A study of Green’s functions for three-dimensional problem in thermoelastic diffusion media

Rajnesh Kumar¹* and Vijay Chawla²

¹Department of Mathematics, Kurukshetra University, Kurukshetra-136119, Haryana, India.
²Department of Mathematics, Maharaja Agrasen Mahavidyalaya, Jagahdri-135003 Haryana, India.

Received 29 July, 2014; Accepted 9 September, 2014

The purpose of the present paper is to study the three-dimensional general solution and Green’s functions in transversely isotropic thermoelastic diffusion media for static problem. With this objective, two displacement functions are introduced to simplify the basic equation and a general solution is then obtained by using the operator theory. Based on the obtained general solution, the three-dimensional Green’s functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. The components of displacement, stress, temperature distribution and mass concentration are expressed in terms of elementary functions and are convenient to use. When the apex angle $2\alpha$ equals to $\pi$, then we obtain the solution for semi-infinite body with a surface point. From the present investigation, a special case of interest is deduced to depict the effect of diffusion on components of stress and temperature distribution.

Key words: Thermoelastic diffusion media, Green’s function, transversely isotropic.

INTRODUCTION

Fundamental solutions or Green’s functions play an important role in the solution of numerous problems in the mechanics and physics of solids. Green’s functions can be used to construct many analytical solutions of boundary value problems. They are essential in boundary element method as well as the study of cracks, defects and inclusion. They are a basic building block of future works. For example, fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed. Ding et al. (1996) derived the general solutions for coupled equations in piezoelectric media. Dunn and Wienecke (1999) investigated the half space Green’s functions in transversely isotropic piezoelectric solid. Pan and Tanon (2000) studied the Green’s functions for three dimensional problems in anisotropic piezoelectric solids. When thermal effects are considered, Sharma (1958) investigated the fundamental solution in transversely isotropic thermoelastic material in an integral form. Chen et al. (2004) derived the three dimensional general solution in transversely isotropic thermoelastic materials. Hou et al. (2008, 2009) investigated the Green’s function for two and three-dimensional problem for a steady point heat source in the interior of a semi-infinite thermoelastic material. Also, Hou et al. (2011) investigated the two dimensional general solutions and fundamental solutions in orthotropic thermoelastic materials.

Diffusion can be defined as random walk of assembly
of particles from a high concentration region to a low concentration region. An example of diffusion is heat transport or movement transport. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermoelastici ty remains a practical process to separate isotopes of noble gases (e.g. xenon) and other light isotopes (e.g. carbon) for research purposes.

Nowacki (1974a, b, c, d) developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief and Saleh (2005) developed the generalized theory of thermoelastic diffusion with one relaxation time which allows finite speeds of propagation of waves. Kumar and Kansal (2008) derived the basic equations for generalized thermoelastic diffusion (G-L model) and discussed the Lamb waves. When diffusion effects are considered, Kumar and Chawla (2011a) derived the Fundamental solution in orthotropic thermoelastic diffusion material. Kumar and Chawla (2011b) discussed the plane wave propagation in the context of anisotropic three-phase-lag and two-phase-lag model of thermoelasticity. Kumar and Chawla (2012) derived the Green’s functions for two-dimensional problem in orthotropic thermoelastic diffusion media. Recently, Kumar and Chawla (2013) discussed the problem of reflection and transmission in thermoelastic media with three-phase-lag model. However, the important Green’s function for three-dimensional problem function in transversely isotropic thermoelastic diffusion material has not been discussed so far.

Keeping in view of these applications, the three dimensional general solution and Green’s function in transversely isotropic thermoelastic diffusion elastic medium for steady state problem was studied. After applying the dimensionless quantities and using the operator theory, the general expression for displacement components, mass concentration and temperature change are derived in terms of four harmonic functions. By virtue of the obtained general solution, the three-dimensional Green’s functions for a study point heat source on the apex of a transversely isotropic thermoelastic cone are constructed by four newly introduced harmonic functions. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

**Basic equations**

Following Sherief and Saleh (2005) the basic governing equations for homogenous anisotropic generalized thermoelastic diffusion solid in the absence of body forces, heat and mass diffusion sources are:

**1) Constitutive relations:**

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl} + a_{ij} T + b_{ij} \phi
\]

**2) Equations of motion:**

\[
c_{ijkl} \varepsilon_{kl,j} + a_{ij} T_{ij} + b_{ij} \phi = \rho \ddot{u}_i
\]

**3) Equation of heat conduction:**

\[
\rho C_E \dot{T} + a_{ij} T_{ij} \phi = K_{ij} T_{ij}
\]

**4) Equation of mass diffusion:**

\[
- \alpha_{ij}^* b_{km} \varepsilon_{km,ij} - \alpha_{ij}^* b C_{ij} + \alpha_{ij}^* a T_{ij} = - \dot{C}
\]

Here, 
\[
c_{ijkl} = (c_{ijkl} = c_{jikm} = c_{ijmk})
\] are elastic parameters;
\[
a_{ij} = (a_{ij}), \quad b_{ij} = (b_{ij})
\] are respectively, the tensor of thermal and diffusion moduli. \( \rho \) is the density and \( C_E \) is the specific heat at constant strain, \( a, b \) are respective coefficients describing the measure of thermoelastic diffusion effects and of diffusion effects, \( T_0 \) is the reference temperature assumed to be such that \( T/T_0 \ll 1 \). \( K_{ij} = (K_{ij}), \sigma_{ij} = (\sigma_{ij}) \) and \( e_{ij} = (u_{ij} + u_{jd})/2 \) denote the components of thermal conductivity, stress and strain tensor respectively. \( T(x,y,z,t) \) is the temperature change from the reference temperature \( T_0 \) and \( C \) is the mass concentration. \( u_i \) is a component of displacement vector while \( \alpha_{ij}^*(=\alpha_{ij}^*) \) are diffusion parameters.

In the above equations, the symbol (.) followed by a suffix denotes differentiation with respect to spatial coordinate and a superposed dot (“.”) denotes the derivative with respect to time respectively. Following Slaughter (2002), applying the transformation, we have:

\[
x' = x \cos \phi + y \sin \phi, \quad y' = -x \sin \phi + y \cos \phi, \quad z' = z,
\]

Where \( \phi \) is the angle of rotation in the \( x - z \) plane. In the Equations (1) to (4), the stress-strain-temperature-concentration relation, equations of motion, heat conduction and mass diffusion equation in homogeneous, transversely isotropic thermoelastic diffusion media in cartesian coordinates \((x,y,z)\) can be written as:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xz} \\
\sigma_{yz} \\
\sigma_{zx}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{yy} \\
e_{zz} \\
e_{xz} \\
e_{yz} \\
e_{zx}
\end{bmatrix} +
\begin{bmatrix}
a_1 \\
a_1 \\
a_1 \\
a_1 \\
a_1 \\
a_1
\end{bmatrix} +
\begin{bmatrix}
h \\
h \\
h \\
h \\
h \\
h
\end{bmatrix}
\begin{bmatrix}
T \\
T \\
T \\
T \\
T \\
T
\end{bmatrix}
\]

\[
\alpha_{ij}^* =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\rho C_E \dot{T} + a_{ij} T_{ij} \phi = K_{ij} T_{ij}
\]

\[
- \alpha_{ij}^* b_{km} \varepsilon_{km,ij} - \alpha_{ij}^* b C_{ij} + \alpha_{ij}^* a T_{ij} = - \dot{C}
\]
\[
\begin{align*}
&c_{11} \frac{\partial^2 u}{\partial x^2} + c_{66} \frac{\partial^2 u}{\partial y^2} + c_{44} \frac{\partial^2 u}{\partial z^2} + (c_{12} + c_{66}) \frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial x \partial z} - \\
&\frac{\partial^2 w}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}, \\
&c_{12} (\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x}) + c_{44} (\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z \partial y}) + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial x \partial z} - \\
&\frac{\partial^2 u}{\partial y^2} = \rho \frac{\partial^2 v}{\partial t^2}, \\
&c_{13} (\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial z \partial x}) + c_{44} (\frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial y \partial z}) + \frac{\partial^2 w}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2}.
\end{align*}
\]

(7)

\[
\begin{align*}
\dot{a}_1 \frac{\partial T}{\partial x} + b_1 \frac{\partial C}{\partial x} &= \rho \frac{\partial^2 u}{\partial y^2}, \\
\dot{a}_2 \frac{\partial T}{\partial y} + b_2 \frac{\partial C}{\partial y} &= \rho \frac{\partial^2 v}{\partial y^2}, \\
\dot{a}_3 \frac{\partial T}{\partial z} + b_3 \frac{\partial C}{\partial z} &= \rho \frac{\partial^2 w}{\partial y^2}.
\end{align*}
\]

(8)

FORMULATION OF THE PROBLEM

We consider a homogenous transversely isotropic thermoelastic diffusion medium. Let us take Oxyz as the frame of reference in Cartesian coordinates.

For three dimensional problems, we assume the displacement vector, temperature distribution and mass concentration are respectively, of the form:

\[
\ddot{u} = (u, v, w), \quad T(x, y, z, t), \quad C(x, y, z, t).
\]

(12)

Moreover, we are discussing steady problem

\[
\begin{align*}
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial T}{\partial t} = \frac{\partial C}{\partial t} = 0.
\end{align*}
\]

(13)

We define the dimensionless quantities as:

\[
\begin{align*}
\dot{x}' = \frac{x}{l}, \quad \dot{y}' = \frac{y}{l}, \quad \dot{z}' = \frac{z}{l}, \quad \dot{u}' = \frac{u}{l}, \quad \dot{v}' = \frac{v}{l}, \quad \dot{w}' = \frac{w}{l}.
\end{align*}
\]

(14)

Applying the dimensionless quantities defined by Equation (14) in Equations (7) to (11), after suppressing the primes, we obtain:

\[
\begin{align*}
\dot{a}_1 \frac{\partial T}{\partial x} + b_1 \frac{\partial C}{\partial x} &= \frac{\rho}{K_1} \frac{\partial^2 u}{\partial y^2}, \\
\dot{a}_2 \frac{\partial T}{\partial y} + b_2 \frac{\partial C}{\partial y} &= \frac{\rho}{K_1} \frac{\partial^2 v}{\partial y^2}, \\
\dot{a}_3 \frac{\partial T}{\partial z} + b_3 \frac{\partial C}{\partial z} &= \frac{\rho}{K_1} \frac{\partial^2 w}{\partial y^2}.
\end{align*}
\]

(15)

\[
\begin{align*}
\dot{a}_1 \frac{\partial T}{\partial x} + b_1 \frac{\partial C}{\partial x} &= \frac{\rho}{K_1} \frac{\partial^2 u}{\partial y^2}, \\
\dot{a}_2 \frac{\partial T}{\partial y} + b_2 \frac{\partial C}{\partial y} &= \frac{\rho}{K_1} \frac{\partial^2 v}{\partial y^2}, \\
\dot{a}_3 \frac{\partial T}{\partial z} + b_3 \frac{\partial C}{\partial z} &= \frac{\rho}{K_1} \frac{\partial^2 w}{\partial y^2}.
\end{align*}
\]

(16)

\[
\begin{align*}
\dot{a}_1 \frac{\partial T}{\partial x} + b_1 \frac{\partial C}{\partial x} &= \frac{\rho}{K_1} \frac{\partial^2 u}{\partial y^2}, \\
\dot{a}_2 \frac{\partial T}{\partial y} + b_2 \frac{\partial C}{\partial y} &= \frac{\rho}{K_1} \frac{\partial^2 v}{\partial y^2}, \\
\dot{a}_3 \frac{\partial T}{\partial z} + b_3 \frac{\partial C}{\partial z} &= \frac{\rho}{K_1} \frac{\partial^2 w}{\partial y^2}.
\end{align*}
\]

(17)

\[
\begin{align*}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T &= 0,
\end{align*}
\]

(18)

\[
\begin{align*}
\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} + \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y} + \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z} + \frac{\partial C}{\partial z}\right)\right] &= 0.
\end{align*}
\]

(19)

Where

\[
\begin{align*}
&\dot{a}_i = \dot{a}_i - b_i, \quad \dot{b}_i = -b_i \dot{a}_i, \\
&K_i = K_i, \quad i \text{ is not summed}
\end{align*}
\]

and

\[
\begin{align*}
c_{66} &= \frac{c_{11} - c_{12}}{2}.
\end{align*}
\]

STATIC GENERAL SOLUTIONS

Two displacements functions \( \Psi \) and \( G \) are introduced as follows:
\[ u = \frac{\partial \Psi}{\partial y} - \frac{\partial G}{\partial x}, v = -\frac{\partial \Psi}{\partial x} - \frac{\partial G}{\partial y} \]  
\[ (20) \]

Using the displacements functions \( \Psi \) and \( G \) in Equations (15) - (19), we obtain
\[
\begin{bmatrix}
\delta_z \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \delta_i \frac{\partial^2}{\partial z^2} \end{bmatrix} \Psi = 0,
\]
\[ (21) \]

\[
D \begin{bmatrix} G \\ w \\ T \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[ (22) \]

where \( D \) is the differential operator matrix given by
\[
\begin{bmatrix}
\Delta + \delta_i \frac{\partial^2}{\partial z^2} & -\delta_i \frac{\partial}{\partial z} & 1 & 1 \\
-\delta_i \frac{\partial}{\partial z} & \delta_i \frac{\partial^2}{\partial z^2} & -\delta_i \frac{\partial}{\partial z} & -\delta_i \frac{\partial}{\partial z} \\
-q_i \frac{\partial^2}{\partial z^2} + q_i \frac{\partial}{\partial z} & q_i \frac{\partial}{\partial z} + q_i \frac{\partial}{\partial z} & \Delta + \delta_i \frac{\partial^2}{\partial z^2} & \\
0 & 0 & 0 & \Delta + \delta_i \frac{\partial^2}{\partial z^2} 
\end{bmatrix}
\]

Equation (22) is a homogeneous set of differential equations in \( G, w, T, C \). The general solution by the operator theory is as follows:
\[
G = A_i F, \quad w = A_i F, \quad C = A_i F, \quad T = A_i F \quad (i = 1, 2, 3, 4).
\]
\[ (23) \]

The determinant of the matrix \( D \) is given as:
\[
|D| = \left( \sigma \frac{\partial^6}{\partial z^6} + \bar{\sigma} \Delta \frac{\partial^4}{\partial z^4} + \bar{c} \Delta^2 \frac{\partial^2}{\partial z^2} + \bar{d} \Delta^3 \right) \times \left( \Delta + \epsilon_i \frac{\partial^2}{\partial z^2} \right)
\]
\[ (24) \]

Where \( \bar{\sigma}, \bar{\sigma}, \bar{c}, \bar{d} \) and \( \Delta \) are given in Appendix A. The function \( F \) in Equation (23) satisfies the following homogeneous equation:
\[
|D| F = 0
\]
\[ (25) \]

It can be seen that if \( i = 1, 2, 3 \) are taken in Equation (23), three general solutions are obtained which \( T = 0 \). These solutions are identical to those without thermal facts and are not discussed here. Therefore if \( i = 4 \) should be taken in Equation (23), the following solution is obtained:
\[
\begin{align*}
\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial z^2} + \bar{\sigma} \Delta^2 + \bar{d} \Delta^3 \right) \times \left( \Delta + \epsilon_i \frac{\partial^2}{\partial z^2} \right) + \bar{c} \Delta^2 \frac{\partial^2}{\partial z^2} + \bar{d} \Delta^3 \right) \times \left( \Delta + \epsilon_i \frac{\partial^2}{\partial z^2} \right) \\
\end{align*}
\]
(26)

(27)

(28)

(29)

(30)

Where \( \bar{\sigma}, \bar{\sigma}, \bar{c}, \bar{d} \) and \( \bar{d} \) are given in Appendix B.

In cylindrical coordinates \( (r, \theta, z) \), the general solution can be easily obtained. In fact, the expression for \( w, T \) and \( C \) are identical to that in Equations (26) to (31), while those radial and circumferential displacements \( u_r \) and \( u_{\theta} \) are, respectively
\[
\begin{align*}
u_r &= \frac{\partial \Psi}{r \partial \theta} \left( \bar{\sigma} \Delta^2 + \bar{\sigma} \Delta^2 + \bar{c} \Delta^2 \right) \frac{\partial F}{\partial \theta}, \\
u_{\theta} &= -\frac{\partial \Psi}{r \partial \theta} \left( \bar{\sigma} \Delta^2 + \bar{\sigma} \Delta^2 + \bar{c} \Delta^2 \right) \frac{\partial F}{\partial r},
\end{align*}
\]
\[ (31) \]

(32)

Here \( \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \) is the Laplacian in polar coordinates.

The general solutions of Equation (25) in terms of \( F \) can be rewritten as:
\[
\prod_{j=1}^{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F = 0,
\]
\[ (33) \]

where
\[
z_j = s_j z, \quad s_j = \sqrt{\frac{K_1}{K_3}}, \quad (j = 1, 2, 3) \text{ are three roots with positive real part}
\]

(34)
As known from the generalized Almansi theorem (Ding et al., 1996) the function \( F \) can be expressed in terms of four harmonic functions:

1) \( F = F_1 + F_2 + F_3 + F_4 \) for distinct \( s_j (j = 1, 2, 3, 4) \),

2) \( F = F_1 + F_2 + F_3 + zF_4 \) for \( s_1 \neq s_2 \neq s_3 \neq s_4 \),

3) \( F = F_1 + F_2 + zF_3 + z^2 F_4 \) for \( s_1 \neq s_2 = s_3 = s_4 \),

4) \( F = F_1 + zF_2 + z^2 F_3 + z^3 F_4 \) for \( s_1 = s_2 = s_3 = s_4 \),

where \( F_j \) satisfies the following harmonic equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F_j = 0 \quad (j = 1, 2, 3, 4).
\]

(35)

The general solution for the case of distinct roots can be derived as follows:

\[
u = \frac{\partial \Psi}{\partial y} - \sum_{j=1}^{4} p_{ij} \frac{\partial^2 F_j}{\partial x \partial z_j}, \quad v = -\frac{\partial \Psi}{\partial x} - \sum_{j=1}^{4} p_{ij} \frac{\partial^2 F_j}{\partial y \partial z_j},
\]

\[
w = \sum_{j=1}^{4} s_j p_{ij} \frac{\partial^2 F_j}{\partial z_j^2}, \quad C = \sum_{j=1}^{4} p_{ij} \frac{\partial^2 F_j}{\partial z_j^2}, \quad T = \sum_{j=1}^{4} p_{ij} \frac{\partial^2 F_j}{\partial z_j^2}.
\]

(36)

Where

\[
p_{ij} = a_k - b_j s_j^2 + c_k s_k^j \quad (k = 1, 2)
\]

\[
p_{3j} = -a_3 + b_j s_j^2 - c_j s_j^j + d_4 s_j^6
\]

\[
p_{4j} = -d + c s_4^2 - b s_j^4 + a s_j^6
\]

In the similar way general solution for the other three cases can be derived. Equation (36) can be further simplified by taking

\[
p_{ij} \frac{\partial^2 F_j}{\partial z_j^2} = \psi_j, \quad (j = 1, 2, 3, 4)
\]

(37)

and writing \( \psi_0 = \psi \).

\[
u = \frac{\partial \psi}{\partial y} - \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial^2 \psi}{\partial x \partial z_j}, \quad v = -\frac{\partial \psi}{\partial x} - \sum_{j=1}^{4} \frac{1}{b_j} \frac{\partial^2 \psi}{\partial y \partial z_j},
\]

\[
w = \sum_{j=1}^{4} s_j \frac{1}{a_j} \frac{\partial^2 \psi}{\partial z_j^2}, \quad C = \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial^2 \psi}{\partial z_j^2}, \quad T = \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial^2 \psi}{\partial z_j^2}.
\]

\[
C = \sum_{j=1}^{4} p_{ij} \frac{\partial^2 \psi}{\partial z_j^2}, \quad T = \sum_{j=1}^{4} \frac{\partial^2 \psi}{\partial z_j^2}.
\]

\[
(38)
\]

Where

\[
P_{ij} = p_{ij}/p_{ij}, \quad P_{2j} = p_{2j}/p_{1j}, \quad P_{34} = p_{34}/p_{14}
\]

The function \( \psi \) satisfies the harmonic equations

\[
\left( \Delta + \frac{\partial^2}{\partial z_j^2} \right) \psi_j = 0 \quad j = 0, 1, 2, 3, 4.
\]

(39)

In which

\[
z_a = s_a z, s_a = \frac{\delta_j}{\delta_j}
\]

In cylindrical coordinates \((r, \theta, z)\), the expression for \( w, T, C \) will remain the same as given in Equation (38), while the components of displacement in cylindrical coordinates are

\[
u = \frac{\partial \Psi}{\partial r} - \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial \psi_j}{\partial r}, \quad u_\theta = -\frac{\partial \Psi}{\partial \theta} - \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial \psi_j}{\partial \theta},
\]

(40)

Introducing the following notations for the components both in Cartesian coordinate \((x, y, z)\) and cylindrical coordinate \((r, \theta, z)\),

\[
U = u + iv = e^{j\theta} (u_r + iu_\theta), \quad \sigma_1 = \sigma_{xx} + \sigma_{xy} = \sigma_r + \sigma_{\theta\theta}, \quad \sigma_2 = \sigma_{xx} - \sigma_{xy} + 2i\sigma_{xy} = e^{2j\theta} (\sigma_r - \sigma_{\theta\theta} + 2i\sigma_{\theta\theta}), \quad \tau_z = \sigma_{zz} + i \sigma_{z\theta} = e^{j\theta} (\sigma_{zz} + i \sigma_{z\theta}).
\]

Upon using the notations, the general solution in Equation (38) in the Cartesian coordinate \((x, y, z)\) can be simplified as

\[
u = -\Gamma \left( i \Psi_0 + \sum_{j=1}^{4} \Psi_j \right), \quad w = \sum_{j=1}^{4} s_j \frac{1}{a_j} \frac{\partial \psi_j}{\partial z_j}, \quad C = \sum_{j=1}^{4} p_{ij} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad T = \sum_{j=1}^{4} \frac{1}{a_j} \frac{\partial^2 \psi_j}{\partial z_j^2},
\]

\[
\sigma_1 = 2 \sum_{j=1}^{4} (e_{66} - r_j s_j^2) \Delta \Psi_j, \quad \sigma_2 = -2e_{66} \Gamma \left( i \Psi_0 + \sum_{j=1}^{4} \Psi_j \right), \quad \sigma_{zz} = -\sum_{j=1}^{4} r_j \Delta \Psi_j, \quad \sigma_{zz} = \Gamma \left[ \sum_{j=1}^{4} s_j \frac{1}{a_j} \frac{\partial \psi_j}{\partial z_j} - i s_a e_{44} \frac{\partial \psi_0}{\partial z_0} \right]
\]

(41)
Where
\[
\Gamma_1 = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y},
\]
\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{in Cartesian coordinates } (x,y,z),
\]
\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial \theta^2} \quad \text{in cylindrical coordinates } (r,\theta,z),
\]
and
\[
r_j = \frac{c_{11}^* + c_{13}^* P_{1 j} s_j^2 - c_{11} P_{2 j} - c_{11}^* P_{34}}{s_j^2} = c_{44}^* (1 - P_{1 j}) =
- c_{13} - c_{33} s_j P_{1 j} + \epsilon_1 c_{11} P_{34} + \gamma_1 c_{11}^* P_{2 j},
\]
\[
(c_{11}^*, c_{13}^*, c_{33}^*, c_{44}^*, c_{66}^*) = \frac{1}{a_i} T_0 (c_{11}, c_{13}, c_{33}, c_{44}, c_{66}).
\]

For non-torsional axisymmetric problem, \(\Psi_0 = 0\) and \(\Psi_j (j=1,2,3,4)\) are independent of \(\theta\), such that \(u_\theta = 0\) and \(\sigma_{z \theta} = \sigma_{r \theta} = 0\).

The general solution given by equations in cylindrical coordinate \((r,\theta,z)\) can be simplified to the following form:
\[
u_r = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial r}, \quad \nu_\theta = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial \theta}, \quad \nu_z = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial z}, \quad T = P_4 \frac{\partial \psi_4}{\partial r},
\]
\[
\sigma_r = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial r}, \quad \sigma_\theta = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial \theta}, \quad \sigma_z = \sum_{j=1}^{4} \frac{\partial \psi_j}{\partial z},
\]
\[
(44)
\]

For torsional axisymmetric problem \(\Psi_j = 0 (j=1,2,3,4)\), \(\Psi_0\) is independent of \(\theta\), so that \(u_r = u_z = 0, T = 0, C = 0\) and \(\sigma_{rr} = \sigma_{r \theta} = \sigma_{z z} = \sigma_{rz} = 0\).

The general solution can be simplified as:
\[
u_\theta = -\frac{\partial \psi_0}{\partial r}, \quad \sigma_{r \theta} = 2c_{66} \left( \frac{1}{c_{44}^*} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \psi_0, \quad \sigma_{z \theta} = -c_{44}^* \frac{\partial \psi_0}{\partial z}.
\]
\[
(45)
\]

**BOUNDARY CONDITIONS OF CONE**

We consider a transversely isotropic thermoelastic diffusion cone \(z = \cot \alpha\), where \(2\alpha\) is the apex angle, whose isotropic plane is perpendicular to \(Z - \)axis. At the origin of the coordinate system, the apex is to be taken.

At the apex, a concentrated force \(P = p_i + p_j + p_k\), a concentrated moment \(M = M_i + M_j + M_k\) and a point heat source \(H\) are applied, where \(i,j,k\) are three unit vectors of Cartesian coordinates \((x,y,z)\).

In addition, the cone is loaded on the surface with prescribed density of normal heat flux \(\tilde{q}_n\) and surface forces \(X = \tilde{X}_r e_r + \tilde{X}_\theta e_\theta + \tilde{X}_z e_z\), where \(e_r, e_\theta, e_z\) are three unit vectors of cylindrical coordinates \((r,\theta,z)\), which are related to \(i,j,k\) by the following relations:
\[
e_r = i \cos \theta + j \sin \theta, \quad e_\theta = i \sin \theta + j \cos \theta, \quad e_z = k.
\]
\[
(46)
\]

The boundary conditions in cylindrical coordinates on the cone \(z/r = \cot \alpha\) are:
\[
\sigma_{rr} \cos \alpha - \sigma_{z z} \sin \alpha = \tilde{X}_r, \quad (47)
\]
\[
\sigma_{r \theta} \cos \alpha - \sigma_{z \theta} \sin \alpha = \tilde{X}_\theta, \quad (48)
\]
\[
\sigma_{z r} \cos \alpha - \sigma_{z z} \sin \alpha = \tilde{X}_z, \quad (49)
\]
\[
K_1 \frac{\partial T}{\partial r} \cos \alpha - K_3 \frac{\partial^2 T}{\partial z^2} \sin \alpha = \tilde{q}_m, \quad (50)
\]
\[
\frac{\partial C}{\partial r} \cos \alpha - K_3 \frac{\partial C}{\partial z} \sin \alpha = \tilde{m}_m. \quad (51)
\]

As shown in Figure 1, when a segment of cone cut off by \(z = b\), its global mechanical concentration and thermal equilibrium equations will be:
\[
P = \int_0^{2\pi} \int_0^b \left( \sigma_{r \theta} \sigma_{r \theta} \cos \alpha - \sigma_{z \theta} \sigma_{z \theta} \sin \alpha / \tan \alpha \right) d\theta d\psi = 0, \quad (52)
\]
\[
M = \int_0^{2\pi} \int_0^b \left( c_{66} \sigma_{r \theta} \sigma_{r \theta} \cos \alpha - c_{44} \sigma_{z \theta} \sigma_{z \theta} \sin \alpha / \tan \alpha \right) d\theta d\psi = 0, \quad (53)
\]
For non-torsional axisymmetric problem, the boundary condition in Equation (48) has been satisfied, and Equations (49) to (51) can be deduced from the global mechanical, impermeable and thermal equilibrium condition in Equations (52). The only boundary condition in Equation (47) and the following equations need to be satisfied:

\[ \int_0^{2\pi} \int_0^b \sigma_{zz} r dr d\theta = 0, \]  
(60)

\[ K_3 \int_0^{2\pi} \int_0^b \frac{\partial T}{\partial z} r dr d\theta = -H, \]  
(61)

\[ \int_0^{2\pi} \int_0^b \frac{\partial C}{\partial z} r dr d\theta = 0, \]  
(62)

Substituting the values of \( \sigma_{rr}, \sigma_{\theta\theta}, C \) and \( T \) from Equation (57) in Equations (47) and (60 to 62) yields

\[ \sum_{j=1}^{4} A_j \left( 2\pi \sum_{j=1}^{4} r^j w_j \frac{1}{W_j} \tan \alpha - s_j \frac{1}{W_j} \frac{1}{V_j} \right) = 0, \]  
(63)

\[ \sum_{j=1}^{4} \frac{r^j}{H_j} A_j = 0, \]  
(64)

\[ \sum_{j=1}^{4} \left( \frac{s_j}{H_j} \tan \alpha - 1 \right) s_j p_{2j} A_j = 0, \]  
(65)

\[ \left( \frac{s_j}{H_j \tan \alpha - 1} \right) s_j p_{2j} A_j = \frac{-H}{2\pi K_3}, \]  
(66)

Where

\[ H_j = \sqrt{1 + s_j^2 / \tan^2 \alpha}, \quad N_j = H_j + s_j / \tan \alpha \quad (j = 1, 2, 3). \]

The constants \( A_j (j = 1, 2, 3, 4) \) can be determined by solving Equations (63) to (66). When the cone has been
reduced to a semi-infinite body, that is, \( \alpha = \frac{\pi}{2} \) then
\[
H_j = N_j = 1 \quad (j=1,2,3,4)
\]
(67)

Using Equation (49) in Equations (45) to (48) can be simplified as:
\[
\sum_{j=1}^{4} A_j s_j r_j = 0, \quad (68)
\]
\[
\sum_{j=1}^{4} r_j A_j = 0, \quad (69)
\]
\[
\sum_{j=1}^{4} s_j P_{2j} A_j = 0, \quad (70)
\]
\[
A_4 = \frac{H}{2\pi K s_4P_{24}} \quad (71)
\]

We have determined four constants \( A_j (j=1,2,3) \) from three equations including Equations (68) to (71) by the method of Cramer's rule.

**Special case**

In the absence of diffusion effects, that is, \( b_1 = b_3 = a = b = 0 \), Equations (57) to (59) yields
\[
u_r = \sum_{j=1}^{4} A_j \frac{r_j}{R_j}, \quad \nu_z = \sum_{j=1}^{4} \frac{\hat{P}_{2j}}{P_{2j}} A_j \text{sign}(z) \log(R_j), \quad \sigma_{zz} = \sum_{j=1}^{4} \frac{A_j}{R_j},
\]
\[
\sigma_{rr} = 2\varepsilon_0 \sum_{j=1}^{4} \frac{A_j}{R_j} + \sum_{j=1}^{4} \frac{A_j}{R_j}, \quad \sigma_{zz} = \sum_{j=1}^{4} \frac{A_j}{R_j},
\]
\[
\sigma_{\theta\theta} = 2\varepsilon_0 \sum_{j=1}^{4} \frac{A_j}{R_j} + \sum_{j=1}^{4} \frac{A_j}{R_j}, \quad \sigma_{\theta r} = \sum_{j=1}^{4} \frac{A_j \text{sign}(z) r_j}{R_j},
\]
(72)

where \( S_1, S_2, S_3, S_4 \) in this case are reduced to \( S_1, S_2, S_3 \)

with \( S_3 = \frac{K_1}{K_3} \) and \( S_1, S_2 \) are two roots (with positive real part) of the equation
\[
\hat{a}s^4 - \hat{b}s^2 + \hat{c} = 0,
\]
(73)

and
\[
\hat{a} = \delta_0 \delta_2, \quad \hat{b} = \delta_0 + \delta_2^2 - \delta_1^2, \quad \hat{c} = \delta_4,
\]
\[
\hat{P}_{2j} = \frac{\hat{P}_{2j}}{P_{2j}}, \quad \hat{P}_{23} = \frac{\hat{P}_{23}}{P_{23}}, \quad \hat{P}_{33} = \frac{\hat{P}_{33}}{P_{33}}, \quad \hat{P}_k = \frac{\hat{P}_k}{P_k}, \quad (k=1,2),
\]
\[
\hat{P}_{33} = \delta_0 - \delta_2 s_3^2, \quad \hat{P}_k = \delta_0 - \delta_2 s_3, \quad \hat{b}_2 = \delta_4 c_1, \quad \hat{b}_3 = (\delta_1^2 - \delta_2^2) - \delta_3, \quad \hat{c}_3 = \delta_4 \delta_2.
\]

Consider the continuity at plane \( z=0 \) for \( u_z \) and \( \sigma_{rr} \) and substituting the values of \( \sigma_{zz}, \sigma_{\theta\theta} \) and \( T \) from Equations (64) with the aid of \( S_3 = \sqrt{K_1/K_3} \) yield the following equations in the absence of diffusion:
\[
\sum_{j=1}^{3} s_j \hat{P}_{2j} A_j = 0. \quad (74)
\]
\[
\sum_{j=1}^{3} s_j A_j = 0. \quad (75)
\]
and
\[
A_3 = \frac{H}{2\pi K s_4 P_{24}} \quad (76)
\]

The constants \( A_j (j=1,2) \) are determined by two Equations (74) and (75) using the method of Cramer's rule.

The above results are similar as obtained by Hou et al. (2005).

**NUMERICAL RESULTS AND DISCUSSION**

Here, the numerical discussions are reported and analysis is conducted for magnesium material. Following Dhaliwal and Singh (2005), the values of physical constants are taken as:
\[
c_{11} = 5.974 \times 10^{10} \text{N.m}^{-2}, c_{12} = 2.624 \times 10^{10} \text{N.m}^{-2}, c_{13} = 2.17 \times 10^{10} \text{N.m}^{-2},
\]
\[
c_{33} = 6.17 \times 10^{10} \text{N.m}^{-2}, c_{44} = 3.278 \times 10^{10} \text{N.m}^{-2}, T_0 = 298 \times 10^3 \text{K},
\]
\[
a_1 = 2.68 \times 10^6 \text{Nm}^{-2} \text{K}^{-1}, a_2 = 2.68 \times 10^6 \text{Nm}^{-2} \text{K}^{-1}, K_1 = 1.7 \times 10^7 \text{Wm}^{-1} \text{K}^{-1},
\]
\[
a_2 = 2.4 \times 10^9 \text{m}^{-2} \text{K}^{-1}, b = 13 \times 10^7 \text{Kgm}^{-1} \text{s}^{-2}, c_1 = 1.95 \times 10^{-8} \text{m}^3 \text{s}^{-2} \text{Kg}^{-1},
\]
\[
K = 9.0 \times 10^{-8} \text{m}^3 \text{s}^{-2} \text{Kg}^{-1},
\]

Figures 2 to 5 depict the variations of radial displacement \( u_r \), axial displacement \( u_z \), temperature change \( T \) and
mass concentration $C$ w.r.t. $r$ for thermoelastic diffusion material. The solid and dotted line respectively, corresponds to thermoelastic theory ($WTD\ z=5$), ($WTD\ z=10$) and centre symbols on these lines, respectively corresponds to thermoelastic theory with mass diffusion ($WD\ z=5$), ($WD\ z=10$).

Figure 2 shows that the values of $u_r$ in case of WTD slightly decrease for smaller values of $r$ and for higher values of $r$, the values of $u_r$ become dispersionless, although for the case of WD, the values of $u_r$ increase for all values of $r$. It is noticed that the values of $u_r$ in case of WD remain more in comparison with WTD.

Figure 3 depicts that the values of $z_u$ in case of WTD decrease for all values of $r$, whereas for the case of WD,
the values of \( u_z \) slightly increase for smaller values of \( r \) and finally becomes constant.

It is evident that the values of \( u_z \) in case of WD remain more in comparison with WTD. Figure 4 shows that the values of \( T \) in case of WTD slightly decreases for all values of \( r \), although for the case of WD, the values of \( T \) increase for all values of \( r \). It is noticed that the values of \( T \) in case of WD remain more in comparison with WTD. Figure 5 depicts that the values of \( C \) in case of \( z = 5 \) slightly decrease for all values of \( r \), whereas for the case of \( z = 10 \) the values of \( C \) increases for all values of \( r \). It is evident that the values of \( T \) in case of \( z = 5 \) remain more in comparison with \( z = 10 \).

### Conclusion

The Green’s functions for three-dimensional problem in transversely isotropic thermoelastic diffusion medium have been derived for static case. After applying the dimensionless quantities and using the operator theory, we have obtained the general expression for components of displacement, temperature distribution, mass concentration and stress components in Cartesian as well as in cylindrical coordinates. Based on the obtained general solution, the three-dimensional Green’s function for a study point heat source on the apex of a transversely isotropic thermoelastic cone in case of steady state problem are derived by four newly introduced harmonic functions. All components of thermoelastic field are expressed in terms of elementary functions and are convenient to use.

From the present investigation, a special case of interest is deduced to depict the effect of diffusion. From numerical results, we conclude that the values of horizontal displacement \( u_r \), axial displacement \( u_z \) and temperature change \( T \) remain more in case of thermoelastic diffusion (WD) in comparison to thermoelastic medium (WTD).

### Conflict of Interest

The authors have not declared any conflict of interest.

### REFERENCES


Appendix A

\[ \pi = \delta_i (\gamma q_0^i - \delta q_0^i), \quad \tau = (\delta_i^2 - \delta_j^2) q_0^i + \delta_j (q_0^i - q_0^j) + \delta_i (\gamma q_0^i - \delta q_0^j), \]

\[ e = (\delta_i^2 - \delta_j^2) q_i^j + q_i^j (\gamma_i - \delta_i) + \delta_i (q_i^j - q_i^j) + \delta_i (q_i^j - q_i^j), \quad d = \delta_i (q_i^j - q_i^j). \]

\[ \Delta = \frac{\delta_i^2}{\delta x} + \frac{\delta_j^2}{\delta y}. \]

Appendix B

\[ \alpha_i = (q_i^j - q_i^j) \delta_i, \quad \bar{\alpha}_i = \delta_i (q_i^j - q_i^j) + \delta_j (q_i^j - q_i^j) + \delta_i (q_i^j - q_i^j) - \gamma_i \delta_i q_i^j, \]

\[ \bar{e}_i = (q_i^j + \delta_i^2 q_i^j) \delta_i + (q_i^j - q_i^j) \delta_i - q_i^j \delta_i, \quad \bar{\alpha}_i = (q_i^j + \delta_i^2 q_i^j) \delta_i + (q_i^j - q_i^j) \delta_i - q_i^j \delta_i, \]

\[ \bar{\alpha}_i = (q_i^j + \delta_i^2 q_i^j) \delta_i, \quad \bar{e}_i = (q_i^j + \delta_i^2 q_i^j) \delta_i + (q_i^j - q_i^j) \delta_i - q_i^j \delta_i, \]

\[ \bar{\alpha}_i = (q_i^j + \delta_i^2 q_i^j) \delta_i, \quad \bar{\alpha}_i = (q_i^j - q_i^j) \delta_i, \quad \bar{e}_i = (q_i^j - q_i^j) \delta_i + (q_i^j + q_i^j) - \delta_i \delta_i q_i^j, \quad \bar{\alpha}_i = (q_i^j - q_i^j) \delta_i. \]