1)^{2k} \ \left| w_v \right|^k \ \left| \Delta_v \ \hat{a}_{n,v} \right| \left| \hat{a}_{n,n} \right|^{k-1} \right|$$

$$= \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{v=1}^{m+1} (v+1)^{2k} \ \left| w_v \right|^k \ \left| \Delta_v \ \hat{a}_{n,v} \right| \ \right|$$

$$= \sum_{v=1}^{m} (v+1)^{2k} \ \left| w_v \right|^k \ \sum_{n=v+1}^{m+1} \left| \Delta_v \ \hat{a}_{n,v} \right|$$

$$= \sum_{v=1}^{m} (v+1)^{2k} \ \left| w_v \right|^k \ \left| a_{vv} \right|$$

using condition 3 (v)

$$= \sum_{v=1}^{m} (v+1)^{2k} \ \frac{1}{(v+1)^{2k+2}}$$

using condition 3 (iv)

$$= \sum_{v=1}^{m} \frac{1}{(v+1)^2} = O(1) \ as \, m \to \infty$$

Finally

$$\sum_{n=1}^{m+1} n^{k-1} \ \left| T_{n,5} \right|^k = \sum_{n=1}^{m+1} n^{k-1} \ \left| \sum_{v=1}^{n-1} \hat{a}_{n,v} (v+1) \ w_v \right|^k$$

$$= \sum_{v=1}^{m} \frac{w_v}{a_{vv}} = O(1), \ as \ \ m \to \infty$$

(proceeding in the lines of $\sum_{n=1}^{m+1} n^{k-1} \ \left| T_{n,2} \right|^k$)

And

$$\sum_{n=1}^{m+1} n^{k-1} \left| T_{n,6} \right|^k = \sum_{n=1}^{m+1} n^{k-1} \left| n \ \hat{a}_{n,n} \ \ w_n \right|^k$$

$$\le \sum_{n=1}^{m+1} (n+1)^{2k} \ \frac{1}{n} \left| w_n \right|^k$$

(proceeding in the line of $\sum_{n=1}^{m+1} n^{k-1} \left| T_{n,3} \right|^k$)

$$= \sum_{n=1}^{m+1} (n+1)^{2k} \ O\left(\frac{1}{(n+1)^{2k+2}} \right)$$

$$\le \sum_{n=1}^{m+1} \frac{1}{(n+1)^2} = O(1) \text{ as } m \to \infty$$
.

This completes the proof of theorem.

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