Norm properties of operators whose norms are Eigenvalues

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In this paper we present properties of a norm-attaining operator on a Hilbert space and there implications. We show that if \( T \) has norm attaining vector then

\[
\left( \sum_{k=0}^{n} \alpha_k T^k \right) x = \sum_{k=0}^{n} \alpha_k \|T\|^k
\]

where the scalars are nonnegative numbers. Thus \( T \) satisfies a generalized Daugavet condition.

Key words: Numerical range, eigenvalue, normalloid operator, Daugavet property.

INTRODUCTION

In this article we extend results obtained by C.S Lin on properties of an operator whose norm is an Eigen value. It is noteworthy that each compact operator on a Hilbert space (Halmos, 1974) has a norm-attaining vector (Shilov, 1965). Thus, these properties are characteristics of compact operators. We will denote operators on a Hilbert space by capital letters. The numerical range of an operator \( T \) is the convex set of complex numbers defined by

\[
W(T) = \{ (Tx, x) : \|x\| = 1, x \in H \}.
\]

We shall denote by \( T^* \) the adjoint of \( T \). We say that \( T \) satisfy the Daugavet (1963) equation if

\[
\|T\| = \|I + T\| - 1
\]

A unit vector \( x \) in \( H \) is a norm attaining vector for \( T \) if \( \|Tx\| = \|T\| \cdot \|x\| \).

Lin (2002) wrote a paper on a bounded operator on a Hilbert space whose norm is an eigenvalue and established the following theorem namely that

\[
1 + \|T\| = \|(I + T)x\|.
\]

\[
\|T\| \text{ is an eigenvalue of } T \text{ and } Tx = \|T\| \cdot x \text{ that is, } \|Tx\| = \|T\| \cdot \|x\|.
\]

\[
\|T\| \text{ is in the numerical range of } T \text{ that is, } \|T\| = (Tx, x).
\]

\[
x \text{ a complete vector for } T, \text{ that is, } \|T\| = (Tx, x) = \|Tx\|.
\]

\[
2 \|T\| \text{ is an eigenvalue of } T + T^* \text{, that is } (T + T^*)x = 2 \|T\| \cdot x.
\]

\[
\|T\| \text{ and } \|T^*\| \text{ are eigen-values of } T \text{ and } T^*T, \text{ respectively, with respect to } x, \text{ that is, } Tx = \|T\| \cdot x \text{ and } T^*Tx = \|T^*\|^2 x.
\]

\[
(1 + \|T\|)\|T\| \text{ is an eigen-value of } (I + T^*)T, \text{ that is, } (I + T^*)Tx = (1 + \|T\|) \cdot \|T\| \cdot x.
\]

\[
\|T\| \text{ is a normal eigenvalue for } T, \text{ that is, } Tx = \|T\| \cdot x = T^*x.
\]

\[
x \text{ a complete vector for } T \text{ and } T^*, \text{ that is, } Tx = (Tx, x) = \|Tx\| = \|T^*x\|.
\]

\[
x \text{ a complete vector for } T \text{ and }
\]
$T^*T$, that is, $\|T\| = (Tx,x) = \|Tx\|$ and $\|T^*\|^2 = \|Tx\|^2 = \|T^*Tx\|

12. $1 + \|T\|^2 = \|(I + T^*T)x\|$.

Now we make a natural extension of part 2 of theorem 1 as follows

**Lemma 2**

Let $x$ be an operator on a Hilbert space $H$ and $x$ be a unit vector in $H$. Then the following statements are equivalent;

(i) $x$ is an eigenvector of $T$ with eigenvalue $\|T\|$, that is, $Tx = \|T\| x$.

(ii) For any sequence $\alpha_1, \ldots, \alpha_n$ of positive numbers

$$\left\| \sum_{k=0}^{n} \alpha_k T^k \right\| = \sum_{k=0}^{n} \alpha_k \|T\|^k$$

**Proof.** $(i) \rightarrow (ii)$ If $\|T\|$ is an eigenvalue of $T$ then it follows that

$$\left\| \sum_{k=0}^{n} \alpha_k T^k \right\| = \left\| \sum_{k=0}^{n} \alpha_k (T^*T)^k \right\| x$$

$(ii) \rightarrow (i)$ Now set $\alpha_k = 0, \alpha_1 = 1, k \neq 1$ to obtain $\|Tx\| = \|T\|$.

Also from $\alpha_0 = \alpha_1 = 1, \alpha_k = 0, k > 1$ we obtain $1 + \|T\| = \|(I + T)x\|$. Hence $\| (I + T) x \|^2 = (1 + \|T\|)^2$.

Consequently, $\| (I + T) x \|^2 = ((I + T)x,(I + T)x) = (x,x) + (Tx,x) + (x,Tx) + (Tx,Tx)$

$$= 1 + (Tx,x) + (x,Tx) + \|T\|^2 = 1 + 2 \|T\| + \|T\|^2$$

This leads to the result $(Tx,x) + (x,Tx) = 2 \|T\|$. To show that $\|T\|$ is an eigenvalue of $T$ we consider the following expansion.

$$\left\| (Tx - \|T\| x) \right\|^2 = (Tx - \|T\| x,Tx - \|T\| x)$$

$$= (Tx,Tx) - \|T\|(Tx,x) - \|T\|(x,Tx) + \|T\|^2 (x,x)$$

$$= \|T\|^2 - \|T\|[(Tx,x) + (x,Tx)] + \|T\|^2$$

$$= \|T\|^2 - \|T\| + \|T\|^2 = 0$$

Hence $Tx - \|T\| x = 0 \iff Tx = \|T\| x$ and so $\|T\|$ is an eigenvalue of $T$.

In the same article Lin proved the theorem below which enumerates the properties of an operator $a$ with norm attaining vector

**Theorem 3**

If $T$ is an operator on a Hilbert space $H$ and $x$ is norm one vector in $H$ then the following are equivalent statements; any of the statements in theorem 1

(ii) $\left\| \sum_{k=0}^{n} \alpha_k T^k \right\| = \sum_{k=0}^{n} \alpha_k \|T\|^k$

**Proof**

It follows immediately from lemma 1

In the same article Lin proved the following

**Theorem 4**

Let $x$ be a unit vector. Then the following are equivalent.

1. $x$ is a norm attaining vector for $T$ that is, $\|T\| = \|Tx\|$.
2. $\|T\|^2$ is an eigenvalue for $T$, that is, $T^*Tx = \|T\|^2 x$.
3. $1 + \|T\|^2 = \|(I + T^*T)x\|$. 

4. x is a complete vector $T' T$, that is, $\|T\|^2 \leq \|(T' T)x\| = \|T x\|^2$

5. $\|T\|^2$ is an eigenvalue for $T' T$, and $T' Tx = \|(T' T)x\|$, that is, $T' Tx = \|T\|^2 x$ and $T' Tx = \|(T' T)x\| x$.

6. $\|T\|^2$ is in the numerical range of $T' T$, that is, $\|T\|^2 = ((T' T)x, x)$.

7. $2\|T\|^2$ is an eigenvalue for $T' T$, and $T' T x = \|T T x\|$, that is, $T' T x = \|T\|^2 x$ and $T' T x = \|(T' T)x\|$.

8. $2\|T\|^2$ is in the numerical range of $T' T$, that is, $2\|T\|^2 = ((T' T)x, x)$.

9. $1 + \|T\|^2 = \|I + T + T' T + (T' T)^2\|$, that is, $1 + \|T\|^2 = \|I + T + T' T + (T' T)^2\|$.

10. $x$ is a complete vector for $T' T$ and $(T' T)^2$, that is, $\|T\|^2 = \|T x\|^2$ and $\|T\|^2 = \|T' T x\|^2 = \|(T' T)^2 x\|$.

We now prove a general result to the above in the following lemma

**Lemma 5**

Let $T$ be an operator on a Hilbert space $H$ and let $x$ be a unit vector in $H$ then the following are equivalent statements:

(i) $X$ is an eigenvector of $T' T$ with Eigen value $\|T\|^2$ that is, $T' T x = \|T\|^2 x$

(ii) For any sequence $\alpha_1, \ldots, \alpha_n$ of positive numbers

$$\sum_{k=0}^{n} \alpha_k \|T\|^2 \leq \left\| \sum_{k=0}^{n} \alpha_k (T' T)^k \right\| x$$

**Proof**

If we replace $T$ with $T' T$ then we obtain the (i) if and only if

$$\sum_{k=0}^{n} \alpha_k \|T' T\| = \left( \sum_{k=0}^{n} \alpha_k (T' T)^k \right) x.$$

But we have that $\|T' T\| = \|T\|^2$. Hence we obtain the result.

**Theorem 6**

Let $T$ be an operator on a Hilbert space $H$ and $x$ be a unit vector then the following are equivalent statements;

Any statement in theorem 6

For any $\alpha_1, \ldots, \alpha_n$ positive numbers

$$\sum_{k=0}^{n} \alpha_k \|T\|^2 = \left\| \sum_{k=0}^{n} \alpha_k (T' T)^k \right\| x$$

**Proof**

The result follows from lemma

The following corollary which shows that if $\|T\|$ is an eigenvalue of $T$ with respect to $x$, then $x$ is a norm attaining vector for $T$ and satisfies the Daugavet property that is,

**Corollary 7**

Let $x$ be a unit vector. Then any statement in theorem 2 implies the following;

Any statement in theorem 4

$T$ satisfy the Daugavet equation, that is, $1 + \|T\| = \|(I + T)x\|$.

$T$ and $T'$ satisfy the generalized Daugavet equation $\|I + T + T'\| = 1 + \|T\|$.

$T$ and $T' T$ satisfy the generalized Daugavet equation $\|I + T + T' T\| = 1 + \|T\| + \|T\|^2$.

$T$ is a normaloid operator, that is, $r(T) = \|T\|$.

$X$ is a norm attaining vector for $I + T + T'$, that is, $\|I + T + T'\| = \|(I + T + T')x\|$.

$X$ is a norm attaining vector for $I + T + T' T$, that is, $\|I + T + T' T\| = \|(I + T + T' T)x\|$.

**Proof**

As in Lin with Theorem 1 and 2 now replaced with 3 and 4. We now consider further results when the operator $T$ is
both self adjoint and compact. In this case the operator has a norm attaining vector as shown by Shilov

**Theorem 8**

If T is a compact self adjoint operator then \( ||T^n|| = w(T^n) = (w(T))^n \).

**Proof**

Since T is self adjoint, \( T^n \) is also self adjoint and so the first equality follows from corollary 2.

Also, if \( x \) is the norm attaining vector for \( T \) we have:

\[
(T^n x; x) = (T^{n-1} x; T x) = ||T|| (T^{n-1} x; x) = ||T|| (T^{n-2} x; x) = ||T|| (w(T))^n.
\]

But we have \( w(T^n) \geq (T^n x; x) \). Consequently \( w(T^n) \geq (w(T))^n \).

For the reverse inequality we note that \( w(T^n) = ||T^n|| \leq ||T||^n = (w(T))^n \) corollary 9.

If \( T \) is a compact self-adjoint operator then

\[
w(\sum_{k=0}^{n} \alpha_k T^k) = \| \sum_{k=0}^{n} \alpha_k T^k \| = \sum_{k=0}^{n} \| \alpha_k T^k \| = \sum_{k=0}^{n} \alpha_k (w(T))^k
\]

Where \( \alpha_k \) are non negative numbers.

**REFERENCES**


