African Journal of Mathematics and Computer Science Research Vol. 5(10), pp. 173-175, August 2012
Available online at http://www.academicjournals.org/AJMCSR
DOI: 10.5897/AJMCSR12.008
ISSN 2006-9731©2012 Academic Journals

Short Communication

Properties of the symmetric groups S_n ($n \leq 7$) acting on unordered triples

Stephen Kipkemoi Kibet[1], Ireri N. Kamuti[1], Gregory Kerich[2] and Albert Kimutai[3]*

[1]Department of Mathematics, Kenyatta University, P. O. Box 43844-00100 Nairobi, Kenya.
[2]Gregory Kerich, Mount Kenya University, Eldoret Campus P. O. Box 6212-30100, Eldoret Kenya.
[3]Kabianga University College, P. O. Box 2030-20200, Kericho, Kenya.

Accepted 6 February, 2012

In this paper, we investigated some properties associated with the action of symmetric group S_n ($n \leq 7$) acting on $X^{(3)}$. If G_x is the stabilizer of $x \in X$, the lengths of the orbits of G_x on X are called sub-degrees and the numbers of orbits are called ranks. Ranks and sub-degrees of symmetric groups S_n (n=1, 2, ----) acting on 2-elements subsets from the set X= (1, 2, ---, n) have been calculated by Higman (1970). He showed that the rank is 3 and the sub-degrees are $1, 2(n-2), \binom{n-2}{2}$. Therefore, we extend these calculations to the specific symmetric groups S_n ($n \leq 7$) acting on $X^{(3)}$.

Key words: Ranks, sub-degrees, suborbits, primitivity.

INTRODUCTION

Sub-degrees of primitive permutation representations of PSL (2, q) have previously been calculated by (Tchuda, 1986; Bon and Cohen, 1989; Kamuti, 1992).They have also gone further to calculate sub-degrees of PGL (2, q) on the cosets of maximal dihedral sub-groups. These sub-degrees have been used by (Bon and Cohen, 1989; Faradžev and Ivanov, 1990) to determine the distance – transitive representations of groups G with PSL (2, q) ≤G≤PΓL (2, q).

In this paper we calculated the ranks and sub-degrees of specific symmetric groups S_n ($n \leq 7$) acting on $X^{(3)}$ (Table 1). We used some geometrical arguments different from those used by (Tchuda, 1986; Bon and Cohen, 1989). Sub-degrees of the transitive symmetric groups S_n ($n \leq 7$) acting on $X^{(3)}$ which do not seem to have been published so far will mainly be calculated by using the methods employed by Higman in 1970.

PRELIMINARY DEFINITIONS

In this aspect, we looked into some results in permutation groups which would be needed later on.

Group actions

Definition 1

Let X be a set; a group G acts on the left on X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that:

(i) $(g_1g_2) = g_1(g_2x)$, $\forall\, g_1, g_2 \in G$ and $x \in X$
(ii) For any $x \in X$, $1x=x$, where 1 is the identity in G.

*Corresponding author. E-mail: kimutaialbert@yahoo.com.

List of notations: S_n**,** Symmetric group of degree n and order n!; **$Stab_G(x)$ or G_x,** The stabilizer of x in G; **| G |,** The order of a group G; **H≤G,** H is a subgroup of G; **{a, b, c},** An unordered triple; **$X^{(3)}$,** The set of all unordered triples from the set; **X =** {1, 2,…,n}; **(p, q),** A graph with p vertices and q edges; $\binom{r}{s}$**,** r combinations; **| Fix (g) |,** The number of elements in the fixed point set of g;

Table 1. Table of marks for ranks and sub-degrees of S_n ($n \leq 7$) acting on $X^{(3)}$.

Symmetric groups	Rank	Sub-degrees
S_3	1	1
S_4	2	1,3
S_5	3	1,3,6
S_6	4	1,1,9,9
S_7	5	1,4,12,18

Definition 2

Let G act on a set X. Then X is partitioned into disjoint equivalent classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is called the orbit of x and is denoted by Orb_G (x).

Definition 3

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g and is denoted by Fix (g). Thus Fix (g) = $\{x \in X \mid gx=x\}$.

Definition 4

Let G act transitively on a set X. Then a subset B of X is a block if gB = B or $gB \cap B = \emptyset$ for $g \in G$. Clearly the set X and the singleton subsets of X form blocks; these blocks are called trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise G acts imprimitively.

We now state some important theorems which will be used later in this project.

Theorem 1 (Harary, 1969: 98) (Cauchy – Frobenius Lemma)

Let G be a finite group acting on a set X. Then the number of orbits of G is;

$$\frac{1}{|G|}\sum_{g\in G}\left|Fix(g)\right|$$

Definition 5

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation б of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2,… α_n cycles of length n, we say that σ and hence g has cycle type $(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Theorem 2 (Krishnamurthy, 1985: 68)

Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in G$ has cycle type $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, then the number of permutations in S_n conjugate to *g* is; $\dfrac{n!}{\prod_{i=1}^{n}\alpha_i!\,i^{\alpha_i}}$.

Graphs

Definition 6

A graph G is an ordered pair (V, E), where V is a non-empty finite set of vertices and E is a set of pairs of (distinct) vertices of G, called edges.

Definition 7

A graph G is connected if every pair of vertices of G are joined by some path; otherwise, G is disconnected.

Suborbits and suborbital graphs

A detailed treatment of the results in this area may be found in Sims (1967) or Neumann (1977); Chapter 5.

Let *G* be transitive on *X* and let G_x be the stabilizer of a point $x \in X$. The orbits $\Delta_o=\{x\},\Delta_1,\Delta_2,\ldots,\Delta_{r-1}$ of G_x on *X* are known as the surborbits of *G.* The rank of *G* in this case is *r.* The sizes $n_i=|\Delta_i|\ (i=0,1,\ldots,r-1)$, often called the 'lengths' of the suborbits, are known as sub-degrees of *G.* It is worth while noting that both *r* and the cardinalities of the suborbits $\Delta_i\,(i=0,1,\ldots,r-1)$ are independent of the choice of $x \in X$.

Definition 8

Let Δ be an orbit of G_x on *X*.

Define $\Delta^*=\{gx \mid g\in G,\quad x\in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or the *G*-suborbit paired with Δ). Clearly $|\Delta|=|\Delta^*|$. If $\Delta^*=\Delta$, then Δ is called a self-paired orbit of G_x.

Theorem 3 (Wielandt, 1964)

G_x has an orbit different from {*x*} and paired with it if and only if *G* has even order.

Observe that *G* acts on $X \times X$ by $g(x,y) = (gx, gy)$, $g \in G, x, y \in X$. If $O \subseteq X \times X$ is a *G*-orbit, then for a fixed $x \in X$, $\Delta = \{y \in X | (x.y) \in O\}$ is a G_x -orbit. Conversely, if $\Delta \subseteq X$ is a G_x -orbit, then $O = \{(gx, gy) | g \in G, y \in \Delta\}$ is a *G*-orbit on $X \times X$. We say Δ corresponds to O. The *G*-orbits on $X \times X$ are called suborbitals. Let $O_i \subseteq X \times X, i = 0,1,...,r-1$ be a suborbital. Then we form a graph Γ_i, by taking *X* as the set of vertices of Γ_i and by including a directed edge from *x* to *y* $(x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x,y) | (y,x) \in O_i\}$ is a *G*-orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbit Δ_i $(i = 0,1,...,r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired.

Theorem 4 (Sims, 1967)

Let G be transitive on X. Then G is primitive if and only if each suborbital graph $\Gamma_{í}$, í = 1, 2,…, r-1 is connected.

SOME PROPERTIES OF THE SYMMETRIC GROUP S_n (n≤7) ACTING ON UNORDERED TRIPLES

Theorem 5

S_n (n≤ 7) acts transitively on $X^{(3)}$.

Proof

We prove this Theorem for n=7. For the other n, the proof is similar. We apply Theorem 5 to show that the action of S_7 on $X^{(3)}$ gives one orbit. Let g∈ S_7 have cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ then the number of permutations in S_7 having the same cycle type as g is given by Theorem 7, and the number of points in $X^{(3)}$ fixed by g is given by |Fix (g) | = $\binom{\alpha_1}{3} + \alpha_2 \quad \alpha_1 + \alpha_3$. Applying Cauchy – Frobenius Lemma (Theorem 1), we find that the number of orbits of G acting on $X^{(3)} = \frac{1}{|G|}\sum_{g \in G}|Fix(g)|$ is 1.

Therefore, S_n (n≤ 7) acts transitively on $X^{(3)}$ but not doubly transitively.

RANKS AND SUB-DEGREES OF S_N (N≤ 7) ACTING ON $X^{(3)}$

The suborbital graphs of S_n (n≤7) acting on $X^{(3)}$ were constructed. We deduced that the resulting suborbital graphs were connected except when n = 6. Therefore S_n (n≤7, n≠6) acts primitively on $X^{(3)}$.

REFERENCES

Bon JV, Cohen AM (1989). Linear groups and distance-transitive graphs, Eur. J. Combin. 10:399-411.

Faradžev IA, Ivanov AA (1990). Distance-transitive representations of groups G with PSL (2,q)≤G≤PΓL(2,q), Eur. J. Combin., 11:347-356.

Harary F (1969). Graph Theory. Addison-Wesley. Publishing Company, New York. pp. 98

Higman DG (1970). Characterization of families of rank 3 permutation groups by subdegrees I., Arch. Math. 21: 151-156.

Kamuti IN (1992). Combinatorial formulas, invariants and structures associated with primitive permutation representations of PSL (2,q) and PGL (2,q). Ph.D. Thesis, University of Southampton, U.K.

Krishnamurthy V (1985). Combinatorics, theory and applications, Affiliatede East- West press Private Limited, New Delhi. pp. 68

Neumann PM (1977). Finite permutation groups, Edge-colored graphs and matrices, Topics in group theory and computation, edited by M.P.J Curran, Academic Press, London.

Sims CC (1967). Graphs and finite permutation groups. Math Z. 95:76-86.

Tchuda FL (1986). Combinatorial-geometric characterizations of some primitive representations of groups PSL(n,q) for n=2,3, Ph.D. Thesis, University of Kiev, U.S.S.R.

Wielandt H (1964). Finite permutation groups. Academic Press New York and London.