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A method for the solution of fractional differential equations using generalized Mittag-Leffler function

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This paper deals with the approximate and analytical solutions of non linear fractional differential equations namely, Lorenz System of Fractional Order and the obtained results are compared with the results of Homotopy Perturbation method and Variational Iteration method in the standard integer order form. The reason for using fractional order differential equations is that, fractional order differential equations are naturally related to systems with memory which exists in most systems and also they are closely related to fractals which are abundant in systems. The derived results are more general in nature. The solution of such equations spread at a faster rate than the classical differential equations and may exhibit asymmetry. A few numerical methods for the solution of fractional differential equation models have been discussed in the literature. However many of such methods are used for very specific types of differential equations, often just linear equations or even smaller classes, but this method shows the high accuracy and efficiency of the approach. Special cases involving the Mittag-Leffler function and exponential function are also considered.

Keywords: *G*eneralized Mittag-Leffler function, Caputo fractional derivative, Lorenz system.

AMS 2010 Subject Classification: 26A33, 33E12.

INTRODUCTION AND MATHEMATICAL PRELIMINARIES

It has been shown by that time fractional derivatives are equivalent to infinitesimal generators of generalized time fractional evolutions arising in the transition from microscopic to macroscopic time scales (Hilfer, 2002; 2003). Hilfer (2000) showed that this transition from ordinary time derivative to fractional time derivative indeed arises in physical problems. The idea on time fractional evolution is presented in detail (Klafter et al., (2011), and in some recent papers on Hilfer-composite time fractional derivative. The connection between the results obtained by solving fractional diffusion and fractional Fokker-Planck equations with those obtained from the continuous time random walk theory is another example of the importance of fractional derivatives (Metzler et al., 1999; Metzler and Klafter, 2000). The time-fractional diffusion-wave equation is obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α ($0<\alpha<2$). Using the similarity method and the method of the Laplace transform, it is shown that the scale-invariant solutions of the mixed problem of signaling type for the time-fractional diffusion-wave

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equation are given in terms of the Wright function in the case 0<α<1 and in terms of the generalized Wright function in the case 1<α<2. The reduced equation for the scale-invariant solutions is given in terms of the Caputo-type modification of the Erdélyi-Kober fractional differential operator (Gorenflo et al., 2000).

Mainardi and Gorenflo (2007) studied the basic theory of relaxation processes governed by linear differential equations of fractional order. The fractional derivatives are intended both in the Riemann-Liouville sense and in the Caputo sense. After giving a necessary outline of the classical theory of linear viscoelasticity, Mainardi and Gorenflo (2007) contrast these two types of fractional derivatives in their ability to take into account initial conditions in the constitutive equations of fractional order and also provide historical notes on the origins of the Caputo derivative and on the use of fractional calculus in viscoelasticity. Shawagfeh (2002), has been considered a class of nonlinear fractional differential equations based on the Caputo fractional derivative and by extending the application of the Adomian decomposition method, the author derived an analytical solution in the form of a series with easily computable terms. For linear equations the method gives exact solution, and for nonlinear equations it provides an approximate solution with good accuracy. Several examples are also mentioned. Some Routh-Hurwitz stability conditions are generalized to the fractional order case. The results agree with those obtained numerically for Lorenz, Rössler, Chua and Chen fractional order equations. The case of coupled map lattice is briefly discussed by Ahmed et al. (2006).

The fractional order state equations are developed to predict the defects of feedback intended to reduce motion in damped structure. The mechanical properties of damping materials are modeled using fractional order time derivatives of stress and strain. These models accurately describe the broadband defects of material damping in the structure's equations of motion. The resulting structural equations of motion are used to derive the fractional order state equations. Substantial difference between the structural and state equation are known to exist. The mathematical form of state equations suggests the feedback of fractional order time derivatives of structural displacements to improve control system performance. Several other advantages of the fractional order state formulation are also discussed by Bagley and Calico (1991).

There are several approaches to the generalization of the notation of differentiation to fractional order such as, The Riemann-Liouville; Grundwald-Letnikov; Caputo and Generalized function approach. The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real-world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the Grunwald-Letnikov formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms. Here we mention the basic definition of the Caputo fractional order differentiation, which is used in this paper and play the most important role in the theory of differential and integral equations of fractional order. The main advantage of Caputo approach are the initial conditions for fractional differential equations with the Caputo derivatives taking on the same form as for integer order differential equations (Rida and Arafa (2011).

Mittag-Leffler functions occur naturally in the solution of fractional order differential and integral equations (Wiman, 1905). The various Mittag-Leffler functions (Mittag-Leffler, 1903; Wiman, 1905) discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of such functions in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power law). Currently, more and more of such phenomena are discovered and studied. It is particularly important for the disciplines of stochastic systems, dynamical systems theory and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly non-equilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This non-equilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion and so forth, and may be governed by fractional calculus. Right now, fractional calculus and generalization of Mittag-Leffler functions are very important in research in physics.

These presentations make the reader familiar with the present trend of research in Mittag-Leffler type functions and their applications. Special functions have contributed a lot to mathematical physics and its various branches. The great use of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of astrophysics/physics.

Throughout this paper, we need the following definitions:

The Swedish mathematician Mittag-Leffler (1903) introduced the function $E_\omega(x)$ and defined as:

$$E_\omega(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\omega n + 1)}, \quad (x, \omega \in C, R(\omega) > 0) \tag{1}$$

A generalization of $E_\omega(x)$ was studied by Wiman (1905) where he defined the function $E_{\omega,\sigma}(x)$ as

$$E_{\omega,\sigma}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\omega n+\sigma)}, \tag{2}$$

Where $x,\omega,\sigma\in C, R(\omega)>0, R(\sigma)>0,$ which is also known as Mittag-Leffler function or Wiman's function.

The fractional derivative of $f(x)$ in the Caputo sense of order α is defined by:

$$D^{\omega} f(x)=J^{m-\omega}D^{m} f(x)=\frac{1}{\Gamma(m-\omega)}\int_{o}^{x}(x-t)^{m-\omega+1} f^{(m)}(t)dt \tag{3}$$

for $m-1<\omega\le m, m\in N, x>0.$

We have $D^{\omega}c=0,$ where c is constant:

$$D^{\omega}t^{n}=\begin{cases}0, & n\le\omega-1\\ \dfrac{\Gamma(n+1)}{\Gamma(n-\omega+1)}t^{n-\omega}, & n>\omega-1\end{cases} \tag{4}$$

For m to be the smallest integer that exceeds ω, the Caputo fractional derivatives of order $\omega>0$ is given as

$$D^{\omega}u(x,t)=\frac{\partial^{\omega}}{\partial t^{\omega}}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(m-\omega)}\int_{0}^{t}(t-\tau)^{m-\omega+1}\dfrac{\partial^{m}}{\partial\tau^{m}}u(x,\tau)d\tau, & m-1<\omega\le m,\\ \dfrac{\partial^{m}}{\partial t^{m}}u(x,t), & \omega=m\in N.\end{cases} \tag{5}$$

Amongst large number of Chaotic systems, it is obvious that the Lorenz model is the most classical and paradigmatic problem because it was the first model of Chaotic behavior. The Lorenz model is a simplified form of a previous more complicated model of Saltsman to describe buoyancy driven convection patterns in the classical rectangular Rayleigh-Benard problem applied to the thermal convection between two plates perpendicular to the direction of the earth's gravitational force (Sparrow, 1982; Lorenz, 1963; Argyris et al.,1994; Viana, 2000; Barrio and Serrano, 2007).

The component of the basic three component model are proportional to the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow is respectively denoted by $x(t), y(t), z(t).$ The famous Lorenz equations (Merdan,2009) are given as:

$$\frac{dx}{dt}=s(y-x),\ \frac{dy}{dt}=rx-y-xz,\ \frac{dz}{dt}=xy-bz. \tag{6}$$

with the initial conditions

$$x(0)=\varepsilon_1, y(0)=\varepsilon_2, z(0)=\varepsilon_3. \tag{7}$$

where s,b are real constants and r so- called bifurcation parameter.

Arafa et al. (2013) introduced the generalized chaotic dynamical system (Lorenz system) which is described by the following equations:

$$D^{\omega_1}x=s(y-z),\ D^{\omega_2}y=rx-y-xz,\ D^{\omega_3}z=xy-bz. \tag{8}$$

where $0<\omega_1,\omega_2,\omega_3\le 1$ and $D^{\omega_i}, i=1,2,3$ is the derivative of order ω_i in the sense of Caputo with the initial conditions

$$x(0)=\varepsilon_1, y(0)=\varepsilon_2, z(0)=\varepsilon_3. \tag{9}$$

and investigated the solution of this system of fractional order by using the generalized Mittag-Leffler function method and in this method, Mittag-Leffler function defined by Mittag-Leffler (1903) is used.

The reason of using fractional differential equations is that the fractional calculus approach provides a powerful tool for the description of memory and hereditary properties of various materials and processes (El-Sayed et al., 2009, 2010; Hairer and Wanner, 1991; Oldham and Spanier, 1974; Miller and Ross, 1993; Raida et al., 2010; Gejji and Jafari, 2007). It has been applied to many fields of Science and Engineering such as Viscoelasticity, Anomalous Diffusion, Fluid Mechanics, Biology, Chemistry, Acoustics, and Control Theory etc. The motivation of this paper is the use of the generalized Mittag-Leffler function introduced by Wiman (1905), which is an extension of the Mittag-Leffler function (Mittag-Leffler,1903), to solve the above mentioned system given by Equations (20) and (21) below. The same method was used by Rida et al. (2010) and Arafa et al. (2013) respectively for solving linear and non linear fractional differential equations.

In this paper, we will investigate the solution of nonlinear fractional differential equations through the imposition of the generalized Mittag-Leffler function introduced by Wiman (1905). Applied method suggests that $y_i(t), i=1,2,,\ldots$ are decomposed by an infinite series of components (Rida and Arafa, 2011):

$$y_i(t) = E_\omega(a_i t^\omega) = \sum_{n=0}^{\infty} a_i^n \frac{t^{n\omega}}{\Gamma(\omega n+1)}, \quad i = 1,2,3,\ldots \tag{10}$$

and

$$D^\omega y_i(t) = D^\omega E_\omega(a_i t^\omega) = \sum_{n=1}^{\infty} a_i^n \frac{t^{(n-1)\omega}}{\Gamma[\omega(n-1)+1]}, \ i = 1,2,3,\ldots \tag{11}$$

The author extends the same technique to define the following infinite series of components:

$$y_i(t) = E_{\omega,\sigma}(a_i t^\omega) = \sum_{n=0}^{\infty} a_i^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)}, \quad i = 1,2,3,\ldots \tag{12}$$

and

$$D^\omega y_i(t) = D^\omega E_{\omega,\sigma}(a_i t^\omega) = \sum_{n=1}^{\infty} a_i^n \frac{t^{(n-1)\omega}}{\Gamma[\omega(n-1)+\sigma]}, \ i = 1,2,3,\ldots \tag{13}$$

if we take $\sigma = 1$ in Equations (12) and (13), when reduced to the Equations (10) and (11) respectively after using the relation

$$E_{\omega,1}(x) = E_\omega(x) \ . \tag{14}$$

RESULTS

Here, the author applied the above said method for solving the fractional differential equations. From Equation (12), we have

$$x(t) = E_{\omega,\sigma}(at^\omega) = \sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)}, \ y(t) = E_{\omega,\sigma}(dt^\omega) = \sum_{n=0}^{\infty} d^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)},$$
$$z(t) = E_{\omega,\beta}(lt^\omega) = \sum_{n=0}^{\infty} l^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)}, \tag{15}$$

By adding Equation (13) and (15) into Equation (8) and taking $\omega_1 = \omega_2 = \omega_3 = \omega$, we get

$$\sum_{n=1}^{\infty} a^n \frac{t^{(n-1)\omega}}{\Gamma[\omega(n-1)+\sigma]} - s\sum_{n=0}^{\infty} d^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + s\sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0,$$
$$\sum_{n=1}^{\infty} d^n \frac{t^{(n-1)\omega}}{\Gamma[\omega(n-1)+\sigma]} - r\sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + \sum_{n=0}^{\infty} d^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + \sum_{n=0}^{\infty} c_1^{\,n} t^{n\omega} = 0,$$
$$\sum_{n=1}^{\infty} l^n \frac{t^{(n-1)\omega}}{\Gamma[\omega(n-1)+\sigma]} - \sum_{n=0}^{\infty} c_2^{\,n} t^{n\omega} + b\sum_{n=0}^{\infty} l^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0 \tag{16}$$

Combining the like terms and replacing n by $(n+1)$ in the first sum, we obtain:

$$\sum_{n=0}^{\infty} a^{n+1} \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} - s\sum_{n=0}^{\infty} d^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + s\sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0,$$
$$\sum_{n=0}^{\infty} d^{n+1} \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} - r\sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + \sum_{n=0}^{\infty} d^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} + \sum_{n=0}^{\infty} c_1^{\,n} t^{n\omega} = 0,$$
$$\sum_{n=0}^{\infty} l^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} - \sum_{n=0}^{\infty} c_2^{\,n} t^{n\omega} + b\sum_{n=0}^{\infty} l^n \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0, \tag{17}$$

where

$$c_1^{\,n} = \sum_{k=0}^{\infty} \frac{a^k l^{n-k}}{\Gamma[\omega k+\sigma]\Gamma[\omega(n-k)+\sigma]}$$
$$c_2^{\,n} = \sum_{k=0}^{\infty} \frac{a^k d^{n-k}}{\Gamma[\omega k+\sigma]\Gamma[\omega(n-k)+\sigma]}. \tag{18}$$

We have

$$\sum_{n=0}^{\infty} (a^{n+1} - sd^n + sa^n) \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0,$$
$$\sum_{n=0}^{\infty} (d^{n+1} - ra^n + d^n + c_1^{\,n}\Gamma[\omega n+\sigma]) \frac{t^{n\alpha}}{\Gamma[\omega n+\sigma]} = 0,$$
$$\sum_{n=0}^{\infty} (l^{n+1} - c_2^{\,n}\Gamma[\omega n+1] + bl^n) \frac{t^{n\omega}}{\Gamma[\omega n+\sigma]} = 0. \tag{19}$$

Equating the coefficient of $t^{n\alpha}$ to zero, we arrive at

$$a^{n+1} - sd^n + sa^n = 0,$$
$$d^{n+1} - ra^n + d^n + c_1^{\,n}\Gamma[\omega n+\sigma] = 0,$$
$$l^{n+1} - c_2^{\,n}\Gamma[\omega n+\sigma] + bl^n = 0. \tag{20}$$

Initially assuming

$$a^0 = \varepsilon_1, d^0 = \varepsilon_2, \ l^0 = \varepsilon_3. \tag{21}$$

If we set $n = 0$ in Equation (20), we get

$$a^1 = s(\varepsilon_2 - \varepsilon_1),$$
$$d^1 = r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3,$$
$$l^1 = \varepsilon_1\varepsilon_2 - b\varepsilon_3. \tag{22}$$

If we put $n = 1$ in Equation (20), we get

$$a^2 = s(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1),$$
$$d^2 = rs(\varepsilon_2 - \varepsilon_1) - r\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_3 - \varepsilon_1^{\,2}\varepsilon_2 + b\varepsilon_3\varepsilon_1 - s\varepsilon_2\varepsilon_3 + s\varepsilon_3\varepsilon_1,$$
$$l^2 = r\varepsilon_1^{\,2} - \varepsilon_1\varepsilon_2 - \varepsilon_1^{\,2}\varepsilon_3 + s\varepsilon_2^{\,2} - s\varepsilon_1\varepsilon_2\varepsilon_1 - b\varepsilon_1\varepsilon_2 + b^2\varepsilon_3. \tag{23}$$

If we put $n=2$ in Equation (20), we get

$$a^3 = r s^2(\varepsilon_2 - \varepsilon_1) - rs\varepsilon_1 + s\varepsilon_1^2\varepsilon_2 + sb\varepsilon_1\varepsilon_3 - s^2\varepsilon_2\varepsilon_3 + s^2\varepsilon_3\varepsilon_1 - s^2 r\varepsilon_1 + s^2\varepsilon_2 + s^2\varepsilon_1\varepsilon_3 + s^3\varepsilon_2 - s^3\varepsilon_1,$$

$$d^3 = rs(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1) - rs(\varepsilon_2 - \varepsilon_1) + r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 + \varepsilon_1^2\varepsilon_2 - b\varepsilon_1\varepsilon_3 + s\varepsilon_2\varepsilon_3 - s\varepsilon_1\varepsilon_3 - r\varepsilon_1^3 + \varepsilon_1^2\varepsilon_2 + \varepsilon_1^3\varepsilon_3 - s\varepsilon_1\varepsilon_2^2 + s\varepsilon_1^2\varepsilon_2 + b\varepsilon_1^2 - b^2\varepsilon_1\varepsilon_3 - s\varepsilon_3(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1) - (s\varepsilon_2 - s\varepsilon_1)(\varepsilon_1\varepsilon_2 - b\varepsilon_3)\frac{\Gamma[2\omega+\sigma]}{\Gamma[\omega+\sigma]},$$

$$l^3 = rs\varepsilon_1(\varepsilon_2 - \varepsilon_1) - r\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1^2\varepsilon_3 - \varepsilon_1(\varepsilon_1^2\varepsilon_2 - b\varepsilon_1\varepsilon_3) - s\varepsilon_1\varepsilon_2\varepsilon_3 + s\varepsilon_1^2\varepsilon_3 + s\varepsilon_2(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1) + s(\varepsilon_2 - \varepsilon_1)(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3)\frac{\Gamma[2\omega+\sigma]}{\Gamma[\omega+\sigma]^2} - br\varepsilon_1^2 + b\varepsilon_1\varepsilon_2 + b\varepsilon_1^2\varepsilon_3 - sb\varepsilon_2^2 + sb\varepsilon_1\varepsilon_2 + b^2\varepsilon_1\varepsilon_2 - b^3\varepsilon_3. \quad (24)$$

and so on.

Now

$$x(t) = E_{\omega,\sigma}(at^{\omega}) = \sum_{n=0}^{\infty} a^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)} = a^0 + a^1\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + a^2\frac{t^{2\omega}}{\Gamma(2\omega+\sigma)} + a^3\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots..$$

$$y(t) = E_{\omega,\sigma}(dt^{\omega}) = \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(\omega n+\sigma)} = d^0 + d^1\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + d^2\frac{t^{2\omega}}{\Gamma(2\omega+\sigma)} + d^3\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots..$$

$$z(t) = E_{\omega,\sigma}(lt^{\alpha}) = \sum_{n=0}^{\infty} l^n \frac{t^{n\omega}}{\Gamma(\omega n+\sigma)} = l^0 + l^1\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + l^2\frac{t^{2\omega}}{\Gamma(2\omega+\sigma)} + l^3\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots.. \quad (25)$$

Hence

$$x(t) = \varepsilon_1 + s(\varepsilon_2 - \varepsilon_1)\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + s(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1)\frac{t^{2\omega}}{\Gamma(2\omega+\sigma)} + [r s^2(\varepsilon_2 - \varepsilon_1) - rs\varepsilon_1 + s\varepsilon_1^2\varepsilon_2 + sb\varepsilon_1\varepsilon_3 - s^2\varepsilon_2\varepsilon_3 + s^2\varepsilon_3\varepsilon_1 - s^2 r\varepsilon_1 + s^2\varepsilon_2 + s^2\varepsilon_1\varepsilon_3 + s^3\varepsilon_2 - s^3\varepsilon_1]\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots.$$

$$y(t) = \varepsilon_2 + [r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3]\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + [rs(\varepsilon_2 - \varepsilon_1) - r\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_3 - \varepsilon_1^2\varepsilon_2 + b\varepsilon_3\varepsilon_1 - s\varepsilon_2\varepsilon_3 + s\varepsilon_3\varepsilon_1]\frac{t^{2\omega}}{\Gamma(2\omega+\sigma)}$$

$$+[rs(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1) - rs(\varepsilon_2 - \varepsilon_1) + r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon + \varepsilon_1^2\varepsilon_2 - b\varepsilon_1\varepsilon_3 + s\varepsilon_2\varepsilon_3 - s\varepsilon_1\varepsilon_3 - r\varepsilon_1^3 + \varepsilon_1^2\varepsilon_2 + \varepsilon_1^3\varepsilon_3 - s\varepsilon_1\varepsilon_2^2 + s\varepsilon_1^2\varepsilon_2 + b\varepsilon_1^2 - b^2\varepsilon_1\varepsilon_3 - s\varepsilon_3(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1)$$

$$-(s\varepsilon_2 - s\varepsilon_1)(\varepsilon_1\varepsilon_2 - b\varepsilon_3)\frac{\Gamma[2\omega+\sigma]}{\Gamma[\omega+\sigma]}]\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots\ldots$$

$$z(t) = \varepsilon_3 + [\varepsilon_1\varepsilon_2 - b\varepsilon_3]\frac{t^{\omega}}{\Gamma(\omega+\sigma)} + [r\varepsilon_1^2 - \varepsilon_1\varepsilon_2 - \varepsilon_1^2\varepsilon_3 + s\varepsilon_2^2 - s\varepsilon_1\varepsilon_2\varepsilon_1 - b\varepsilon_1\varepsilon_2 + b^2\varepsilon_3]\frac{t^{2\omega}}{\Gamma(2\alpha+\sigma)}$$

$$+[rs\varepsilon_1(\varepsilon_2 - \varepsilon_1) - r\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1^2\varepsilon_3 - \varepsilon_1(\varepsilon_1^2\varepsilon_2 - b\varepsilon_1\varepsilon_3) - s\varepsilon_1\varepsilon_2\varepsilon_3 + s\varepsilon_1^2\varepsilon_3 + s\varepsilon_2(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3 - s\varepsilon_2 + s\varepsilon_1) + s(\varepsilon_2 - \varepsilon_1)(r\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_3)\frac{\Gamma[2\omega+\sigma]}{\Gamma[\omega+\sigma]^2}$$

$$-br\varepsilon_1^2 + b\varepsilon_1\varepsilon_2 + b\varepsilon_1^2\varepsilon_3 - sb\varepsilon_2^2 + sb\varepsilon_1\varepsilon_2 + b^2\varepsilon_1\varepsilon_2 - b^3\varepsilon_3]\frac{t^{3\omega}}{\Gamma(3\omega+\sigma)} + \ldots.. \quad (26)$$

SPECIAL CASES

From the main results, we can easily obtain the following:

(1) If we set $\sigma = 1$ in the above equations, it reduces to the results given recently by Arafa et al. (2013).

(2) If we set $\omega = \sigma = 1$ in Equation (26), then we have the same results with the results of Homotopy perturbation method and Variational iteration method for the solution of Lorenz system as seen in Merdan (2009).

Conclusion

In this survey, a new generalization of the generalized Mittag-Leffler function method has been developed to investigate the solution of nonlinear fractional differential equations that is, the Lorenz system, the new generalization is based on the Caputo fractional derivative, and the results was compared with the results of Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM), the results of HPM and VIM in the standard integer order form when $\omega = \sigma = 1$ in Equation (26). The new generalization is based on the Caputo fractional derivative. It may be concluded that this technique is very powerful and efficient for finding approximate solutions for large classes of nonlinear differential equations of fractional order.

Conflict of Interests

The author(s) have not declared any conflict of interests.

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