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Full Length Research Paper

Relationship between continuity and momentum equation in two dimensional flow

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In this paper a quantitative discussion on a theory describing the relationship between the continuity and momentum equation in two dimensional flow together with the momentum equation in vectorial form: $\rho \, {}^{dq}\!/_{dt} = -\nabla p + \rho g + \mu \nabla^2 q$ is reported. Via expanding $\nabla * (\nabla . q)$ in cylindrical polar coordinates, the end result is proved to be Euler equation.

Key word: Continuity equation, momentum equation, cylindrical coordinates, polar coordinate.

INTRODUCTION

A more detailed view of the fluxes across the parcel can be obtained within a reasonable space of time of which our attention is restricted to two dimensions. We can then write the equations for the component and look closely at the change in these components. Donna (2003) Considered the transformations permit, the determination of pressure distributions and fluid film thickness for any orientation of the hemispherical shell including the horizontal position, for which the conventional description, of Reynolds equation is well suited. Serre et al. (2001) Studied the configuration of cylindrical cavities subjected to a radial through flow or to a differential rotation of the walls are relevant to rotating machinery devices. Van Doormal and Raithby (1982) describe an iteration technique for the solution of the set of coupled algebraic equation that represent the mass and momentum conservation equations in an incompressible fluid flow formulation and the proposed method solves for continuity and momentum simultaneously along lines through the calculation domain. Phillip and Liu (2008) in two phase flow models continuity and momentum equations are established for a sediment phase and a fluid phase. The model we present here solves concentration weighted averaged equations for both phase. Leggett and Liu (1984) Studied Applications of boundary element methods to fluid mechanics topics in boundary element research, Karnidakis et al. (1991) studied the high order splitting methods for the incompressible Navier-Stokes Equations. Also Tan et al. (1998) considered simulated flow around long rectangular plates under cross flow perturbations and Amoudry (2008) consider planar view of a parcel unit depth. Assume ρ is constant across the parcel, so we can write for the mass of the parcel, $\delta\mu = \rho\delta v = \rho\delta x\delta y\delta z$. In the two-dimensional flow, each component of velocity can vary in both x and y directions. We can approximate those velocity changes across our incremental parcel by a Taylor expansion. In this case we will consider the base values of qualities such as pressure and velocity to be the value of the center of the parcel and expand around these values. Note that value of the corner, $x = y = z = 0$, could also be assured as base values. Since the parcel is infinitesimal with respect to mean flow scales. The magnitudes of these values are uniform across the parcel in the limit $\delta v \to 0$. We write the incremental changes at the point, we need not be zero. Again we look at the total change in the density and the scope of the parcel as it instantaneously occupies the point (x, y). We can derive the continuity equation in a slightly different manner, by considering a specific infinitesimal parcel in a Largrangian sense. The derivation will illustrate the close connection between Largrangian and Eulerian perspectives and we will send up with the familiar Eulerian expression. Starting with the Langrangian perspectives we consider a very small parcel such that $\delta v \to 0$, with no sinks or sources. We then follow the particular

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parcel that experiences volume and density changes with respects to five only field varcash will vary infinitesimal across the small dimensions of the parcel. Then the statement for the constant mass of fluids parcel, then the statement, for the constant mass of this parcel $\rho\delta v$ is completely expressed in the five derivatives, D(($\rho\delta v$)/$\partial t) = 0$. However, when the parcel moves through the fluid, to volume must distorts and changes due to the changing forces in the thus field. The derivative separated into density and volume changes by using the chain rule for differentiation. In the end, the derivative can be converted to the Eulerian expression.

MATHEMATICAL ANALYSIS

The differences between the various derivatives can be explained in a more formal manner as follows:

Consider a fluid particle moving with a load velocity;

$$q = i\upsilon + j\nu + \kappa\omega \qquad 1.0$$

and investigate the change of the property $b = b(x, y, z, t)$ of the particle. The change in b with time and position may be expressed as

$$db = \left(\frac{\partial b}{\partial t}\right)\partial t + \left(\frac{\partial b}{\partial x}\right)\partial x + \left(\frac{\partial b}{\partial y}\right)\partial y + \left(\frac{\partial b}{\partial z}\right)\partial z \qquad 2.0$$

The rate of change of s in time $\partial b/\partial t$ equation become

$$\frac{\nabla b}{\nabla t} = \frac{\partial b}{\partial t} + U\frac{\partial s}{\partial x} + V\frac{\partial b}{\partial y} + W\frac{\partial b}{\partial z}\} \qquad 3.0$$

$$\frac{\nabla b}{\nabla t} = \frac{\partial b}{\partial t} + z \cdot \partial b \qquad 4.0$$

By equation (3) we can direct operation of ∂b in new coordinates.

$$\frac{\nabla b}{\nabla t} = \frac{\partial b}{\partial t} + q_r\frac{\partial b}{\partial r} + q_\vartheta\frac{\partial b}{r\partial\theta} + q_z\frac{\partial b}{\partial z},$$

5.0

$$\frac{\nabla b}{\nabla t} = \frac{\partial b}{\partial t} + q_r\frac{\partial b}{\partial r} + q_\vartheta\frac{\partial b}{r\partial\theta}q_\theta\frac{\partial b}{r\sin\theta\partial\theta}$$

The law of conservation of mass has already been presented in a form applicable to a control volume may be rewritten as:

$$\int_v \frac{\partial R}{\partial t}\partial v + \int_s \partial q \cdot nds \qquad 6.0$$

Hence, $\frac{\partial p}{\partial t} + \Delta(pq) = 0$ 7.0

The equation (7.0) is known as the equation of continuity. It is the differential form of the law of conservation of mass written in form of the flow field.

Equation (7.0) is now rewritten in detail in the three most continuity used coordinate systems.

In Cartesian coordinates

$$\frac{\partial p}{\partial t} + \frac{\partial(pu)}{\partial x} + \frac{\partial(pw)}{\partial y} + \frac{\partial(pw)}{\partial z} = 0 \qquad 8.0$$

In cylindrical coordinates

$$\frac{\partial p}{\partial t} + \frac{d}{r}\frac{\partial(rpqr)}{\partial r} + \frac{1}{r}\frac{\partial(pq\theta)}{\partial\theta} + \frac{\partial(pqz)}{\partial z} = 0 \qquad 9.0$$

In spherical coordinates

$$\frac{\partial p}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2pq)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(pq\theta\sin\theta)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial(pq\theta)}{\partial\theta} = 0 \qquad 10$$

In some particular cases equation of continuity assumes simpler form given in Cartesian coordinates.

$$\left.\begin{array}{l}\Delta\cdot(pq) = 0\\ or\\ \Delta\cdot q = 0\end{array}\right\}. \qquad 11$$

Now, for momentum, Newton's second law of motion states that the rate of change of momentum of a thermodynamics system equals the sum total of the forces acting on the system.

$$\frac{D}{Dr}\int_v pqdv = \int_v gpdv + \int_s Tds \qquad 12$$

When g is a general body force per unit mass, and T is the system boundary for *x*-component Equation 12 becomes

$$\frac{D}{Dt}\int_v pxdv = \int_v gxpdv + \int_s Tnxds \qquad 13$$

The Reynolds transport theorem may now be applied to the left-hand side of this equation

$$\frac{D}{Dt}\int_v pudv = \int_v \rho\left[\frac{du}{\partial t} + \frac{u\partial u}{\partial u} + \frac{v\partial u}{\partial y} + \frac{w\partial u}{\partial z}\right]dv \qquad 14$$

The stress term *Tu* inside the surface integral is now written in terms of its components to yield

$$\int_s T_{nn}ds = \int_s \left[T_{xxi} + T_{ynj} + T_{zx}k\right]\cdot nds = \int_v \left[\frac{\partial \tau_{xx}}{\partial u} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau t}{\partial_x}\right]dv \qquad 15$$

where the divergence theorem has been used again. By subtraction of equation 14 and 15 into equation 13 yields

$$\int_v \left\{\rho\left[\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial u} + \frac{v\partial u}{\partial t} + \frac{w\partial u}{\partial t} - g_x\right] - \left[\frac{\partial T_{xx}}{\partial u} + \frac{\partial T_{yn}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right]\right\}\partial v = 0 \qquad 16$$

Becomes

$$\rho = \frac{T_{xx} + T_{yy} + T_{zz}}{3} \qquad 17$$

It is customary to separate out the pressure terms from the total stress

$$\mathrm{Tij} + \text{-}\, \mathrm{p}\partial ij + \tau ij \qquad 18$$

And $\tau_{ij} = \partial\mu\Sigma ij$ equation 18 is written in tensor form as

T=-p + τ 19

$$P = \begin{vmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{vmatrix}$$

Equation 19 is used to modify the momentum.

$$\rho\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial t}\right) = -\frac{\partial p}{\partial x} + \rho_{gy} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad 20$$

$$\rho\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial x} + \rho_{gy} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad 21$$

$$\rho\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + V\frac{\partial w}{\partial y} + W\frac{\partial w}{\partial t}\right) = -\frac{\partial p}{\partial x} + \rho_{gy} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad 22$$

This may be put in symbolic compact form.

$$\rho\frac{\Delta q}{\Delta t} = -\nabla_p \rho g + \nabla.\tau \qquad 23$$

The expression for the stress and the rate of strain component in several coordinate systems are now written down.

In Cartesian coordinates of q = iu + jv + kw

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \varepsilon_{zz} = \frac{\partial w}{\partial r},$$

$$\ell_{xx} = \partial\mu\frac{\partial u}{\partial x},\ \ell_{yy} = \partial\mu\frac{\partial r}{\partial y},\quad \ell_{zz} = \partial\mu\frac{\partial w}{\partial z}, \qquad 24$$

In cylindrical coordinates $\mathrm{q} = \mathrm{e_r}q_r + e_9q_9 + e_zq_z$

$$\varepsilon_{rr} = \frac{\partial qr}{\partial r}\ \varepsilon_{9ee} = \left(\frac{1}{r}\frac{\partial q\vartheta}{\partial\vartheta} + \frac{qr}{r}\right),\ \varepsilon_{zz} = \frac{\partial qz}{\partial z}$$

$$\tau_{rr} = 2\mu\frac{\partial qr}{\partial r},\ \tau_{99} = 2\mu\left(\frac{1}{r}\frac{\partial q\vartheta}{\partial\vartheta} + \frac{qr}{r}\right),\ \tau_{zz} = 2\mu\frac{\partial qz}{\partial z} \qquad 25$$

In spherical coordinates q $= e_rq_r + e_9q_9 + e_\phi q_\phi$

$$E\theta\theta = \left(\frac{1}{R}\frac{\partial qR}{\partial R} + \frac{qr}{R}\right),\ \tau\theta\varphi = 2\mu\left(\frac{1}{R}\frac{\partial qR}{\partial R} + \frac{qr}{R}\right)$$

$$E\varphi\varphi = \left(\frac{1}{RSin\varphi}\frac{\partial q\varphi}{\partial\varphi} + \frac{qR}{R}\frac{q\theta Cot\theta}{R}\right),$$

$$\tau\varphi\varphi = 2\mu\left(\frac{1}{RSin\varphi}\frac{\partial q\varphi}{\partial\varphi} + \frac{qR}{R}\frac{q\theta Cot\theta}{R}\right) \qquad 26$$

Using 29-31 to eliminate the stress components from the differential momentum and equation 20-22 Becomes

$$\rho\frac{Dq}{Dt} = -\nabla\rho + \rho g - \mu\nabla x(\nabla xq) = -\nabla\rho + \rho g - \mu\nabla^2 q \qquad 27$$

By using equation (11), if ∇^2 is the Laplacian operator applied to the velocity vector in Cartesian coordinates, expanding $\nabla\times(\nabla\times q)$ in cylindrical polar coordinates we obtain

$$= \rho_{gr} - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)(rq_r)\right] + \frac{1}{r^2}\frac{\partial^2 q_r}{\partial z^2} - \frac{p}{r^2}\frac{\partial q\theta}{\partial\theta} \qquad 28$$

$$= \rho_{g\theta} - \frac{\partial p}{r\partial\theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rq\theta)\right) + \frac{1}{r^2}\frac{\partial^2\theta q_r}{\partial z^2} + \frac{\partial^2\theta q_r}{\partial z^2} + \frac{2}{r^2}\frac{\partial q\theta}{\partial\theta}\right] \quad 29$$

$$= \rho_{yz} - \frac{\partial p}{\partial y} + \mu\left[\frac{I}{r}\frac{\partial}{\partial r}\left(r\frac{\partial qz}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 qz}{\partial\theta^2} + \frac{\partial^2 qz}{\partial z^2}\right] \quad 30$$

By repeating the for spherical coordinate, we obtain

$$= \rho_{y\varphi} - \frac{1}{RSin^2\theta}\frac{\partial p}{\varphi} + \mu\left[\frac{I}{R}\frac{\partial}{\partial R}\left(R\frac{\partial q\theta}{\partial R}\right) + \frac{1}{R^2 Sin\theta}\frac{\partial^2}{\partial\theta^2} + \left(Sin\theta\frac{\partial q\varphi}{\partial\theta}\right)\right]$$
$$+ \mu\left[\frac{I}{RSin^2\theta}\frac{\partial^2 q\varphi}{R^2 Sin^2\theta} + \frac{2}{R^2 Sin\theta}\frac{\partial qr}{\partial\varphi} + \frac{2Cos\theta}{R^2 Sin^2\theta}\frac{\partial q\varphi}{\partial\varphi}\right] \quad 31$$

By substituting $\mu = 0$ in the navier-stokes equating which is called momentum equation (27) – (31) we obtain an equation

$$\rho\frac{Dq}{Dt} = \rho_g - \nabla\rho \quad 32$$

This is called the Euler equation

DISCUSSION

Solutions of the momentum equation result in velocity vectors q and pressure ρ which satisfy both the momentum equation and the continuity equation. Given such a combination, [$q.\rho$], we can check whether it constitutes a solution by substitution into the equations. How to find such a solution is another matter and any general step leading toward this goal is useful. For two dimensional flows it is possible to eliminate the continuity equation from the system of equations by using only functions which satisfy the continuity equation. This elimination is a formal step toward a solution and functions which affect this elimination and the stream functions and if the flow is defined as two dimensional when its description in Cartesian coordinates shows no z-component of the velocity and no dependence on the z-coordinate. Such a flow can be described in the $z = 0$ plane, by shown a flow pattern identical to that in the $z = 0$ plane. The $z = 0$ plane is therefore called representative plane.

Figure 1 shows a representative plane for two-dimensional flow, with four streamlines denoted by the letters A, B, C, D. which the whole pattern may be shifted in the z-direction parallel to itself. Thus the streamlines also represent stream sheets, that is barriers which are not crossed by the flow. The Mass flux entering at the left, between, say, streamlines A and B must therefore come out at the right side without change. Because the distance between the two streamlines accommodating this mass flux seems in the drawing to increase, the mass flux seems per unit Cross section $\rho..q$, must decrease from left to right. There is therefore some relation between the convergence and divergence of stream lines and the vector $\rho.q$. Furthermore, because stream sheets are not crossed by the flow, each sheet represents a certain mass flux per unit depth of stream sheet taking place below it, flowing between it and some particular stream sheet representing zero flux.

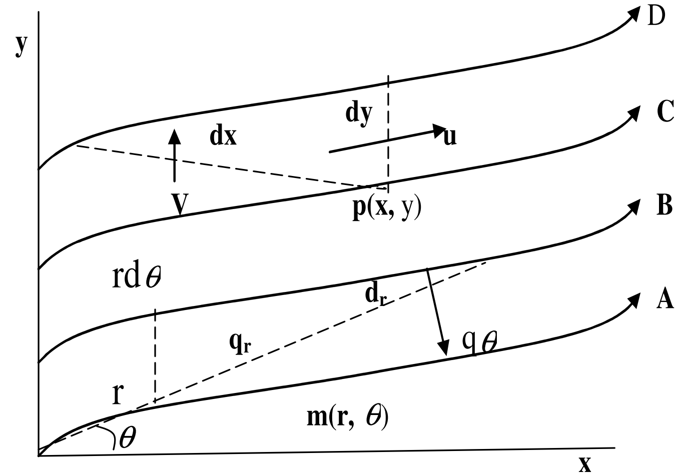

Figure 1. Plane for two-dimensional flow, with four streamlines denoted by the letters A, B, C and D.

This mass flux is called the stream function and it is denoted by φ

$$\partial\varphi = (\partial y)(up) = (-\partial x)(vp)$$

of which follows

$$u\ell = \frac{\partial x}{\partial y},\ v\rho = \frac{-\partial\varphi}{\partial x}$$

By using planned polar coordinate in the representative plane and letting.

$$\varphi B = \varphi A + d\varphi$$

$$d\varphi = (rdq(q_r\rho)$$

$$d\varphi = (dr)(-qq\rho)$$

Which becomes;

$$q_r \rho = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}, \quad q\theta\rho = \frac{-\partial \varphi}{\partial r}$$

REFERENCES

Amoudry H (2008). Two – phase modeling of grander sediment for sheet flow and scour. Ph.D. Thesis Cornell University.

Donna M (2003) Reynolds Equation for Spherical Bearings J. Tribol. 125(1): 203-207.

Karnidakis GE, Israeli M, Orszag SA (1991). High-Order Splitting Methods for the Incompressible Navier-Stokes Equations, J. Comput. Phys. 97: 414-443.

Leggett JA, Liu PLF (1984). Applications of boundary element methods to fluid mechanics topics in boundary element research, led by C Brebbia), Springer – ver lag. 1(1).

Phillip L, Liu F (2008). Two-Dimensional, Two-Phase sediment Transport Modeling Laurent Amoudry, Proudman Oceanographic Laboratory, Liverpool, UK.

Serrie EBP, Kofarba R (2001). Numerical Simulation of the transition is three – dimensional rotating flows with walls. Boundary layers instability. Int. Formal Fluid Dyn. 5(2): 17-30.

Tan BT, Thompson MC, Hourigan K (1998). Simulated Flow around Long Rectanglular Plates under Cross Flow Pertubations. Int. J. Fluid Dyn. 2(1).

Van Doornail JP, Raithby GD (1982).The Simultaneous Solution along lines of the Continuity and Momentum equations.