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Full Length Research Paper

A dimension result for polar sets of Brownian path in n-spaces

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Let $T(x,r)$ denote the occupation measure of the ball of radius r centered at x for Brownian motion $\{X_t\}, 0 \le t \le 1$ in $R^n, n \ge 3$. We consider the set $E_a = \left\{ x \in R^n : \limsup_{r \downarrow 0} \frac{T(x,r)}{h(r)} = a \right\}$ and show that for a = 0, $h(r) = r^2 (\log r)^\lambda, \lambda > 1$. Moreover, $E_0 \cap E_a = \phi$ for $a > 0$. We deduce that the Hausdorff dimension of E_0 is *2* for n > 3

Key words: Brownian motion, Hausdorff dimension, multifractal analysis, polar sets, occupation measure.

INTRODUCTION

Let $X_t = \{X(t), t > 0\}$ be a Brownian motion defined on a probability space (Ω, Σ, P) taking values in R^n. This is a special case of a more general symmetric stable processes $X_{\alpha,t}$ of index $\alpha, \alpha \in (0,2]$ defined in Taylor (1986).

A Borel set $E \subset R^n$ is said to be polar for X_t if;

$$P\{X_t \in E \quad \text{for some } t > 0\} = 0 \qquad 1.1$$

These sets are "thin" in the sense that they have zero Lebesgue measure.

In the study of geometric properties of polar sets, one is often interested in the fractal dimensions of the sets (e.g. the Hausdorff dimension and the packing dimension, see for example by Taylor (1986) and Xiao (2004).

These properties also provide information about the underlying geometric structure of the set of points where the solution of a Laplace equation fails to be bounded in n-dimensional space.

Hausdorff dimension is the most commonly used tool in analyzing the geometry properties of such sets.

The well known relationship between Riesz - Bessel capacity and Hausdorff dimension is often used to measure the dimension of any Borel set *E* in R^n, using the range of a symmetric stable process.

Following the results of Hawkes (1971a, b): if

$$R_{\alpha,t} = \{x \in R^n : x = X_{\alpha,t} \text{ for some } t > 0\}$$

is the range of a symmetric stable process of index α, and dim denotes Hausdorff dimension, then for a Borel set *E* in R^n

$$P\{R_{\alpha,t} \cap E \ne \phi\} = \begin{cases} 1 & \text{if } \dim E \ge n - \alpha \\ 0 & \text{if } \dim E < n - \alpha \end{cases} \quad \text{and}$$

$$\dim(E \cap R_{\alpha,t}) = \dim E \;\; \text{for } \alpha > n \qquad 1.2$$

More details were given, see (for example by Taylor (1986) and Xiao (2004)) that for any Borel set $E \subset R^n$ with $\dim E \ge n - 2$.

$$\dim E = n - \inf\{\alpha > 0 : E \text{ is not polar for } X_{\alpha,t}\}$$

Our particular interest in the present note is to use the

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more recent tool of multifractal analysis to obtain more information about the possible values of $\dim E \ for \ n > 3$.

TOOLS FROM MULTIFRACTAL ANALYSIS

We summarize some useful techniques for determining the Hausdorff dimension. We consider the random probability measure u on R^n defined by:

$$u(B) = \left|\{t \in [0,1] : X_t \in B\}\right|$$

For any Borel set $B \subset R^n$

Where;
X_t = Brownian motion in R^n
$|\cdot|$ = Lebesgue measure in R^1.

Thus, $u(B(x,r)) = \int_0^1 I_{B(x,r)}(X_t)dt$ is the length of time from zero up to 1 spent in the ball $B(x,r)$ of radius r centered at x by the process X_t and $I_{B(x,r)}$ is the indicator function of $B(x,r)$.

If $R_1 = \{x \in R^n : x = X_t \ for \ 0 \le t \le 1\}$ denote the range of the Brownian motion $\{X_t\}, 0 \le t \le 1$. u is supported by R_1 and uniformly spread on R_1.

If u is a locally finite Borel measure on R^n, then the pointwise Holder exponent of u at x is $\lim_{r \downarrow 0} \log \frac{u(B(x,r))}{\log r}$, if it exists.

It is well known that for almost all Brownian paths X_t, in the range R_T, the pointwise Holder exponent $\lim_{r \to 0} \log \frac{u(B(x,r)}{\log r}$ is 2. But the Hausdorff dimension of the exceptional sets E_a where $E_a = \left\{x \in R^n : \lim_{r \downarrow 0} \frac{\log u(B(x,r))}{\log r} = a\right\}$, vanishes $\forall \quad a \ne 2, a > 0$

One of the aims of multifractal analysis is to characterize the geometric properties of a measure u by giving the "size" of the set E_a, more precisely their fractal dimensions.

The Hausdorff dimension $\dim E_a \ of \ E_\alpha$ is called the singularity or multifractal spectrum of u and we say that u is a multifractal measure when $\dim(E_\alpha) > 0$ for general α's.

To capture the dedicate fluctuation of the occupation measure of Brownian motion, the set of "thick" points of the measure u were considered by Dembo et al. (2000).

A point $x \in R^n (n \ge 3)$ is called a thick point of u if

$$\limsup_{r \downarrow 0} \frac{u(B(x,r))}{r^2 |\log r|} = a, \quad a > 0$$

They obtained the Hausdorff dimension of the set of thick points of the occupation measure of Brownian motion as follows:

Let X_t be a Brownian motion in $R^n, n \ge 3$, then for all $0 < a \le \frac{4}{q_n^2}$,

$$\dim\left\{x \in R^n : \limsup_{r \downarrow 0} \frac{u(B(x,r))}{r^2 |\log r|} = a\right\} = 2 - a \ q_{n/2}^2 \text{ a.s.} \qquad 2.1$$

Where;
q_n is the first positive zero of the Bessel function $J_{n/2-2(x)}$

Let h be a gauge function that is $h : (0,1) \to (0,1)$ is a continuous monotone increasing function satisfying $h(0+) = 0 \quad and \quad h(2r) \le ch(r), c > 0$. It is clear that for any $x \notin R_1$, the limit $\limsup_{r \downarrow 0} \frac{u(B(x,r))}{h(r)}$ gives zero, while for a fixed point $x \in R_1$, $\limsup_{r \downarrow 0} \frac{u(B(x,r))}{h(r)}$ may give a finite positive limit for a suitable choice of h. (2.1) can be strengthened to hold for $0 \le a \le \frac{4}{q_n^2}$ (Xiao (2004), theorem 12.6). Thus;

if

$$E = \left\{x \in R^n : \limsup_{r \to 0} \frac{u(B(x,r))}{h(2r)} = 0\right\}$$

Then $E \cap R_t = \phi$.

HAUSDORFF DIMENSION OF THE POLAR SET FOR A BROWNIAN MOTION

Now we are ready to prove the main result. We show that

$h(r) = r^2\left(\left|\log\frac{1}{r}\right|\right)^{\lambda}, \lambda > 1$ is the correct guage function such that

$$\limsup_{r\downarrow 0}\frac{u(B(x,r))}{h(2r)} = 0 \text{ a.s}$$

3.1

Note that for the occupation measure associated with Brownian motion in $n \geq 3$, (3.1) has a simple meaning, for it becomes;

$$\limsup_{r\to 0}\frac{T(x,r)}{h(2r)}$$

Where;

$$T(x,r) = \int_0^1 I_{B(x,r)}(X_t)dt$$

is the total time spent in $B(x,r)$ from zero up to time 1. It is sufficient to consider $T(0,r) = T(r)$.

The key to the proof of the main result is the following result in Taylor (1967) which we state as;

Lemma 1

For a Brownian motion in $R^n, n > 2$ there exists a positive constant c such that for $z \geq z_0 > 0$

$$P\{T(r) \geq zr^2\} \leq \exp(-cz)$$

Then we state:

Lemma 2

Let X_t be a Brownian motion in $R^n, n \geq 3$

Suppose $h(r) = r^2\left(\log\frac{1}{r}\right)^{\lambda}, \lambda > 1$.

Then $\limsup_{r\downarrow 0}\frac{T(r)}{h(2r)} = 0$ a.s

Proof

For a fixed $\in > 0$ and $a_k \to 0$ as $k \to \infty$

Define

$$E_\lambda = \left\{T(a_k) \geq \in a_k^2\left(\log\frac{1}{a_k}\right)^{\lambda}\right\}$$

By Lemma 1,

$$P(E_\lambda) \leq \exp\left\{-c \in \left(\log\frac{1}{a_k}\right)^{\lambda}\right\}$$

$$\leq \exp\left\{-c\log\left(\log\frac{1}{a_k}\right)^{\lambda\in}\right\}$$

$$= \left(\log\frac{1}{a_k}\right)^{-\lambda\in c}$$

Hence,

$$\sum P(E_\lambda) < \infty \quad if \quad \lambda > \frac{1}{c\in} > 1$$

Thus by Borel Cantelli Lemma, we have $P(E_i, i.0) = 0$
Therefore there exists a_0 such that

$$\left\{T(a_k) < \in a_k^2\left(\log\frac{1}{a_k}\right)^{\lambda}, i.0\right\}$$

For some $a_k \leq a_0$ a.s
Hence,

$$\limsup_{a_k\to 0}\frac{T(a_k)}{a_k^2\left(\log\frac{1}{a_k}\right)^{\lambda}} \leq \in \quad for \quad \lambda > 1.$$

Allowing $\in \to 0$, shows that

$$P\left(\limsup\frac{T(a_k)}{a_k^2\left(\log\frac{1}{a_k}\right)^{\lambda}} = 0\right) > 0, \quad \lambda > 1 \quad , \quad \text{by}$$

Blumental zero one law.

Then $P\left(\limsup_{a_k \to 0} \frac{T(a_k)}{a_k^2\left(\log \frac{1}{a_k}\right)^{\lambda}} = 0\right) = 1, \ \lambda > 1$

Hence by monotonicity of T and h, we have

$\limsup_{r \downarrow 0} \frac{T(r)}{h(r)} \le (1+\in) \limsup_{k \to \infty} \frac{T(a_k)}{\left(a_k^2 \log \frac{1}{a_k}\right)^{\lambda}}$, $\lambda > 1$ and the result is established.

It then follows from 6, theorem 12.6.i that $\dim E = 2$ a.s.

But $\dim E < n-2$ a.s., by (1.2)

Hence, $\dim E = 2 \quad for \quad n > 3$.

Conclusion

Characterizing the polar sets for Brownian motion is related to determining the sizes of sets $A \subset R^h$ for which there are nontrivial bounded harmonic functions on R^n.

It is well known that there exist bounded harmonic functions on R^n if and only if A is polar for Brownian motion.

Such a set is called removable for a bounded harmonic functions and it is big enough to hide a pole of the harmonic function inside.

But for all Borel sets with $\dim(A) < n-2$, A is polar. In this note we have shown that, for a polar set A for Brownian motion, $\dim(A) = 2$ for $n > 3$. This means that the size of A is substantial in this case. Also we established a class of gauge functions for Hausdorff dimension results.

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