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Full Length Research Paper

A study on hypo hamiltonian graphs

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A graph is said to be Hamiltonian if it contains a spanning cycle. The spanning cycle is called a Hamiltonian cycle of G and G is said to be a Hamiltonian graph. A Hamiltonian path is a path that contains all the vertices in V(G) but does not return to the vertex in which it began. A graph G is said to be hypohamiltonian if for each v$\in$V (G), the vertex sub graph G-v is Hamiltonian. This paper shall prove that every hypohamiltonian graph G is Hamiltonian if we make the degree of removable vertex V exactly equal to n - 1, that is, $\delta_v = n-1$ and illustrate it by some counter examples.

Key words: Graphs vertex, Hamiltonian cycle, degree.

INTRODUCTION

A path p of a graph G is a hamiltonian path if "P" visits every vertex of G once. Similarly, a cycle "C" is a hamiltonian cycle if it visits each vertex once. A graph is hamiltonian if it has a hamiltonian cycle.

Note that if C: $U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow \ldots \rightarrow U_n$ is a hamiltonian cycle, and then so is $U_i \rightarrow \ldots U_n \rightarrow U_1 \rightarrow \ldots U_{i-1}$ for each i$\in$[1,n] and thus we can choose where to start a cycle. (Tero, 2011)

Hypohamiltonian graphs first appeared in the literature in response to a problem cited by Sousselier (1963). The solution, proposed by Gaudin et al. (1964), established that the Petersen graph is the smallest hypohamiltonian graph. Ever since Herz et al. (1966) have exhaustively searched computer to show that there are no hypohamiltonian graphs of 11 or 12 vertices. A later exhaustive search by Collier and Schmeichel (1978) showed that there is no hypohamiltonian graph of 14 vertices. In addition, Aldred et al. (1997) proved that there is no hypohamiltonian graph of 17 vertices. If n = 10, 13, 15, 16 and n ≥ 18, then there are hypohamiltonian graphs of order n (Doyen 1975; Thomassen, 1974).

Bondy (1972) found an infinite sequence of hypohamiltonian graphs of 12k+10 vertices. Chv´atal (1973) asked if planar hypohamiltonian graphs exist. An infinite family was subsequently found by Thomassen (1979, 1976; 1981), with the smallest among them having 105 vertices. Thomassen improved this to 94 vertices and Hatzel (1979) improved the lower bound to 57 vertices. Zamfirescu (2007) found a planar hypohamiltonian graph of 48 vertices; and Wiener and Araya (2009) found a planer hypohamiltonian graph of 42 vertices, known as Wiener-Araya graph.

Hypohamiltonain graphs arise in integer programming solutions to the traveling salesman's problem Charles (2005). Every hypohamiltonian graph must be 3-vertex connected as the removal of any two vertices leaves a hamiltonian path which is connected. There exists an n-vertex hypohamiltonian graph whose maximum degree is n/2 and has approximately $n^2/4$ edges.

Lemma 1

For every graph G of order n, $\Delta(G) \le n-1$

Proof

In graph a vertex can be joined to at most n – 1 of other vertices; therefore, the maximum degree of a vertex can have n – 1. So,

$$\Delta(G) \le n-1$$

Lemma 2

If a hypohamiltonian graph has a vertex V of degree3, then V lies on no triangles.

Definition

Let G be a non hamiltonian graph, with maximum degree D. Let top (G) denote any graph T obtainable by the following process:

(i) Set T = G,
(ii) Add to T every edge uv such that uv is not an edge of G, G + uv is non-hamiltonian u and v has degree < D in G. Note that T needs not be non-hamiltonian,
(iii) Repeat these any number of times you please; choose one vertex of degree 3 in T; delete from T any edges joining two of its neighbors.

Lemma 3

If H is a super graph of G, that is, hypohamiltonian and has maximum degree D, then It is a sub graph of top (G)

Proof

Clearly the value of the graph T after Step (ii) is a super graph of every non-hamiltonian super graph of G having maximum degree D. If after that vertex V has degree three in T, then the neighborhood of V in T must be the neighborhood of V in any hypohamiltonian graph between G and T (since hypohamiltonian graphs cannot have vertex of degree 2). In that case, Lemma 2 is violated unless we remove the edges between the neighbors of V. it must be noted that top (G) might not be a super graph of G because Step (iii) might remove some edges that are in G. The lemma still holds, implying that G has no hypohamiltonian super graphs of maximum degree D.

Theorem 4

According to Dirac's theorem, if G is a simple graph with n vertices where n$\geq$ 3 and $\partial(G) \geq \frac{n}{2}$ then G is hamiltonian.

Theorem 5

According to Ore's theorem, let G be a graph with n vertices and u, v be distinct nonadjacent vertices of G with d(u)+d(v) $\geq$n. Then G is hamiltonian if and only if G + (u,v) is hamiltonian.

Theorem 6

If G is simple graph with n vertices, then G is hamiltonian if and only if its closure is hamiltonian (Bondy-Chvata, 1972).

The hamiltonian closure of a graph G, denoted by C(G) is the super graph of G on V(G) obtained by iteratively adding edges between pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. Fortunately, the closure does not depend on the order in which we choose to add edges as more than one is available; that is, the closure of G is well defined.

Theorem 7

A graph G is hamiltonian if and only if CL(G) is hamiltonian.

RESULT

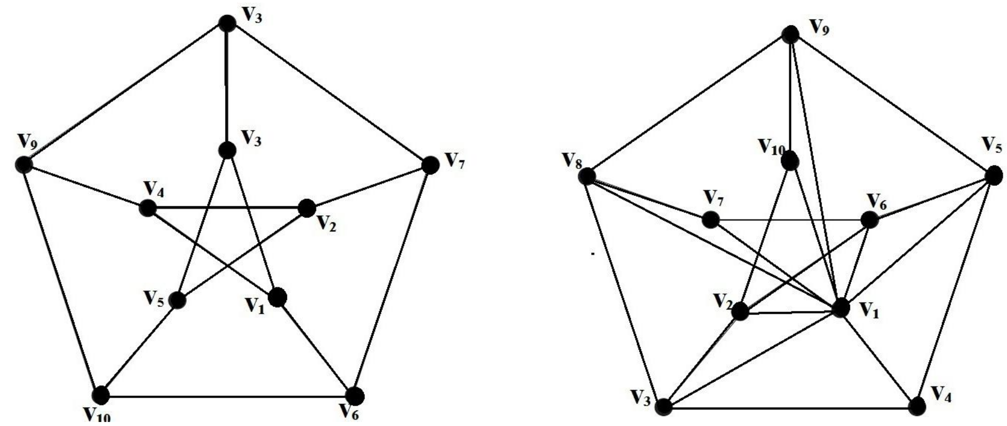
Theorem 8

Every hypohamiltonian graph G is hamiltonian if we make the degree of removable vertex V exactly equal to n – 1; that is, $\delta_v = n-1$ and illustrate it by some counter examples.

Proof

Let G be a hypohamiltonian graph of n vertices say v_1 v_2 v_3 . . . $v_{n;}$ let us suppose that by removing any vertex V_I from G, the graph so obtained is hamiltonian based on the definition of hypohamiltonian graphs. A graph G is said to be hypohamiltonian if G itself does not have a hamiltonian cycle, but every graph formed by removing a single vertex from G is hamiltonian. Since the hypohamiltonian graphs are connected if any vertex is deleted in G the graph so obtained is hamiltonian; but no hamiltonian graph is disconnected because it has a path to travel in every vertex once and reaches back to the starting vertex, which is not possible in disconnected graphs. Thus, we conclude that every hypohamiltonian graph is connected. It is known that the length of a hamiltonian path, if it exists, in a connected graph of n vertices is n-1. In considering hamiltonian circuit, we need only simple graphs because a hamiltonian circuit traverses every vertex exactly once. Hence it cannot include a self loop or a set of parallel edges.

Thus, a general graph may be made simple by removing parallel edges and self loops before looking for a hamiltonian circuit in it. When we remove the vertex v_1 say from G all the edges which are incident to v_1 are removed and the remaining graph is hamiltonian; we can choose any vertex and go through all the vertices of G such that no vertex is repeated except the initial vertex. Now, if we again add the removable vertex "V_1" of the

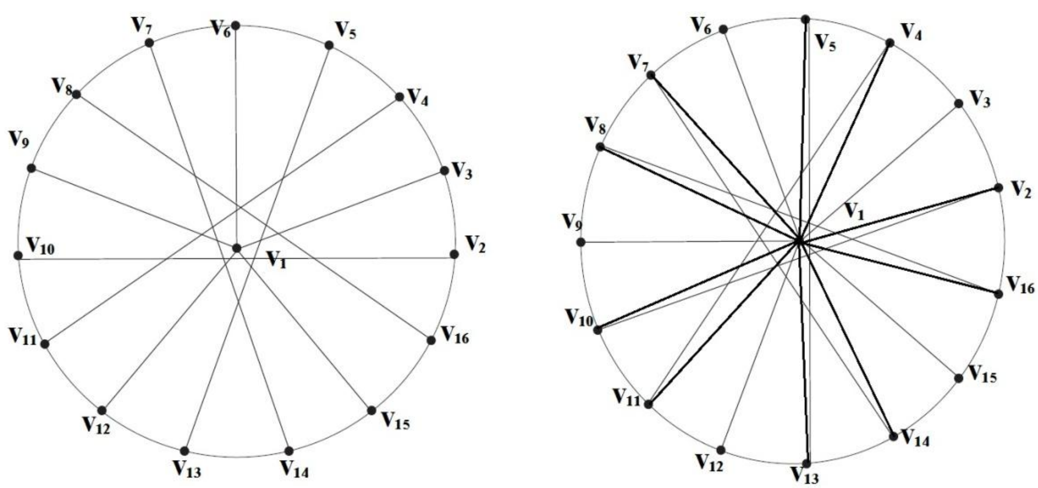


Hypohamiltonian

Hamiltonian

$V_1\,V_2\,V_3\,V_4\,V_5\,V_6\,V_7\,V_8 V_9\,V_{10}\,V_1$

Figure 1. Simple hypohamiltonian graphs of 10 vertices.



Hypohamiltonian

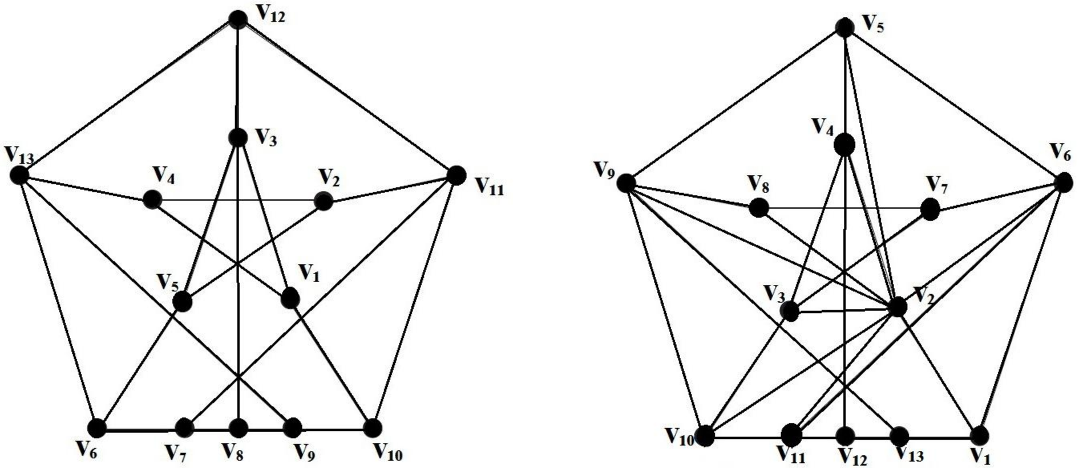
Hamiltonian

$V_1V_2V_3V_4V_5V_6V_7V_8V_9V_{10}V_{11}V_{12}V_{13}V_{14}V_{15}\,V_{16}\,V_1$

Figure 2. Simple hypohamiltonian graphs of 16 vertices.

graph G in such a way that degree of $V_1 = n-1$ where $\delta v_1 = n-1$, with graph G having n vertices, then we construct step by step super graphs of G to get the degree of removable vertex V_1 exactly equal to n-1 by adding an edge at each step between two vertices that are not already adjacent to V_l and finally V_1 has a path to all other vertices in a graph as in Lemma 1. For every graph G of order n, $\Delta(G) \leq n-1$. Thus, from any vertex in G there is a path P from all other vertices. Let that path P be the longest path in G; if P is in a cycle then P is a hamiltonian path. Let us suppose that P = {U = U_0, U_1, U_2 . . . U_t = V} of length t and p is contained in a cycle C = {U = U_0 ,U_1 . . . U_t = V} since vertices of "C" and "P" are equal, that is, V(C) = V(P). Otherwise, p would be a part of a longest path a contradiction assumes for the sake of contradiction t < n-1, that is, P is not hamiltonian path. Since G is connected there must be an edge of the form ($\alpha\ \beta$) such that $\alpha \in$V (p) = V(c) and $\beta \in$V (G) – V(C). If $\alpha \in$Ui then there is a path P = {β, α = Ui, Ui+1, . . . U1 U2 . . . Ui-1} with length t + 1 which is a contradiction, since P is the longest path in G. Thus, the graph G_1 obtained by making the degree of removable vertex equal to n-1 is hamiltonian. That proves the result.

Now above theorems shall be illustrated by counter



Hypohamiltonian

Hamiltonian

$V_1\ V_2\ V_3\ V_4\ V_5V_6V_7V_8V_9V_{10}V_{11}V_{12}\ V_{13}\ V_1$

Figure 3. Simple Hypohamiltonian graphs of 13 vertices.

examples. In Figure 1, the graph G of 10 vertices is hypohamiltonian but can be made hamiltonian by making the degree of vertex V_1 equal to n - 1 = 10 - 1 = 9; the graph so obtained is hamiltonian having hamiltonian cycle as $V_1 V_2 \ldots V_{10} V_1$

In Figures 2 and 3, graph with 16 and 13 vertices are both hypohamiltonian.

Both can be made hamiltonian by making the degree of vertex V_1 = n-1 = 16-1 = 15 (Figure 2) and vertex of V_1 = n - 1 = 13 - 1 = 12 (Figure 3).

Conclusion

In this study, a new condition for a graph to be Hamiltonian is presented. The conditions given in the form of Theorem 8 seem to be significant and interesting; and also explore new idea. Since the condition is applied to degree of vertices, it is proved in Theorem 8 that every hypohamiltonian graph is hamiltonian if we make the degree of removable vertex exactly equal to n-1 and illustrate it with simple hypohamiltonian graphs of 10, 16 and 13 vertices as shown in Figures 1, 2 and 3.

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