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Full Length Research Paper

A model teaching for the cycloid curves by the use of dynamic software with multiple representations approach

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This study proposes the use of dynamic software that will enable students to explore a specific kind of parametric equation at the tertiary level. Firstly, it was aimed at visualizing the basic cycloid curve which is the trajectory of a point on the circle rolling along x-axis. Then, the mathematical meaning of the design process, which is mostly based on the syntax, was tried to be drawn out. It was explained however, that this meaning, produced the parametric equation of basic cycloid. Lastly, it explained the abstraction of more specific cycloids, which are called epicycloids and hypocycloid, from the mathematical analysis of first design. The entire abstraction process is defined as an example of how a model plays cyclic role between mathematics and the real world.

Key words: Dynamic mathematics software, abstraction process, cycloids, mathematical modeling, multiple representations, tertiary level.

INTRODUCTION

A dynamic software should not be seen as a presenter just more dynamic than any regular presentation software. When the opportunity of examining the mathematical background of the dynamism is assessed, valuable mathematical abstraction may be reached. At this point, Fischbein's (1987) comment should be remembered, which states that: "a visual image not only organizes the data at hand in meaningful structures, but is also an important factor guiding the analytical development of a solution". Bishop (1989), also advocated that emphasizing visual representations in all aspects of mathematics is important. Furthermore, Duval (1998) highlighted the need of differentiating between different visual processes in the curriculum and proposes three cognitive processes which are involved in geometrical reasoning as visualization processes, construction processes (using tools) and reasoning processes. Presmeg (1986) also pointed out that some visual thinkers are able to think by using dynamic mental images.

By the above point of view, Gorgorio and Jones (1996), determined that the use of dynamic geometry software such as Cabri-Geometry can support the development of visual skills and understanding the visual phenomena behind the mathematical concepts. Dynamically capable software GeoGebra, also provides us an innovative opportunity to investigate and understand the curves described as dynamic. GeoGebra can be thought as an innovative mathematical modeling tool. Doerr and Pratt (2008), proposed two kinds of modeling according to the learners' activity; exploratory modeling and expressive modeling after a comprehensive literature synthesis about modeling with technology.

A learner uses a ready model, which is constructed by

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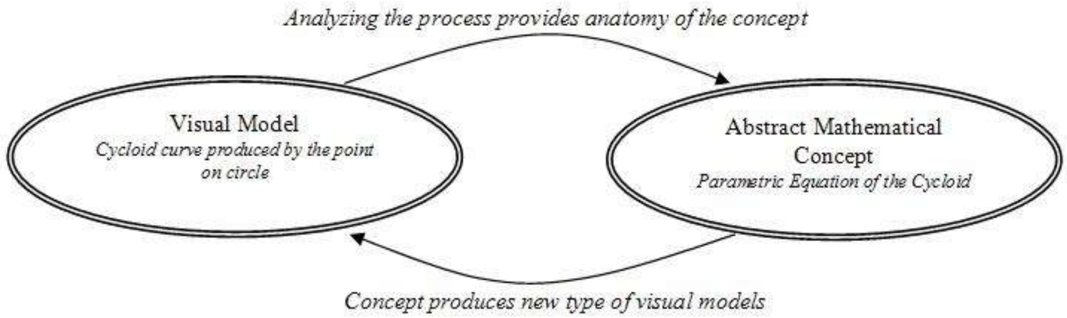

Figure 1. Cyclic structure of modeling task.

an expert, in exploratory modeling, while he or she shows his/her own performance to construct the model in expressive modeling. During the process of constructing the model, learners can find the opportunity to reveal the way of understanding the relationship between the real world and the model world (Doerr and Pratt, 2008).

In Greek geometry, curves were defined as objects which are geometric and static. For example, a Parabola, is defined as the intersection of a cone and plane like other conics, which are first introduced by Apollonius of Perga (262 BC – 190 BC). Alternatively, 17th century European mathematicians preferred to define the curves as the trajectory of a moving point. In his "Dialogue Concerning Two New Science of 1638", Galileo, found the trajectory of a canon ball. Assuming a vacuum, the trajectory is a parabola (Barbin, 1996). We can understand that some of the scientists, who studied on curves, were actually interested in the problems of applied science, like Galileo, as an astronomer and a physician; Nicholas of Cusa as an astronomer and so on and so forth. Some of the scientist, who lived approximately in the same century, took further, the research on the curves as a mathematician (e.g. Roberval, Mersenne, Descartes and Wren).

This historical process also points out the different representations for the curves. Multiple representations of a mathematical concept, provide the opportunity of seeing concepts in different ways and establishing the connections among them (Even, 1998; Hiebert and Carpenter, 1992). Furthermore, it provides a constructive learning environment by enriching the mathematical learning environments (Chrysanthour, 2008). Most importantly, concepts, which also have dynamic views, can be made more understandable by a suitable modeling approach.

The view of expressive modeling approach, defined by Doerr and Pratt (2008) and dynamic property of the curves, a modeling task, which is constructing the basic cycloid curve without using its well known parametric equation, was chosen in this study. A cycloid is a curve which is the trajectory of a point on the circle rolling along any axis. The basic cycloid curve is produced by a circle rolling along *x*-axis. The process of producing visual model was analyzed using geometry and algebra. During this analysis, every step was defined parametrically in terms of the software's language (syntax). At the end of the process, an atomic structure of a parametric equation of cycloid was obtained. More so, how this anatomic structure helps us to produce more complicated cycloid curves such as hypocycloids and epicycloids was described. Figure 1 represents the cyclic structure of this modeling task.

In other words, this study displayed the mathematical anatomy of the equation providing us the chance of producing new kinds of visual models and proposes, using multiple representation approach in tertiary level in the light of using a diverse and dynamic visualization process (Duval, 1998; Presmeg, 1986). The following sections describe this process and readers are recommended to experience and see what is going on.

METHODS

In accordance with the purpose of explaining the use of a technological tool in making an advanced mathematic concept clearly understandable, the entire process was defined by the sections below. Basically, three steps were used in illustrating the anatomy of the curve.

First, it is shown how standard units circle can be used in producing the standard cycloid, both graphically and analytically. Second, the parametric equation, which is abstracted primitively at the first step, is analyzed and converted into a formal version. Lastly, after understanding the effects of several parameters to the cycloid visually, how to obtain hypocycloid and epicycloids curves was investigated. This step was also a check on the feasibility of the anatomy of the cycloid curve. The free accessible mathematics software GeoGebra, has been chosen as a dynamic software. GeoGebra is one of the easiest software that can make the relation between algebraic and graphical representations clearer (Edwards and Jones, 2006).

Construction of the process

First, we need a generator circle with radius r on the x-axis. The variable r, will control the radius and should be constructed as a slider changing from 0 to 5 on GeoGebra. The coordinates of the

Table 1. Description of visual model (In terms of GeoGebra).

Construction of objects	Name of objects on GeoGebra	Description
r=1	Slider r	radius of generator circle with the initial value of 1
M=(0,r)	Point M	Center point of generator circle (should be hidden)
Circle[M, r]	Circle c	Generator circle with the center of M and radius *r* (should be hidden)
A=Point[c]	Point A	Leg point for the rotating point on generator circle (should be hidden). Initial position is making 0 angles with *x*-axis.
t=1	Slider t	Angle t that controls the rotation in terms of radian
B=Rotate[A,t,M]	Point B	Rotated point according to point A on generator circle (should be trace on)

Table 2. Description of visual model (In terms of GeoGebra).

Objects of visual model		Mathematical model (parametric equation of the trace)	
Construction of Objects	Description	Equation	Equation command on GeoGebra
M=(0, r)	The movement of generator circle will be controlled by this point.	$x=r*\cos(u)$, $y=r*\sin(u) + r$	Curve[r cos(u), r sin(u) + r, u, 0, t]
B=Rotate[A, t, M]	This point is generating the curve by rotating on the circle whose center is M. Parameter (slider) *t* controls the rotation.		

circle must be (0, *r*). Then, we need a point (named as A) rotating on the generator circle with the circular angle *t*. *t* will be a parameter controlling the rotation size in terms of radian. Lastly, the point A must be rotated by angle the *t*. The trace of this rotated point will generate the cycloid curve. The generator point will be named as *A′* by GeoGebra automatically. The name is changed as *B* to utilize the descriptions in Table 1. When the slider *t* is animated, the generator point B will trace a curve. Hereafter, we focused on the shaded rows of Table 1 and inserted a column to identify the mathematical model corresponding to the visual model. The mathematical model was also checked by the command "Curve" which is a GeoGebra command producing parametric curve and its syntax is Curve[x-component, y-component, parameter, initial value, end value].

More so, we obtained the curve represented by the equation independent from its visual model. So we see both the trace of generator point and the curve. When the end value is defined as same as slider *t*, the curve will be plotted synchronously with the curve obtained by the trace.

Table 2 shows a summary of the mathematical meaning of visual GeoGebra objects. We have a circle with radius *r* and center M. The center point M is free of the slider *t* which only controls the rotation of the point B. While pondering on a famous parametric equation $\{x=r\cos(u),\ y=r\sin(u)\}$, where *u* is the counter clockwise angle on the origin with the *x*-axis representing the unit circle and in the interval $[0, 2\pi]$; the visual model means the circle which is centered at (0, *r*).

Therefore, our new equation must be $\{x=r\cos(u),\ y=r\sin(u)+r\}$ where *u* is changing from 0 to *t*. That is, when the slider *t* is animated, the point B generates the circle as in Figure 2. While the point B is rotating around the circle, the circle also should move on *x*-axis horizontally, in order that the center point's coordinates must be controlled by the slider *t* with the new coordinates as (*t*, *r*). According to this change on visual construction, the *x* component of parametric equation, must also be revised as $x=r*\cos(u) + u$ to translate the circle horizontally by the slider *t* as seen in Table 3.

Now, we have a premature cycloid curve as seen in Figure 3. This curve does not start from origin and the rotation is not a natural rolling on axis as a true cycloid should be. This can be understood better by displaying the generator circle and the reference point A you will easily see that the rotation is not a real rolling. To obtain the real rolling move, firstly, we need to fix the rotation direction of the point B and the rotation angle should be set as –t. Second, the starting point of the rotation should be changed to the origin. To provide this, we can easily drag the point A, which generates the point B by rotation, to the (0, 0) as it is on the beginning position of generating circle. Table 4 summarizes the changes producing the basic cycloid curve.

Now, we are sure that the trace of the point B generates the basic cycloid. To obtain the curve mathematically (the curve obtained by the *curve* command), we have to focus on the role of variables in the parametric equation. We can abstract that, the independent variable *u* in the trigonometric function controls the rotation, while the other independent variable *u* controls the horizontal translation. Therefore, the mathematical arrangement should be on the angle of the trigonometric function as $(-u-\pi/2)$. After this editing, we will have the cycloid both visually (trace) and mathematically (curve) as seen in Figure 4.

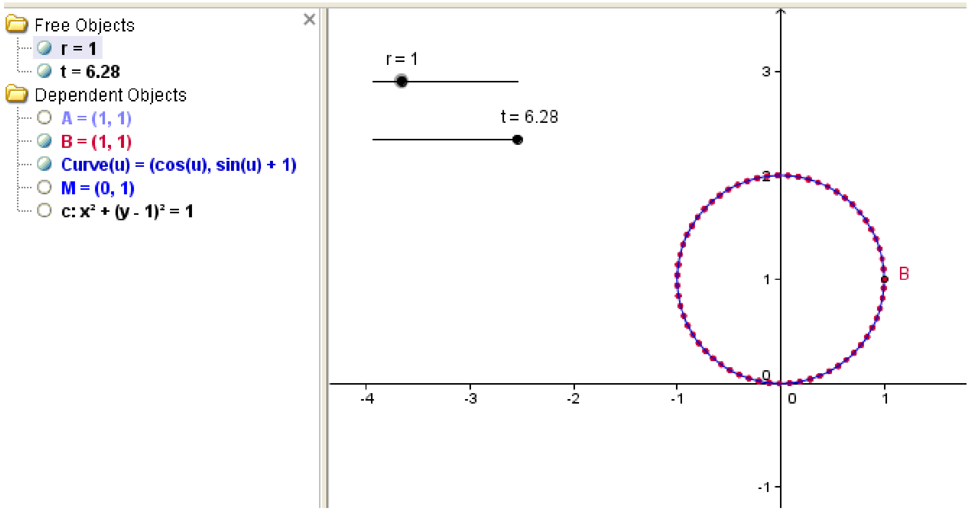

Figure 2. Obtaining the circle.

Table 3. Changes on the model towards cycloid.

Objects of visual model (In terms of GeoGebra)		**Mathematical model (parametric equation of the trace)**	
Construction of Objects	Description	Equation	Equation command on GeoGebra
M=(t, r)	Generator circle is being moved horizontally with the center (t, r)	x=r*cos(u) + u y=r*sin(u) + r	Curve[r cos(u)+u, r sin(u) + r, u, 0, t]
B=Rotate[A,t,M]	The generator point is rotating counterclockwise with the angle t		

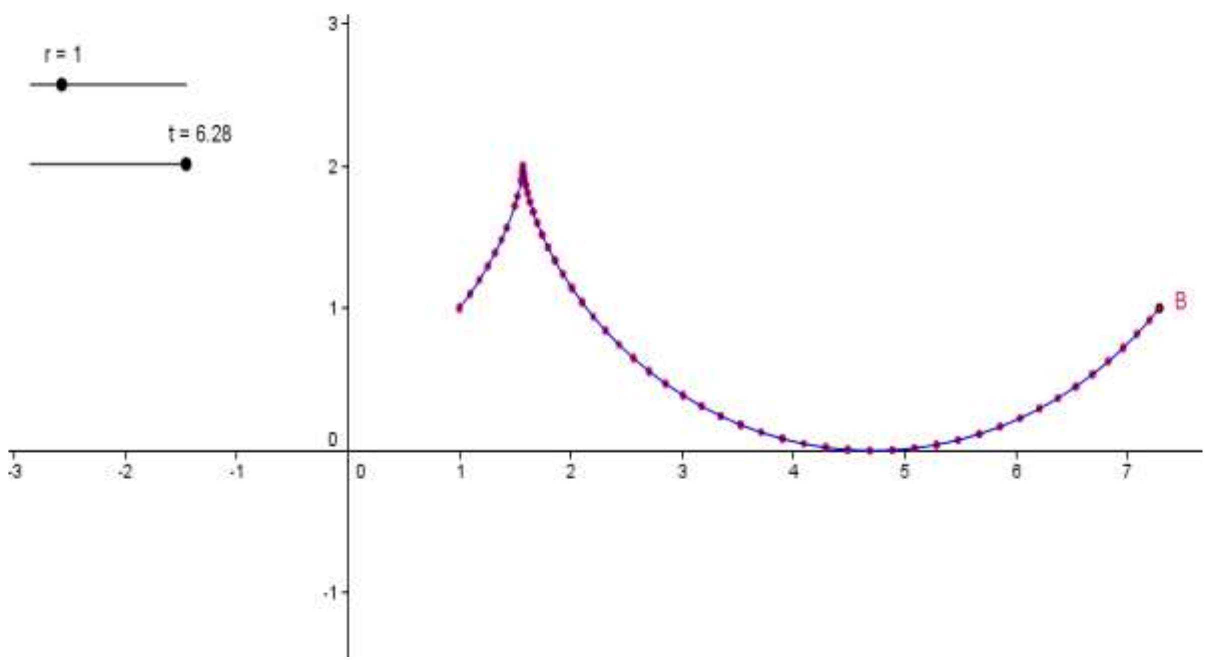

Figure 3. Premature cycloid.

Table 4. Changes producing the basic cycloid.

Objects of visual model (In terms of GeoGebra)		**Mathematical model (parametric equation of the trace)**	
Construction of objects	Description	Equation	Equation command on GeoGebra
M=(t, r)	Generator circle is being moved horizontally with the centre (t, r)	$x=r*\cos(-u-\pi/2) + u$ $y=r*\sin((-u-\pi/2) + r$	Curve[r cos(-u - π / 2) + u, r sin(-u - π / 2) + r, u, 0, t]
B=Rotate[A,-t, M]	The generator point is rotating counterclockwise with the angle -t		
Starting position of point B should be changed	Drag the point A, which generates the point B by rotation to the (0,0) at the initial position of the circle		

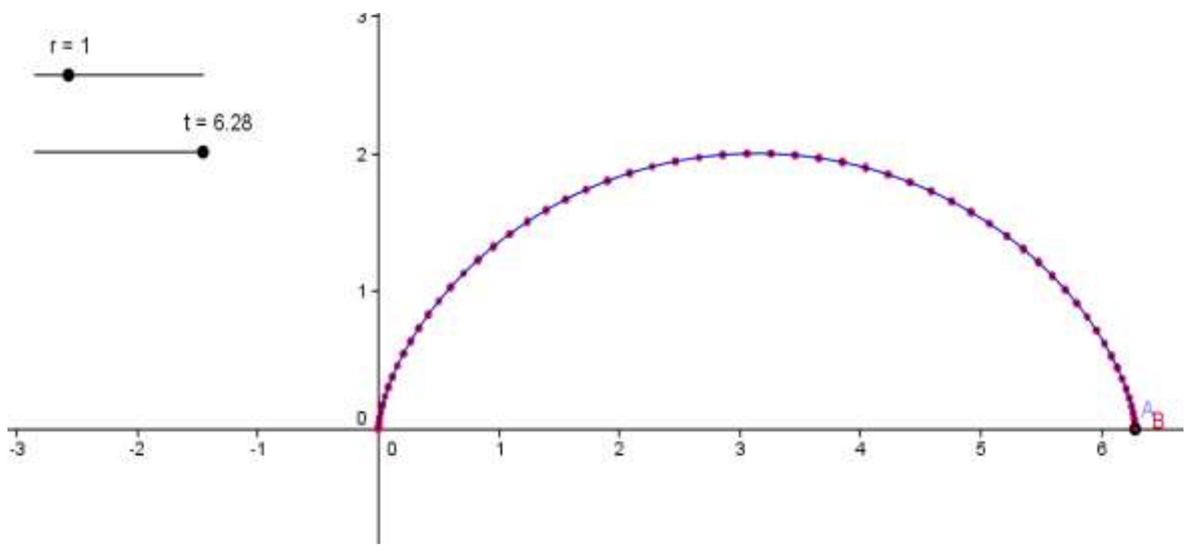

Figure 4. The basic cycloid.

Lastly, let's learn the effect of the generator circle's radius to the cycloid curve. When the parameter *r* changed into a different value than 1, the following curve will be obtained as seen in Figure 5. In the Figure 5, the cycloid tends to create a knot. This problem needs to be solved. The size of horizontal translation of the center of the generator circle and the size of rotation of the generator point are not equal. We need to enlarge the horizontal translation by *r*. So coordinates of the center must be (*rt*, *r*) and the horizontal component of the equation must be $x=r*\cos(-u-\pi/2) + r*u$. Finally, we will have the basic cycloid even if the radius of generator circle is different from 1 as seen again like in Figure 4.

Abstraction of the formal equation

So far, we reached the following parametric equation by writing with a little algebraic arrangement as follows:

$$\begin{aligned} x &= r(u + \cos(-(u+\pi/2))) \\ y &= r(1 + \sin(-(u+\pi/2))) \end{aligned} \quad (1)$$

Since cosine and sine functions are even and odd functions respectively, the Equation (1) can be revised as thus:

$$\begin{aligned} x &= r(u + \cos(u+\pi/2)) \\ y &= r(1 - \sin(u+\pi/2)) \end{aligned} \quad (2)$$

Now, consider the unit circle to make a revision on sine and cosine function of the Equation (2) as seen in Figure 6. Note that, the length of the line segment AC can be stated as $\cos(u+\pi/2)$ or $\sin(u)$. Because of the sign of the cosine function in the second quadrant, it must be multiplied by the negative unit.

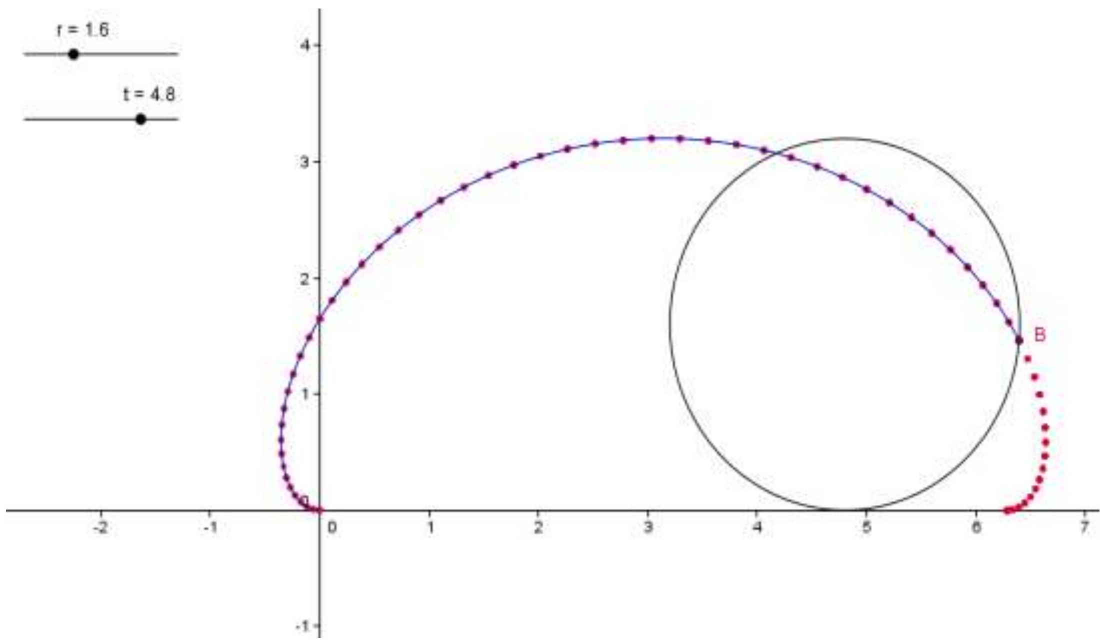

Figure 5. Effect of the radius to the curve.

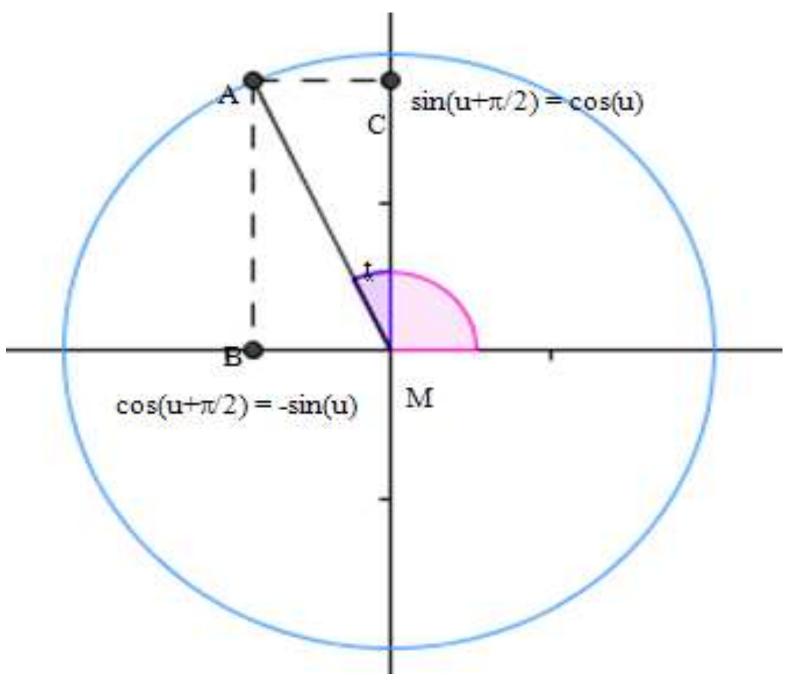

Figure 6. Analyzing the trigonometric arrangement.

Furthermore, the length of the line segment AB can be stated as sin ($u+\pi/2$) or cos (u).

After this last revision, you will have the parametric Equation (3), which is completely same with the formal equation of the cycloid:

$$x = r(u - \sin(u))$$
$$y = r(1 - \cos(u)) \tag{3}$$

It is expected that we completely understood the equation and visual construction of the cycloid. So, we can make desired manipulations to obtain other types of cycloids. Before proceeding, we should understand the Equation (3) in terms of visual effect. Both x and y components have two summative terms. When these terms are written as ordered pairs correspondingly, the first and second terms will be (ru, r) and ($-r\sin(u)$, $-r\cos(u)$), respectively. The ordered pair (ru, r) controls the generator circle's center, so the parameter r means the circle's radius and the ordered pair

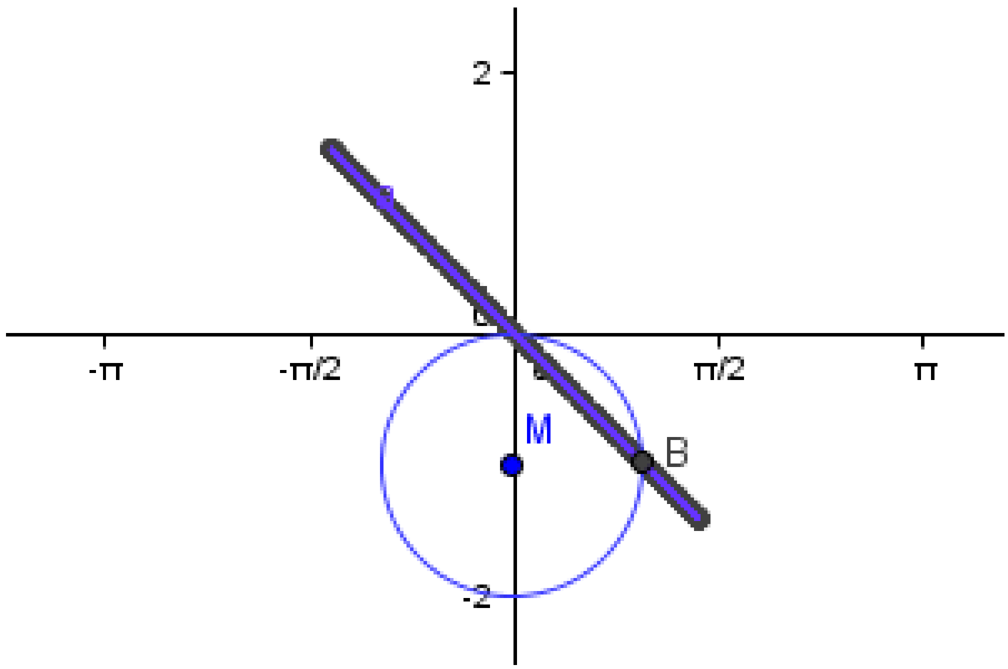

Figure 7. Strange epicycloid.

($-r\sin(u)$, $-r\cos(u)$) controls the rotation of the generator point on the generator circle.

Hypocycloid and epicycloids

If the generator circle is rotated on or under another circle, the curve obtained this way is named as Epicycloids and Hypocycloid respectively. This means that the center of the generator circle must make a circular movement. Since the ordered pair (ru, r) controls the linear translation of the center, we need to think on the arrangement on that ordered pair to make the movement circular. As well known, ($r\cos(u)$, $r\sin(u)$) creates a circle centered at origin. Therefore, the coordinates of the generator circle's center should be ($r\cos(t)$, $r\sin(t)$) and the corresponding parametric equation should be as follows:

$$x = r(\cos(u) - \sin(u))$$
$$y = r(\sin(u) - \cos(u)) \quad (4)$$

After this arrangement, the following curve is obtained both visually and mathematically. This however, looks strange. After thinking on the role of summative terms in the equation, it can be easily concluded that the center's rotation radius and the generator circle's radius are equal. So the generator circle is turning around just a point which is (0, 0) (Figure 7).

We need to create a new slider to control the radius of orbit of the generator circle. Let's name it as R and rearrange the generator circle's coordinates as ($R\cos(t)$, $R\sin(t)$).

Correspondingly, the parametric equation must be {$R\cos(u)-r\sin(u)$, $R\sin(u)-r\cos(u)$}. This arrangement also does not produce a regular epicycloids or hypocycloid (Figure 8).

To understand the situation clearly, let's display the circles under and on the generator circle. We can create the circles centered at origin and with radiuses R-r and R+r with the commands Circle [(0, 0), R-r] and Circle[(0,0), R+r] respectively. This provides us to see that we need to re-edit the rotation speed of the generator circle as R/r times of the rotation speed of its center. So, we need to re-write the command controlling the generator circle's rotation as Rotate [A,-t*R/r,M], while the parametric equation is re-edited as follows:

$$x = R\cos(t) - r\sin(Rt/r)$$
$$y = R\sin(t) - r\cos(Rt/r) \quad (5)$$

After this editing, the following curve, which is oblique according to a regular hypocycloid, is obtained as seen in Figure 9. To obtain the regular hypocycloid, which is named as Asteroid for the R=3 and r=1, it is enough to drag the point A as 90° on the generator circle in a counterclockwise direction.

Correspondingly the equation must be changed as follows to produce the regular asteroid curve (Figure 10):

$$x = R\cos(t) - r\sin(R(t+\pi/2)/r)$$
$$y = R\sin(t) - r\cos(R(t+\pi/2)/r) \quad (6)$$

After a trigonometric arrangement the asteroid curve's parametric equation can be stated as follows:

$$x = R\cos(t) + r\cos(Rt/r)$$
$$y = R\sin(t) - r\sin(Rt/r) \quad (7)$$

Finally, to obtain a hypocycloid, we need to make the generator circle, rolling on a circle. To give this effect to the generator circle it is enough to change the rotation direction, shown in Figure 11. That is, we need to re-state command as Rotate[A,t*R/r,M], which produces the generator point, by removing the minus sign.

Consequently, the parametric equation should be restated with required trigonometric arrangements also:

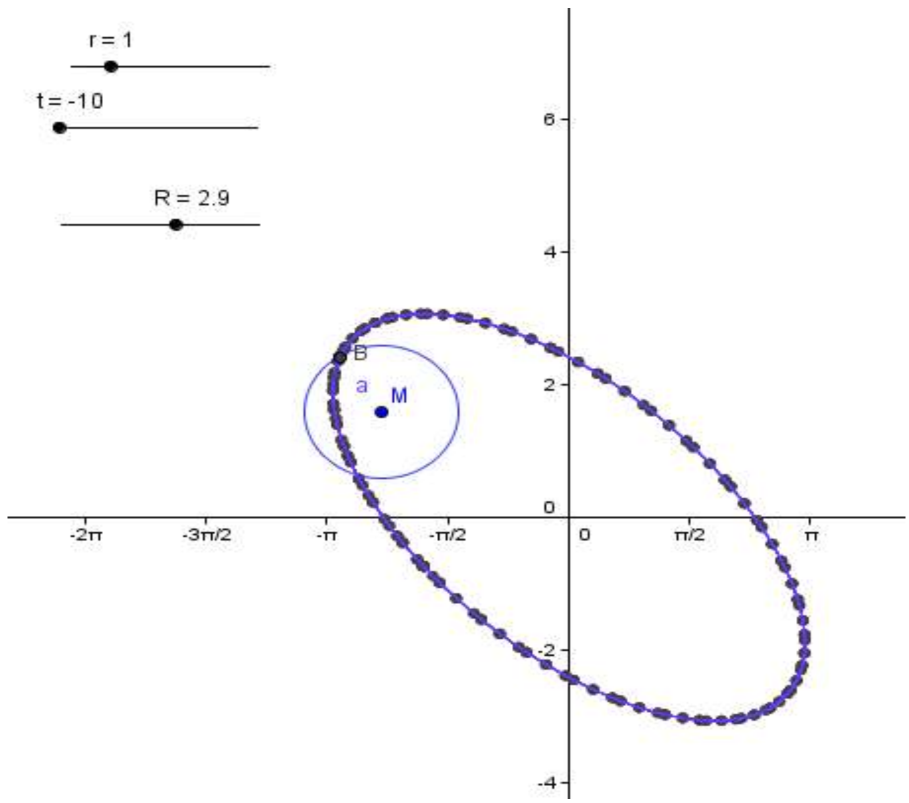

Figure 8. Antother strange epicycloid.

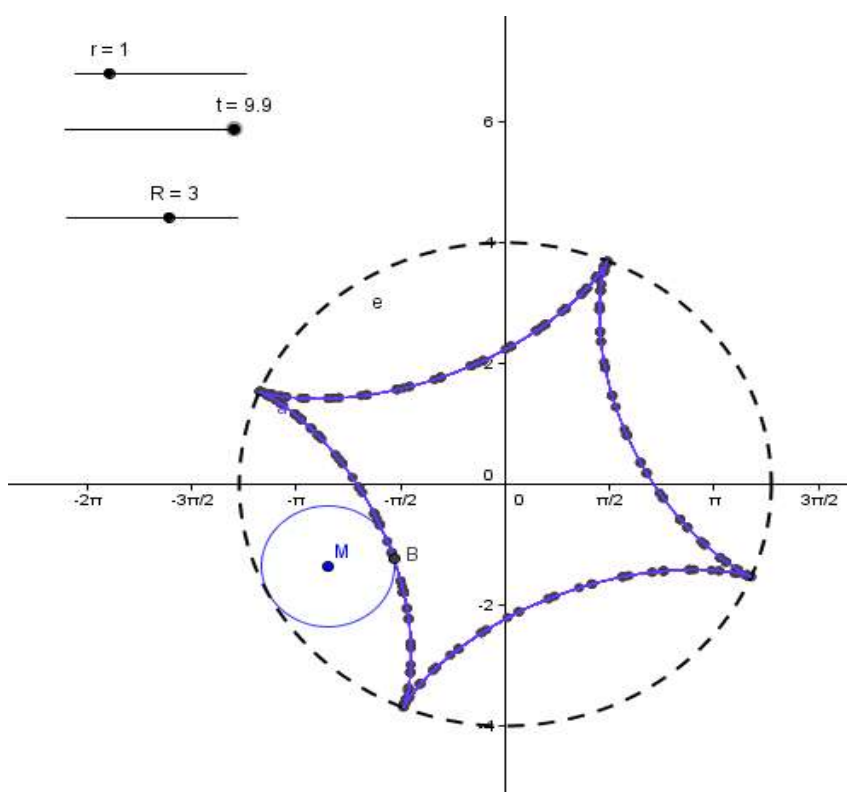

Figure 9. Oblique hypocycloid.

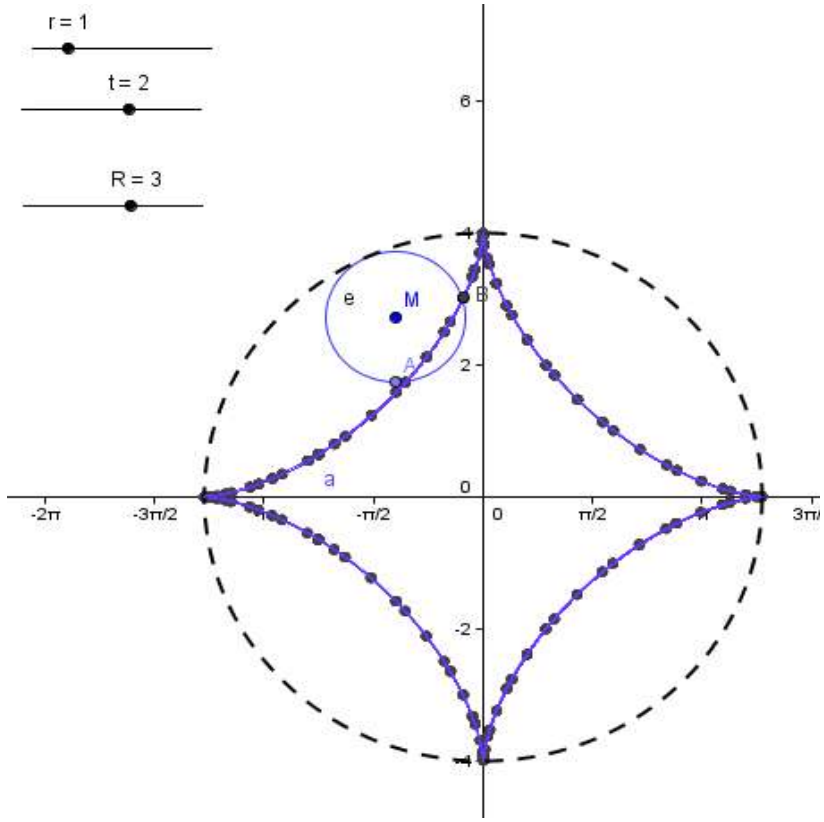

Figure 10. Regular asteroid.

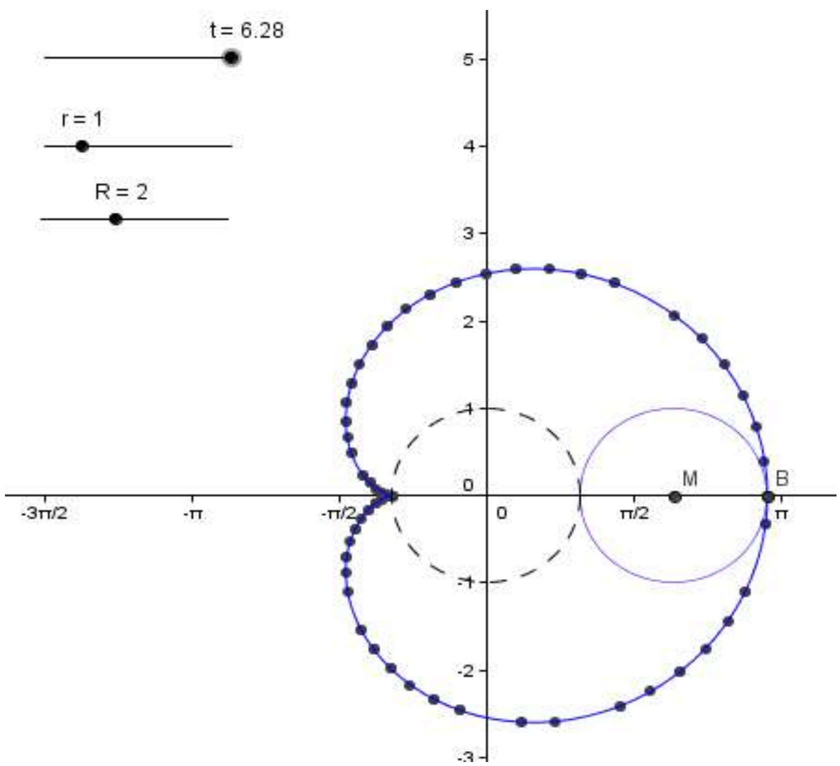

Figure 11. Epicycloid (Cardioid for R=2 and r=1).

$$x = R\cos(t) + r\cos\left(Rt/r\right)$$
$$y = R\sin(t) + r\sin\left(Rt/r\right) \quad (8)$$

RESULTS AND DISCUSSION

In view of the expressive modeling approach, given the task of "*constructing the cycloid curve without using its formal equation"*, the students may be more thorough in this, than presenting a ready cycloid graph, even if the ready graph is also dynamic. Students may encounter some problems, most of which are mathematical, while designing the task and in the process of solving them the students may have several learning opportunities for some special cases that are sometimes unexpected.

Somehow, we can construct the dynamic cycloid curve with any dynamic geometry software. But, thanks to an eligible synthesis of algebra and geometry in GeoGebra, we can assess an innovative abstraction processes even for advanced mathematics concepts. The proposal in this paper is just one of the cases.

In summarizing teaching and learning possibilities, which are presented by this case, we have seen that the Cartesian coordinates is not sufficient to define the point, moving on the circle. This is also an opportunity to point out that we need an alternative definition of a point on the coordinate system like parametric definition. Thanks to the parametric definition of a point as an ordered paired. We can use a parameter to define different manipulations like rotation and translation. Especially, easy construction of the equation of the cycloid along *y*-axis with parametric equation is a good additional example of why we need parametric equations rather than Cartesian, because, this vertical cycloid is not actually a function y or x. It is more than one y correspond for all *x* in its domain.

The epicycloids and hypocycloid curves' parametric equations are given as {(R-r)cosθ+rcos((R-r)θ/r), (R-r)sinθ-rsin((R-r)θ/r)} and {(R+r)cosθ-rcos((R+r)θ/r), (R+r)sinθ-rsin((R+r)θ/r)} respectively while *R* represents the base circle radius and *r* represents the generator circle radius (url, 2012). It can be easily seen that these are completely same with the Equations (7) and (8) by taking into account that the *R* and *r* represent the radius of the orbit of the generator circle center and the radius of the generator circle and some trigonometric identities.

An additional learning opportunity, which is appearing in the construction process of the cycloid model, is supporting the comprehension of the radian unit, which is a way of measuring an angle. Radian is the measure of an angle *M*, which is corresponding to the length of circular arc of a unit circle, whose center is *M*. When you have to be sure that the circular distance of the point rotating on the circle and the horizontal distance of the movement of the circle is equal, you will have a chance to strengthen the meaning of the radian in your mind.

This study also showed that trying to explain even an advanced mathematical concept by using technology and multiple representations approach can provide a chance to understand the connections as it is stated in the literature (Even, 1998; Hiebert and Carpenter, 1992). Furthermore, the teaching model is an operable example of the suggestions of Duval (1998) and Presmeg (1986) about visualization and the use of dynamic tools.

Besides using GeoGebra as an expressive modeling tool, it has a great potential of reflecting the iterative and cyclic view of modeling (Doerr and Pratt, 2008). While the modeling process is defined as an examination of the relationship between a model and a real world experiment, modeling has two different epistemological perspectives. First, it is separate from the world to be modeled; second, modeling is a cyclic process. The cyclic process view of a model refers to co-construction of the real world and model world by the ways in which models are projected back into the real world. By this point of view, the formal equations of the curves can be seen as a mathematical real world. The capability of enlightening the relationship between algebraic representation and geometric representation of GeoGebra allows us to discover the anatomy of the parametric equation of the cycloid as Edwards and Jones (2006) also determined; and Gorgorio and Jones (1996) determined the same approach by the use of Cabri-Geometry. There is only one parameter, named as *t*, used to define both *x* and *y* components in the cycloid equation. Our struggle of controlling two different movements by a unique slider on GeoGebra reveals that parameter *t* has two different meaning. Decoding this meaning of the equation allows us to create new kinds of cycloids. Accordingly, the cycloid model, modeled by GeoGebra with its algebraic capabilities, has constructed the cycloids' equation in mathematical real world. This construction can be advanced to the other dynamic curves like epicycloids, hypocycloid or asteroid by creative manipulations on coordinates of the center of generating circle, and the rotation of the point on this circle.

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